Econ 510a (second half)
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Fall 2004
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Solutions for Final Exam

1 Question 1

a) A Competitive Equilibrium with date-0 trading for this economy is a vector of prices $\{p_t\}_{t=0}^2$ and a vector of quantities $\{c_{it}^*\}_{t=0}^2$ for $i = A, B$ such that

(1) For $i = A, B$,

$$\begin{aligned} \{c_{it}^*\}_{t=0}^2 &= \arg\max \sum_{t=0}^{2} \beta^t u(c_{it}) \\ & s.t. \\ & \sum_{t=0}^{2} p_t c_{it} = \sum_{t=0}^{2} p_t \omega_{it} \end{aligned}$$

(2) $c_{At} + c_{Bt} = \omega_{At} + \omega_{Bt} \; for \; t = 0, 1, 2$

b) From the f.o.c. of the consumer's problem, we get:

$$\frac{\beta u'(c_{i,t+j})}{u'(c_{i,t})} = \frac{p_{t+j}}{p_t} \;\; \forall t, j$$

This together with budget constraint and market clearing condition determines the competitive equilibrium. Now there are 2 ways of solving this problem. The first is writing down all the f.o.c. for each of the 2 agents, the market clearing conditions for each of the 3 time periods and the 2 budget constraints and solve out for the prices and quantities (you won't need to use all of the equations to solve the system). The second and easiest is to use the fact that each agents consumption in every period is going to be a constant share of the aggregate endowment. This follows from the homotheticity of preferences. In other words we have that:

$$\begin{aligned} c_{At} &= \gamma (w_{At} + w_{Bt}) \;\; \forall t \\ c_{Bt} &= (1 - \gamma)(w_{At} + w_{Bt}) \;\; \forall t \end{aligned}$$

Now we are going to use the f.o.c. and we get (normalizing $p_0 = 1$):

$$
\begin{aligned}
\frac{\beta u'(c_{A1})}{u'(c_{A0})} &= p_1 \Rightarrow \\
\frac{\beta c_{A0}}{c_{A1}} &= p_1 \Longrightarrow \\
\frac{\beta \gamma w_o}{\gamma w_1} &= p_1 \Rightarrow \\
\frac{1}{2}\frac{4}{16} &= p_1 \Leftrightarrow \\
p_1 &= \frac{1}{8}
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\frac{\beta^2 u'(c_{A2})}{u'(c_{A0})} &= p_2 \Rightarrow \\
\frac{\beta^2 c_{A0}}{c_{A2}} &= p_2 \Longrightarrow \\
\frac{1}{4}\frac{\gamma 4}{\gamma 4} &= p_2 \Leftrightarrow \\
p_2 &= \frac{1}{4}
\end{aligned}
$$

We can now plug in the prices we found in consumer A's budget constraint and solve out for his share:

$$
\begin{aligned}
\gamma 4 + \frac{1}{8}\gamma 16 + \frac{1}{4}\gamma 4 &= \left(4 + \frac{4}{8} + 1\right) \Leftrightarrow \\
\gamma (4 + 2 + 1) &= \frac{11}{2} \Leftrightarrow \\
7\gamma &= \frac{11}{2} \Leftrightarrow \\
\gamma &= \frac{11}{14}
\end{aligned}
$$

and therefore the consumption in each period will be:

$$\begin{aligned} c_{0A} &= \frac{11}{14}4 = \frac{22}{7} \\ c_{1A} &= \frac{11}{14}16 = \frac{88}{7} \\ c_{2A} &= \frac{11}{14}4 = \frac{22}{7} \\ c_{0B} &= \frac{3}{14}4 = \frac{6}{7} \\ c_{1B} &= \frac{3}{14}16 = \frac{24}{7} \\ c_{2B} &= \frac{3}{14}4 = \frac{6}{7} \end{aligned}$$

Just to be sure lets verify that the budget constraint for consumer B holds:

$$\begin{aligned} 1 \cdot \frac{6}{7} + \frac{1}{8} \cdot \frac{24}{7} + \frac{1}{4} \cdot \frac{6}{7} &= \frac{1}{8} \cdot 12 \Leftrightarrow \\ \frac{21}{14} &= \frac{12}{8} \end{aligned}$$

and thus it does hold.

c) A Competitive Equilibrium with sequential trading for this economy is a sequence $\{c_{it}^*\}_{t=0}^2, \{a_{i,t+1}^*\}_{t=0}^2, \{R_t^*\}_{t=0}^2$ (where R_t^* means interest rate from t to $t+1$) for $i = A, B$ such that

(1) For $i = A, B$,

$$\begin{aligned} \{c_{it}^*, a_{i,t+1}^*\}_{t=0}^2 &= \arg\max \sum_{t=0}^2 \beta^t u(c_{it}) \\ & s.t. \\ & c_{it} + a_{i,t+1} = R_t^* a_{i,t} + \omega_{it} \\ & a_{i,3} \geq 0 \text{ (no-Ponzi condition)} \\ & a_{i,0} = 0, c_{it} \geq 0 \end{aligned}$$

(2) $c_{At}^* + c_{Bt}^* = \omega_{At} + \omega_{Bt}$ for $t = 0, 1, 2$
(3) $a_{A,t}^* + a_{B,t}^* = 0$ for $t = 0, 1, 2$

It will be the case that:

$$\begin{aligned} p_t R_t &= p_{t-1} \Leftrightarrow \\ R_t &= \frac{p_{t-1}}{p_t} \end{aligned}$$

Therefore:

$$\begin{aligned} R_1 &= \frac{p_0}{p_1} = \frac{1}{1/8} = 8 \\ R_2 &= \frac{p_1}{p_2} = \frac{1/8}{1/4} = \frac{1}{2} \end{aligned}$$

2 Question 2

a) A Recursive Competitive Equilibrium for the economy is a set of functions:

$$\begin{aligned} price\ function &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\ policy\ function &: k' = g\left(k, \overline{k}\right) \\ value\ function &: v\left(k, \overline{k}\right) \\ transition\ function &: \overline{k}' = G\left(\overline{k}\right) \end{aligned}$$

such that:
(1) $k' = g\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \overline{k}\right) &= \max_{\{c,k'\}} u\left(c, \overline{s}\right) + \beta v\left(k', \overline{k}'\right) \\ & s.t. \\ & c + k' = r\left(\overline{k}\right) k + (1-\delta)k + w\left(\overline{k}\right) \\ & \overline{k}' = G\left(\overline{k}\right) \\ \overline{s} &= \theta f\left(\overline{k}, \overline{n}\right) \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r\left(\overline{k}\right) &= f_1\left(\overline{k}, 1\right) \\ w\left(\overline{k}\right) &= f_2\left(\overline{k}, 1\right) \end{aligned}$$

(3) Consistency:

$$G\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

b) Solving for consumer's problem in the usual way, we get the Euler equation:

$$\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)} \left(r\left(\overline{k}_{t+1}\right) + 1 - \delta\right) = 1$$

Imposing equilibrium conditions and substituting in for the resource constraint we get:

$$\frac{\beta u'\left((1-\delta)\overline{k}_{t+1} + f\left(\overline{k}_{t+1}\right) - \overline{k}_{t+2}\right)}{u'\left((1-\delta)\overline{k}_t + f\left(\overline{k}_t\right) - \overline{k}_{t+1}\right)} \left(f'\left(\overline{k}_{t+1}\right) + 1 - \delta\right) = 1$$

Setting $\overline{k}_t = \overline{k}^*$ for every t, we obtain:

$$\begin{aligned} f'\left(\overline{k}^*\right) + 1 - \delta &= 1 \Leftrightarrow \\ f'\left(\overline{k}^*\right) &= \beta^{-1} - 1 + \delta \end{aligned}$$

which does not depend on θ. The reason is that the firms don't internalize the (negative) externality caused by their production.

c) The important thing to realize here is that the speed of convergence equals the inverse of the slope of the decision rule. The way one could calculate the speed of convergence of the aggregate capital stock to its steady state in a neighborhood of the steady state, is the following. We write the Euler equation as follows:

$$\beta u'\left((1-\delta)\,g(k)+f(g(k))-g(g(k))\right)\left(1-\delta+f'(k)\right)=u'\left((1-\delta)\,k+f(k)-g(k)\right)$$

Then we differentiate with respect to k.

We evaluate at the steady state $(g(k^*)=k^*)$

What we will end up with is a quadratic equation is $g'(k^*)$. One of the solutions can be proven to be in the interval $(-1,1)$. Given that we have an expression for $g'(k)$ we can easily approximate $g(k)$ by linear approximation around k^*.

d) The planning problem for this economy will be:

$$\max_{\{c_t,k_{t+1},s_t\}}\sum_{t=0}^{\infty}\beta^t u(c_t,s_t)$$

s.t.

$$\begin{aligned} c_t+k_{t+1} &= (1-\delta)\,k_t+f(k_t) \\ s_t &= \theta f(k_t) \end{aligned}$$

After substituting in the constraints and taking the f.o.c., we are able to derive the planner's Euler equation:

$$\beta\frac{\left(1-\delta+f'(k_{t+1})\right)u_1(c_{t+1},s_{t+1})+\theta f'(k_{t+1})\,u_2(c_{t+1},s_{t+1})}{u_1(c_t,s_t)}=1$$

It easy to see that the steady state level of capital from the planning problem will depend on θ and thus will differ from the competitive equilibrium. Therefore the competitive equilibrium is Pareto inferior. The reason is that the social planner takes into account the negative externality and internalizes it in his problem.

3 Question 3

a) Note: there is more than one way to state the planning problem for this economy.

We know that:

$$\begin{aligned} k' &= (1-\delta)\,k+Bk_i \Leftrightarrow \\ k_i &= B^{-1}\left[k'-(1-\delta)\,k\right] \end{aligned}$$

Noting the above, we can write the social planning problem for this economy (after normalizing $n = 1$) as:

$$v(k) = \max_{k'}\{\log\left[k - B^{-1}(k' - (1-\delta)k)\right]^a + \beta v(k')\}$$

b) The f.o.c. is:

$$\begin{aligned} -\frac{aB^{-1}}{k - B^{-1}(k' - (1-\delta)k)} + \beta v'(k') &= 0 \Leftrightarrow \\ \beta v'(k') &= \frac{aB^{-1}}{k_c} \end{aligned}$$

The envelope condition is:

$$v'(k) = \frac{a\left(1 + B^{-1}(1-\delta)\right)}{k_c}$$

Updating the envelope condition one period and substituting in the f.o.c. we get:

$$\begin{aligned} \beta\frac{a\left(1 + B^{-1}(1-\delta)\right)}{k'_c} &= \frac{aB^{-1}}{k_c} \Leftrightarrow \\ \frac{k'_c}{k_c} &= \beta(B + 1 - \delta) \Leftrightarrow \\ k'_c &= \beta(B + 1 - \delta)k_c \end{aligned}$$

Thus if g is the growth rate of k_c along the balanced growth path, then:

$$\begin{aligned} e^g &= \beta(B + 1 - \delta) \Leftrightarrow \\ g &= \ln(\beta(B + 1 - \delta)) \Leftrightarrow \\ g &\simeq \beta(B + 1 - \delta) - 1 \end{aligned}$$

where the last equation follows from the fact that $\ln(1 + x) \simeq x$. Since:

$$k_{t+1} = (1-\delta)k_t + B(k_t - k_{ct})$$

k is growing at rate g and so is k_i (since $k_i = k - k_c$). Notice that consumption is growing at lower rate:

$$\begin{aligned} c &= Ak_c^a \\ c' &= Ak_c'^a \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{c'}{c} &= \left(\frac{k'_c}{k_c}\right)^a \Rightarrow \\ e^{g_c} &= (e^g)^a \Leftrightarrow \\ g_c &= ga \end{aligned}$$

Alternative you can set up the planner's problem as follows:

$$v(k) = \max_{k_c} \left(\log Ak_c^a + \beta v\left((1-\delta)k + B(k-k_c)\right)\right)$$

Take the f.o.c and get:

$$\begin{aligned} ak_c^{-1} - \beta Bv'(k') &= 0 \Leftrightarrow \\ v'(k') &= \frac{aB^{-1}\beta^{-1}}{k_c} \end{aligned}$$

The envelope condition is:

$$v'(k) = \beta(1-\delta+B)v'(k')$$

Substituting the f.o.c. condition into the envelope condition above gives us:

$$\begin{aligned} v'(k) &= \beta(1-\delta+B)\frac{aB^{-1}\beta^{-1}}{k_c} \Leftrightarrow \\ v'(k) &= (1-\delta+B)\frac{aB^{-1}}{k_c} \end{aligned}$$

We update one period:

$$v'(k') = (1-\delta+B)\frac{aB^{-1}}{k'_c}$$

And finally we plug into the f.o.c. to get:

$$\begin{aligned} (1-\delta+B)\frac{aB^{-1}}{k'_c} &= \frac{aB^{-1}\beta^{-1}}{k_c} \Leftrightarrow \\ k'_c &= \beta(1-\delta+B)k_c \end{aligned}$$

which is exactly what we got with our original setup. From now on we just proceed in exactly the same way.

4 Question 4

a) The consumer's problem is:

$$\max_{\{c_{it}, c_{2t+1}, s_t\}} u(c_{1t}) + \beta u(c_{2t+1})$$

s.t.

$$\begin{aligned} w_t &= c_{1t} + s_t \\ c_{2t+1} &= R_{t+1}s_t + \lambda w_{t+1} \end{aligned}$$

We substitute in for c_{1t} and c_{2t+1} and take the f.o.c. w.r.t. s_t:

$$-u'(w_t - s_t) + \beta u'(R_{t+1}s_t + \lambda w_{t+1})R_{t+1} = 0$$

Assuming that prices are competitively determined, we have $w_{t+1} = (1-a)\, k_{t+1}^{a} (1+\lambda)^{-a}$ and $R_{t+1} = a k_{t+1}^{a-1} (1+\lambda)^{1-a}$. Moreover, given log utility and the fact that $s_t = k_{t+1}$ we have:

$$
\begin{aligned}
& -\frac{1}{(1-a)\, k_t^{a} (1+\lambda)^{-a} - k_{t+1}} \\
+\frac{\beta a k_{t+1}^{a-1} (1+\lambda)^{1-a}}{a k_{t+1}^{a-1} k_{t+1} (1+\lambda)^{1-a} + \lambda (1-a)\, k_{t+1}^{a} (1+\lambda)^{-a}} &= 0 \Leftrightarrow \\
\frac{\beta a k_{t+1}^{a-1} (1+\lambda)}{a k_{t+1}^{a} (1+\lambda) + \lambda (1-a) k_{t+1}^{a}} &= \frac{1}{(1-a)\, k_t^{a} (1+\lambda)^{-a} - k_{t+1}} \\
\frac{\beta a (1+\lambda)}{a k_{t+1} (1+\lambda) + \lambda (1-a) k_{t+1}} &= \frac{1}{(1-a)\, k_t^{a} (1+\lambda)^{-a} - k_{t+1}} \\
\beta a (1+\lambda)(1-a)\, k_t^{a} (1+\lambda)^{-a} - \beta a (1+\lambda)\, k_{t+1} &= k_{t+1} \left(a (1+\lambda) + \lambda (1-a) \right) \\
k_{t+1} \left(a + a\lambda + \lambda - a\lambda + \beta a (1+\lambda) \right) &= \beta a (1-a)(1+\lambda)^{1-a} k_t^{a} \Leftrightarrow \\
k_{t+1} &= \frac{\beta a (1-a)(1+\lambda)^{1-a}}{a + \lambda + \beta a (1+\lambda)} k_t^{a}
\end{aligned}
$$

and thus the steady state level of capital is:

$$
\overline{k} = \left(\frac{\beta a (1-a)(1+\lambda)^{1-a}}{a + \lambda + \beta a (1+\lambda)} \right)^{\frac{1}{1-a}}
$$

b) The steady state is dynamically efficient if:

$$
\begin{aligned}
f'\left(\overline{k}\right) &> 1 \Longrightarrow \\
a \left(\frac{\beta a (1-a)(1+\lambda)^{1-a}}{a + \lambda + \beta a (1+\lambda)} \right)^{\frac{a-1}{1-a}} &> 1 \Leftrightarrow \\
a \frac{a + \lambda + \beta a (1+\lambda)}{\beta a (1-a)(1+\lambda)^{1-a}} &> 1 \Leftrightarrow \\
\frac{a + \lambda + \beta a (1+\lambda)}{\beta (1-a)(1+\lambda)^{1-a}} &> 1
\end{aligned}
$$

Let's take the extreme case where $\lambda = 1$. Then the steady state level of capital will be:

$$
\begin{aligned}
\frac{a + 1 + 2\beta a}{\beta (1-a)\, 2^{1-a}} &> 1 \Leftrightarrow \\
a + 1 + 2\beta a &> \beta (1-a)\, 2^{1-a}
\end{aligned}
$$

Since $\beta < 1$, $(1-a) < 1$ and $2^{1-a} < 1$, it will be the case that the l.h.s is less than the r.h.s. (since the r.h.s. is clearly larger than 1). Therefore, when $\lambda = 1$, the steady state level of capital is dynamically efficient for *any* value of a

and β. If we define $h(\lambda) \equiv \frac{a+\lambda+\beta a(1+\lambda)}{\beta(1-a)(1+\lambda)^{1-a}}$, and notice that $h(\cdot)$ is a continuous function, it becomes apparent that there will be some "cutoff" value $\overline{\lambda} \in (0,1)$, above which the $h(\lambda) > 1$, for all a and β (and thus the steady state level of capital is dynamically efficient). The intuition behind this result is that the higher the labor income the old generation receives, the less they need to save when they are young and thus the capital stock will be lower as well. Thus the probability that the steady state level of capital exceeds the golden rule, will also be lower.