1. (a) \( V(w) = \max \left[ U(T-w', w) + \beta V(w') \right] \)

\( b = T-w' = \) time spent making bread

= amount of bread consumed today

\( w' = \) time spent making grape juice (tomorrow's wine)

\( w = \) stock of grape juice carried over from last period (it is now wine, whose only use is in consumption)

(b) f.o.c. \(-U_b(T-w', w) + \beta V'(w') = 0\)

envelope condition \( V'(w) = U_w(b, w) \)

\[ \Rightarrow V'(w') = U_w(b', w') \]

\[ \Rightarrow U_b(b, w) - \beta U_w(b', w') = 0 \]

(c) steady state:

\[ U_b(\bar{b}, \bar{w}) = \beta U_w(\bar{b}, \bar{w}) \]

\( \bar{b} = T-\bar{w} \Rightarrow U_b(T-\bar{w}, \bar{w}) = \beta U_w(T-\bar{w}, \bar{w}) \)

Solve this equation for \( \bar{w} \).
decision rule: \( w' = h(w) \)
\[ b = T - w' = T - h(w) \]
\[ b' = T - w'' = T - h(h(w)) \]

Plugging these into Euler equation yields:

\[ u_b(T-h(w), w) = \beta u_w(T-h(h(w)), h(w)) \]

an identity in \( w \). Now take derivatives of both sides with respect to \( w \), evaluate at \( \bar{w} \), and solve for \( h'(\bar{w}) \):

\[
-u_{bb}(\bar{b}, \bar{w}) h'(\bar{w}) + u_{bw}(\bar{b}, \bar{w}) = -\beta u_{bw}(\bar{b}, \bar{w}) (h'(\bar{w}))^2 + \beta u_{ww}(\bar{b}, \bar{w}) h'(\bar{w})
\]

or, dropping arguments,

\[-u_{bb} h' + u_{bw} = -\beta u_{bw} (h')^2 + \beta u_{ww} h' \]

\[
\frac{1}{\beta} \frac{u_{bb}}{u_{bw}} h' - \frac{1}{\beta} = (h')^2 - \frac{u_{bw}}{u_{ww}} h'
\]

\[ \Rightarrow (h')^2 - \left( \frac{u_{ww}}{u_{bw}} + \frac{1}{\beta} \frac{u_{bb}}{u_{bw}} \right) h' + \frac{1}{\beta} = 0 \]
2. (b) \[ \frac{u'(c^1_t)}{u'(c^2_t)} = \frac{\lambda^1}{\lambda^2} \]

\[ \Rightarrow \frac{c^2_t}{c^1_t} = \frac{\lambda^1}{\lambda^2} \]

\[ c^1_t = \frac{\lambda^2}{\lambda^1} c^2_t \]

\[ c^1_t + c^2_t = \bar{w}_t = \left(1 + \frac{\lambda^2}{\lambda^1}\right) c^2_t \]

\[ \Rightarrow c^i_t = \theta^i \bar{w}_t \]

\[ q_{k\ell} = \beta \frac{u'(c^i_{t+1})}{u'(c^i_t)} \left( \frac{p(s_{t+1} = k | s_t = \ell)}{p(s_{t+1} = k | s_t = \ell)} \right) \]

\[ = \beta \frac{\theta^i \bar{w}_t}{\theta^i \bar{w}_{t+1}} \left( \frac{p(s_{t+1} = k | s_t = \ell)}{p(s_{t+1} = k | s_t = \ell)} \right) \]

\[ \Rightarrow q_{11} = \beta \frac{2}{2} \Pi = \beta \Pi \]
\[ q_{12} = \beta \frac{2}{1} (1-\pi) = 2 \beta (1-\pi) \]

\[ q_{21} = \beta \frac{1}{2} \pi = \frac{\beta \pi}{2} \]

\[ q_{22} = \beta \frac{1}{1} (1-\pi) = \beta (1-\pi) \]

(c) \[ p_i^k = \text{price of a } k\text{-period-ahead bond in state } i \text{ today} \]

\[ q(i) = \text{Arrow security price} \]

\[ p_i^2 = q(i_1) p_1^1 + q(i_2) p_2^1 \]

\[ p_i^1 = q(i_1) + q(i_2) \]

\[ = q(i_1) (q_{11} + q_{12}) + q(i_2) (q_{21} + q_{22}) \]
3. (b) Typical Euler equations:

\[ q_t u'(c_t^i) = \beta u'(c_{t+1}^i) \]

\[ q_t (c_t^i)^{-\sigma_i} = \beta (c_{t+1}^i)^{-\sigma_i}, \quad i = A, B \]

Conjecture \( c_t^i = \bar{w} \quad \forall \ i \) and \( t \)

Then both Euler equations can be satisfied if \( q_t = \beta \quad \forall \ t \).

The eq. gross interest rate is then \( q^{-1} \).

(c) Suppose we are in a steady state \( \phi \)/

\[ c_{t+1}^i = (1 + g) c_t^i, \quad i = A, B \]

In a steady state \( q_t = q \quad \forall \ t \)

Euler equation of type-A consumer:

\[ q_t (c_t^A)^{-\sigma_A} = \beta ( (1 + g) c_t^A )^{-\sigma_A} \]

\[ \Rightarrow q = \beta (1 + g)^{-\sigma_A} \quad (\star) \]
But the Euler equation of the type-B consumer yields:

\[ q = \beta (1+g)^{-\sigma_B} \quad (**) \]

(**) and (***) cannot both be satisfied unless either \( g = 0 \) or \( \sigma_A = \sigma_B \).

(Note that \( c_t^i = w_t^i \) is a feasible allocation \( i \) and \( t \).)
budget balance requires: \( g = \pi^t \cdot w(K^t, h^t) \cdot h^t \)

prices: \( p(K, h), w(K, h) \) (but not \( h \) could be eliminated using \( h = F_h(K) \))

agg. ft.: \( R' = F_K(K) \)
\( h = F_h(K) \)

ind. decision rules: \( k' = f_K(k, K) \)
\( h = f_h(k, K) \)

tax rule: \( \tau = T(K) \)

s.t.: (a) prices are competitive (i.e. marginal products)

(b) govt. budget is balanced in every period

(i.e., \( T(K) = \frac{g}{w(K, h) \cdot h} \), where \( h = L(K) \))

(c) individuals optimize: \( f_k + f_h \) solve the consumer's problem, taking the functions \( T, F_K, F_h, \) and \( (r, w) \) as given.

(d) \( f_K(K, K) = F_K(K) \& K \)
\( f_h(K, K) = F_h(K) \& K \)
individual solve:

$$V(k, \bar{e}) = \max_{k', h} \left[ \log(c) + \alpha \log(1-h) + \beta V(k', \bar{e}') \right],$$

where

$$c + k' - (1-r)k = r(\bar{e}, \bar{h})k + (1-T(\bar{e}))\bar{w}(\bar{e}, \bar{h})h,$$

taking as given \( \bar{k}' = F_{k}(\bar{k}) \) and \( \bar{h} = F_{h}(\bar{k}) \).

(b) solution to individual consumer problem:

f. o. c. \( k' : -\frac{1}{c} + \beta V_{1}(k', \bar{e}') = 0 \)

\[ h : \frac{(1-c)w}{c} = \frac{A}{1-h} \]

w.r.t. \( V_{1}(k, \bar{e}) = \frac{1}{c} \left( r(\bar{e}, \bar{h}) + 1 - \delta \right) \)

in steady state:

$$-\frac{1}{c} + \beta \frac{1}{b} \left( r(\bar{e}, \bar{h}) + 1 - \delta \right) = 0$$

so \( r(\bar{e}, \bar{h}) \) does not depend on \( g \); and

\( r(\bar{e}, \bar{h}) \) depends on \( \frac{k}{h} \), so this ratio does not depend on \( g \).
(c) In steady state,\[ g = \overline{h} \, w(\overline{k}, \overline{h}) \, \tau \]
\[ = \overline{h} \, (1-\alpha) \, \overline{k}^\alpha \, \overline{h}^{-\alpha} \, \tau \]
\[ = (1-\alpha) \, \overline{k}^\alpha \, \overline{h}^{1-\alpha} \, \tau \]

from intertemporal f.o.c.:
\[ \frac{(1-\tau)\, w}{c} = \frac{A}{1-h} \]

\[ \frac{w}{c} = \frac{(1-\alpha) \, \overline{k}^\alpha \, \overline{h}^{-\alpha}}{\overline{k}^\alpha \, \overline{h}^{1-\alpha} - \delta \overline{k} - g} \]

\[ = \frac{(1-\alpha) \, (\overline{k}/\overline{h})^\alpha}{\overline{h} \left( (\overline{k}/\overline{h})^\alpha - \delta \left( \overline{k}/\overline{h} \right) - \frac{g}{\overline{h}} \right) } \]
\[ g = \frac{1}{\bar{h}} (1 - \alpha) \left( \frac{k}{\bar{h}} \right)^{\alpha} \]

\[ \tau = \frac{g}{\bar{h}} \frac{1}{(1 - \alpha) \left( \frac{k}{\bar{h}} \right)^{\alpha}} \]

\[ (1 - \tau) \left( (1 - \alpha) \left( \frac{k}{\bar{h}} \right)^{\alpha} \right) = (1 - \alpha) \left( \frac{k}{\bar{h}} \right)^{\alpha} - \frac{g}{\bar{h}} \]

\[ \Rightarrow \frac{(1 - \alpha) \left( \frac{k}{\bar{h}} \right)^{\alpha} - \frac{g}{\bar{h}}}{\left( \frac{k}{\bar{h}} \right)^{\alpha} - \delta \left( \frac{k}{\bar{h}} \right) - \frac{g}{\bar{h}}} = \frac{A \bar{h}}{1 - \bar{h}} \]

\[ \frac{k}{\bar{h}} \text{ does not depend on } g. \]

Take the total derivative of this equation with respect to \( g \) and \( \bar{h} \) in order to calculate \( \frac{d\bar{h}}{dg} \).