Suggested Solutions to Midterm Exam
Econ 511b (Part I), Spring 2004

1. **Consider a competitive equilibrium neoclassical growth model populated by identical consumers whose preferences over consumption streams are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Consumers do not value leisure and are endowed with k_0 units of capital in period 0 and with one unit of labor in every period. Consumers rent the services of capital and labor in competitive markets to profit-maximizing firms with identical constantreturns-to-scale production functions. Capital depreciates fully in one period. The government, which balances its budget in every period, taxes capital income at a (time-invariant) proportional rate τ and returns the proceeds to consumers in the form of a lump-sum subsidy to income.**

   (a) **Carefully define a recursive competitive equilibrium for this economy.**

   A recursive competitive equilibrium for the economy with capital income taxation is a set of functions:

$$\begin{aligned}
price\ function &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\
policy\ function &: k' = g\left(k, \overline{k}\right) \\
value\ function &: v\left(k, \overline{k}\right) \\
Taxation &: T\left(\overline{k}\right) \\
transition\ function &: \overline{k}' = G\left(\overline{k}\right)
\end{aligned}$$

   such that:

   (1) Given $r\left(\overline{k}\right), w\left(\overline{k}\right)$, $k' = g\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned}
v\left(k, \overline{k}\right) &= \max_{\{c,k'\}} u(c) + \beta v\left(k', \overline{k}'\right) \\
& s.t. \\
& c + k' = (1-\tau) r\left(\overline{k}\right) k + w\left(\overline{k}\right) + T\left(\overline{k}\right) \\
& \overline{k}' = G\left(\overline{k}\right)
\end{aligned}$$

   (2) Price is competitively determined:

$$\begin{aligned}
r\left(\overline{k}\right) &= F_1\left(\overline{k}, 1\right) \\
w\left(\overline{k}\right) &= F_2\left(\overline{k}, 1\right)
\end{aligned}$$

   (3) Government balances its budget

$$T\left(\overline{k}\right) = \tau r\left(\overline{k}\right) \overline{k}$$

   (4) Consistency:

$$G\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

(b) Show that the competitive equilibrium allocation for this economy is identical to the allocation chosen by a social planner whose preferences over consumption streams are given by $\sum_{t=0}^{\infty} \widetilde{\beta}^t u(c_t)$, where $\widetilde{\beta}$ is a "distorted" discount rate that differs from the discount rate of a typical consumer. Express the distorted discount rate in terms of τ and β.

We can solve for F.O.C. as

$$u'(c) = \beta v_1\left(k', \overline{k}'\right)$$

Envelope condition gives us

$$v_1\left(k, \overline{k}\right) = u'(c)(1-\tau) r\left(\overline{k}\right)$$

Therefore, we get the Euler equation as

$$\begin{aligned} & u'(c) = \beta(1-\tau) u'(c') r\left(\overline{k}'\right) \\ \Rightarrow\ & u'\left((1-\tau) r\left(\overline{k}\right) k + w\left(\overline{k}\right) + T\left(\overline{k}\right) - \overline{k}'\right) \\ =\ & \beta(1-\tau) u'\left((1-\tau) r\left(\overline{k}'\right) k' + w\left(\overline{k}'\right) + T\left(\overline{k}'\right) - \overline{k}''\right) r\left(\overline{k}'\right) \end{aligned}$$

After plug into price $\left(r\left(\overline{k}\right), w\left(\overline{k}\right)\right)$, taxation $\left(T\left(\overline{k}\right)\right)$, and equilibrium condition $\left(k = \overline{k}\right)$, it becomes

$$\begin{aligned} & u'\left((1-\tau) F_1\left(\overline{k}, 1\right) \overline{k} + F_2\left(\overline{k}, 1\right) + \tau F_1\left(\overline{k}, 1\right) \overline{k} - \overline{k}'\right) \\ =\ & \beta(1-\tau) u'\left((1-\tau) F_1\left(\overline{k}', 1\right) \overline{k}' + F_2\left(\overline{k}', 1\right) + \tau F_1\left(\overline{k}', 1\right) \overline{k}' - \overline{k}''\right) F_1\left(\overline{k}', 1\right) \\ \Rightarrow\ & u'\left(F\left(\overline{k}, 1\right) - \overline{k}'\right) = \beta(1-\tau) u'\left(F\left(\overline{k}', 1\right) - \overline{k}''\right) F_1\left(\overline{k}', 1\right) \end{aligned}$$

Compare this to the Euler equation for central planning problem

$$u'\left(F\left(\overline{k}, 1\right) - \overline{k}'\right) = \widetilde{\beta} u'\left(F\left(\overline{k}', 1\right) - \overline{k}''\right) F_1\left(\overline{k}', 1\right)$$

We have

$$\widetilde{\beta} = \beta(1-\tau)$$

Therefore, the competitive equilibrium allocation for this economy is identical to the allocation chosen by a social planner whose preferences over consumption streams are given by $\sum_{t=0}^{\infty} \widetilde{\beta}^t u(c_t)$, where $\widetilde{\beta} = \beta(1-\tau)$.

(c) Does the result from part (b) continue to hold if consumers value leisure and the government taxes both labor income and capital income at a proportional rate τ? Explain why or why not.

We start from defining an equilibrium with valued leisure.

A recursive competitive equilibrium for the economy with valued leisure is a set of functions:

$$\begin{aligned} price\ function &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\ policy\ function &: k^{'} = g_k\left(k, \overline{k}\right), l = g_n\left(k, \overline{k}\right) \\ value\ function &: v\left(k, \overline{k}\right) \\ Taxation &: T\left(\overline{k}\right) \\ transition\ function &: \overline{k}^{'} = G\left(\overline{k}\right) \end{aligned}$$

such that:
(1) Given $r\left(\overline{k}\right), w\left(\overline{k}\right)$, $k^{'} = g_k\left(k, \overline{k}\right), n = g_n\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \overline{k}\right) &= \max_{\left\{c,n,k'\right\}} u\left(c, 1-n\right) + \beta v\left(k^{'}, \overline{k}^{'}\right) \\ & s.t. \\ & c + k^{'} = \left(1-\tau\right)\left(r\left(\overline{k}\right)k + w\left(\overline{k}\right)n\right) + T\left(\overline{k}\right) \\ & \overline{k}^{'} = G\left(\overline{k}\right) \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r\left(\overline{k}\right) &= F_1\left(\overline{k}, g_n\left(\overline{k}, \overline{k}\right)\right) \\ w\left(\overline{k}\right) &= F_2\left(\overline{k}, g_n\left(\overline{k}, \overline{k}\right)\right) \end{aligned}$$

(3) Government balances its budget

$$T\left(\overline{k}\right) = \tau\left(r\left(\overline{k}\right)\overline{k} + w\left(\overline{k}\right)g_n\left(\overline{k}, \overline{k}\right)\right)$$

(4) Consistency:

$$G\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

Solve for this, we can get the F.O.C. as

$$\begin{aligned} \{k'\} &: u_1\left(c, 1-n\right) = \beta\left(1-\tau\right)u_1\left(c', 1-n'\right)r\left(\overline{k}'\right) \\ \{n\} &: u_1\left(c, 1-n\right)\left(1-\tau\right)w\left(\overline{k}\right) = u_2\left(c, 1-n\right) \end{aligned}$$

After plug into c, price $(r\left(\overline{k}\right), w\left(\overline{k}\right))$, taxation $(T\left(\overline{k}\right))$, and equilibrium condition $(k = \overline{k})$, the F.O.C. becomes

$$\begin{aligned} \{k'\} &: u_1\left(F\left(\overline{k}, n\right) - \overline{k}', 1-n\right) = \beta\left(1-\tau\right)u_1\left(F\left(\overline{k}', n'\right) - \overline{k}'', 1-n'\right)F_1\left(\overline{k}', n'\right) \\ \{n\} &: \left(1-\tau\right)u_1\left(F\left(\overline{k}, n\right) - \overline{k}', 1-n\right)F_2\left(\overline{k}, n\right) = u_2\left(F\left(\overline{k}, n\right) - \overline{k}', 1-n\right) \end{aligned}$$

where $\overline{k}' = g_k\left(\overline{k}, \overline{k}\right)$ and $\overline{k}'' = g_k\left(\overline{k}', \overline{k}'\right)$. Compare this to the Euler equation for central planning problem

$$\begin{aligned}
\{k'\} &: u_1\left(F\left(\overline{k}, n\right) - \overline{k}', 1-n\right) = \widetilde{\beta} u_1\left(F\left(\overline{k}', n'\right) - \overline{k}'', 1-n'\right) F_1\left(\overline{k}', n'\right) \\
\{n\} &: u_1\left(F\left(\overline{k}, n\right) - \overline{k}', 1-n\right) F_2\left(\overline{k}, n\right) = u_2\left(F\left(\overline{k}, n\right) - \overline{k}', 1-n\right)
\end{aligned}$$

we can see that the result from part (b) won't hold since the F.O.C. with respect to n is different, even if $\widetilde{\beta} = \beta(1-\tau)$.

2. **Consider a real-business-cycle model with variable capital utilization. There is a representative consumer whose preferences are given by:**

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where u is strictly increasing and strictly concave. The aggregate resource constraint reads:$c_t + x_t = f(k_t, n_t, z_t, h_t) \equiv \exp(z_t)(h_t k_t)^{\alpha} n_t^{1-\alpha}$. **The variable $h_t \geq 0$ measures the "utilization level" of machines, and it is a choice variable at all points in time. Capital accumulates according to:**

$$k_{t+1} = (1 - \delta(h_t)) k_t + x_t,$$

where the function $\delta(h_t)$ is given by:

$$\delta(h_t) = \delta_0 + \delta_1 \frac{h_t^{\omega}}{\omega}.$$

The parameters δ_0 and δ_1 are positive and the parameter ω is greater than 1. Thus, the depreciation rate of capital in period t is an increasing and convex function of the utilization level h_t.

Individuals do not value leisure and supply labor inelastically. Without loss of generality, set labor resources n_t equal to 1. The productivity variable z_t is stochastic and evolves according to:

$$z_{t+1} = \rho z_t + \epsilon_{t+1},$$

where $\{\epsilon_{t+1}\}_{t=0}^{\infty}$ is an independent and identically distributed sequence of shocks drawn from a $N(0, \sigma_\epsilon^2)$ distribution and $|\rho| < 1$.

(a) **Carefully define a recursive competitive equilibrium for this economy. Assume that consumers own the factors of production and rent their services to firms in every period in competitive markets. (Hint: Let the firm's production function be** $F(k_t, n_t, z_t, h_t) \equiv f(k_t, n_t, z_t, h_t) + (1 - \delta(h_t)) k_t$)

A recursive competitive equilibrium for the stochastic economy with variable capital utilization is a set of functions:

$$\begin{aligned} price\ function &: r\left(\overline{k},z\right),w\left(\overline{k},z\right) \\ policy\ function &: k^{'}=g\left(k,\overline{k},z\right) \\ value\ function &: v\left(k,\overline{k},z\right) \\ capital\ utilization &: h\left(\overline{k},z\right) \\ transition\ function &: \overline{k}^{'}=G\left(\overline{k},z\right),z'=\rho z+\epsilon' \end{aligned}$$

such that:

(1) Given $r\left(\overline{k},z\right),w\left(\overline{k},z\right)$, $k^{'}=g\left(k,\overline{k},z\right)$ and $v\left(k,\overline{k},z\right)$ solve consumer's problem:

$$\begin{aligned} v\left(k,\overline{k},z\right) &= \max_{\{c,k'\}} u\left(c\right)+\beta E_{z'|z}v\left(k^{'},\overline{k}^{'},z'\right) \\ & s.t. \\ & c+k^{'}=r\left(\overline{k},z\right)k+w\left(\overline{k},z\right) \\ & \overline{k}^{'}=G\left(\overline{k},z\right) \\ & z'=\rho z+\epsilon' \end{aligned}$$

(2) Firm solves the problem

$$\begin{aligned} & \max_{\{k,n,h\}} f\left(k,n,z,h\right)+\left(1-\delta\left(h\right)\right)k_t-r\left(\overline{k},z\right)k-w\left(\overline{k},z\right)n \\ & s.t. \\ & f\left(k,n,z,h\right)\equiv\exp\left(z\right)\left(hk\right)^{\alpha}n^{1-\alpha} \\ & \delta\left(h\right)=\delta_0+\delta_1\frac{h^{\omega}}{\omega} \end{aligned}$$

or

$$\max_{\{k,n,h\}} \exp\left(z\right)\left(hk\right)^{\alpha}n^{1-\alpha}+\left(1-\left(\delta_0+\delta_1\frac{h^{\omega}}{\omega}\right)\right)k-r\left(\overline{k},z\right)k-w\left(\overline{k},z\right)n$$

which leads to the equilibrium function

$$\begin{aligned} \{h\} &: \alpha k^{\alpha}\exp\left(z\right)h^{\alpha-1}n^{1-\alpha}=\delta_1 h^{\omega-1}k \\ &\Rightarrow h\left(\overline{k},z\right)=\left(\frac{\alpha}{\delta_1}\right)^{\frac{1}{\omega-\alpha}}\exp\left(\frac{z}{\omega-\alpha}\right)\overline{k}^{\frac{\alpha-1}{\omega-\alpha}} \\ \{k\} &: r\left(\overline{k},z\right)=\alpha h\left(\overline{k},z\right)\exp\left(z\right)\left(h\left(\overline{k},z\right)\overline{k}\right)^{\alpha-1}+1-\left(\delta_0+\delta_1\frac{h\left(\overline{k},z\right)^{\omega}}{\omega}\right) \\ \{n\} &: w\left(\overline{k},z\right)=\left(1-\alpha\right)\exp\left(z\right)\left(h\left(\overline{k},z\right)\overline{k}\right)^{\alpha} \end{aligned}$$

(3) Consistency:

$$G\left(\overline{k},z\right)=g\left(\overline{k},\overline{k},z\right)$$

(b) What is the deterministic steady-state value of the aggregate capital stock in the competitive equilibrium of this economy? (You need to find an equation that determines the steady-state capital stock but you do not have to solve it.)

The Euler equation for the consumer is

$$u'(c_t) = \beta E_t \left[u'(c_{t+1}) \, r\left(\overline{k}_{t+1}, z_{t+1}\right) \right]$$

In deterministic steady state, it becomes

$$r\left(\overline{k}^*, 0\right) = \beta^{-1}$$

Plug in the equilibrium price function from firm's F.O.C., we have

$$\begin{aligned}
&\alpha h\left(\overline{k}, \overline{z}\right) \exp\left(\overline{z}\right) \left(h\left(\overline{k}, \overline{z}\right) \overline{k}\right)^{\alpha-1} + 1 - \left(\delta_0 + \delta_1 \frac{h\left(\overline{k}, \overline{z}\right)^{\omega}}{\omega}\right) = \beta^{-1} \\
\Rightarrow \quad & \alpha h\left(\overline{k}, 0\right) \left(h\left(\overline{k}, 0\right) \overline{k}\right)^{\alpha-1} + 1 - \left(\delta_0 + \delta_1 \frac{h\left(\overline{k}, 0\right)^{\omega}}{\omega}\right) = \beta^{-1}
\end{aligned}$$

where $h\left(\overline{k}, 0\right) = \left(\frac{\alpha}{\delta_1}\right)^{\frac{1}{\omega-\alpha}} \overline{k}^{\frac{\alpha-1}{\omega-\alpha}}$. This equation defines the deterministic steady-state value of the aggregate capital stock $\overline{k}^*$ in the competitive equilibrium of this economy.

(c) Explain how to use linearization methods to obtain an approximation to the stochastic behavior of the competitive equilibrium. You do not have to carry out any explicit computations, but you should provide a careful, detailed description of how to perform the required computations.

We need to proceed in several steps to obtain a linear approximation to the stochastic behavior of the competitive equilibrium.

Step1. Express the stochastic Euler equation as a function of $\overline{k}_{t+2}$, $\overline{k}_{t+1}$, $\overline{k}_t$, z_{t+1}, z_t, which can be done by plugging in the equilibrium price ($r\left(\overline{k}, z\right)$ and $w\left(\overline{k}, z\right)$) and capital utilization $h\left(\overline{k}, z\right)$. For an illustration, we can find the stochastic equation as

$$\begin{aligned}
& u'(c_t) = \beta E_t \left[u'(c_{t+1}) \, r\left(\overline{k}_{t+1}, z_{t+1}\right) \right] \\
\Rightarrow \quad & u'\left(r\left(\overline{k}_t, z_t\right) k_t + w\left(\overline{k}_t, z_t\right) - \overline{k}_{t+1} \right) \\
= \quad & \beta E_t \left[u'\left(r\left(\overline{k}_{t+1}, z_{t+1}\right) k_{t+1} + w\left(\overline{k}_{t+1}, z_{t+1}\right) - \overline{k}_{t+2} \right) r\left(\overline{k}_{t+1}, z_{t+1}\right) \right] \\
\Rightarrow \quad & u'\left(\exp(z_t) \left(h\left(\overline{k}_t, z_t\right) \overline{k}_t\right)^{\alpha} + 1 - \left(\delta_0 + \delta_1 \frac{h\left(\overline{k}_t, z_t\right)^{\omega}}{\omega}\right) - \overline{k}_{t+1} \right) \\
= \quad & \beta E_t \left[\begin{array}{c} u' \left(\begin{array}{c} \exp(z_{t+1}) \left(h\left(\overline{k}_{t+1}, z_{t+1}\right) \overline{k}_{t+1}\right)^{\alpha} + \\ 1 - \left(\delta_0 + \delta_1 \frac{h\left(\overline{k}_{t+1}, z_{t+1}\right)^{\omega}}{\omega}\right) - \overline{k}_{t+2} \end{array} \right) \\ \left(\begin{array}{c} \alpha h\left(\overline{k}_{t+1}, z_{t+1}\right) \exp(z_{t+1}) \left(h\left(\overline{k}_{t+1}, z_{t+1}\right) \overline{k}_{t+1}\right)^{\alpha-1} + \\ 1 - \left(\delta_0 + \delta_1 \frac{h\left(\overline{k}_{t+1}, z_{t+1}\right)^{\omega}}{\omega}\right) \end{array} \right) \end{array} \right]
\end{aligned}$$

where $h\left(\overline{k},z\right)=\left(\frac{\alpha}{\delta_1}\right)^{\frac{1}{\omega-\alpha}}\exp\left(\frac{z}{\omega-\alpha}\right)\overline{k}^{\frac{\alpha-1}{\omega-\alpha}}$.

Step 2. Take the first-order Taylor approximation of stochastic Euler equation around the deterministic steady state $\left(\overline{k}^*,\overline{z}\right)$, which transforms the Euler equation into an expectational linear second-order difference equation with respect to deviation from the steady state $(\widehat{k}_{t+2},\widehat{k}_{t+1},\widehat{k}_t,\widehat{z}_{t+1},\widehat{z}_t)$. For an illustration, we can obtain the following equation:

$$\begin{aligned} LHS &= a_{k0}\widehat{k}_t+a_{k1}\widehat{k}_{t+1}+a_{z0}\widehat{z}_t \\ RHS &= E_t\left(b_{k1}\widehat{k}_{t+1}+b_{k2}\widehat{k}_{t+2}+b_{z1}\widehat{z}_{t+1}\right) \end{aligned}$$

Step 3. Conjecture the solution form as

$$\begin{aligned} \widehat{k}_{t+1} &= c_k\widehat{k}_t+c_z\widehat{z}_t \\ \widehat{z}_{t+1} &= \rho\widehat{z}_t+\epsilon_{t+1} \end{aligned}$$

and plug this into the expectational linear difference equation. Now our equation becomes

$$\begin{aligned} LHS &= a_{k0}\widehat{k}_t+a_{k1}\left(c_k\widehat{k}_t+c_z\widehat{z}_t\right)+a_{z0}\widehat{z}_t \\ RHS &= E_t\left(b_{k1}\left(c_k\widehat{k}_t+c_z\widehat{z}_t\right)+b_{k2}\left(c_k\widehat{k}_{t+1}+c_z\widehat{z}_{t+1}\right)+b_{z1}\widehat{z}_{t+1}\right) \\ &= b_{k1}\left(c_k\widehat{k}_t+c_z\widehat{z}_t\right)+b_{k2}\left(c_k\left(c_k\widehat{k}_t+c_z\widehat{z}_t\right)+c_zE_t\left(\widehat{z}_{t+1}\right)\right)+b_{z1}E_t\left(\widehat{z}_{t+1}\right) \\ &= b_{k1}\left(c_k\widehat{k}_t+c_z\widehat{z}_t\right)+b_{k2}\left(c_k\left(c_k\widehat{k}_t+c_z\widehat{z}_t\right)+c_z\rho\widehat{z}_t\right)+b_{z1}\rho\widehat{z}_t \end{aligned}$$

which is a deterministic equation with respect to $\widehat{k}_t$ and $\widehat{z}_t$. Since this equation is an identity, after collecting terms we should equate the coefficients for $\widehat{k}_t$ and $\widehat{z}_t$. This allows us to solve for the coefficients c_k and c_z.

Step 4. After getting the evolution law of state variable, we can get the behavior of other endogenous variables (control variables) by exploiting the optimal choice of individual agents. For example, we can linearize the capital utilization $h\left(\overline{k},z\right)$ around the steady state and get

$$\widehat{h}_t=d_k\widehat{k}_t+d_z\widehat{z}_t,$$

which is the evolution law of $\widehat{h}_t$ as a function of $\widehat{k}_t$ and $\widehat{z}_t$.

Step 5. We can analyze the behavior of the economy by finding impulse response behavior or simulating the economy.

3. **Consider an exchange economy with two infinitely-lived consumers with identical preferences given by:**

$$E_0\sum_{t=0}^{\infty}\beta^t\log\left(c_t\right).$$

Both of the consumers have random endowments that depend on the (exogenous) state variable s_t. The s_t's are statistically independent random variables with identical distributions. Specifically, for each t, $s_t = H$ with probability π and $s_t = L$ with probability $1-\pi$, where π does not depend on time or on the previous realizations of the states. If $s_t = H$, then the first consumer's endowment is 2 and the second consumer's endowment is 1; if $s_t = L$, then the first consumer's endowment is 1 and the second consumer's endowment is 0. Markets are complete.

(a) Carefully define a competitive equilibrium with date-0 trading for this economy. (Assume that consumers make decisions before observing the realization of the state in period 0.)

A competitive equilibrium with date-0 trading is a pair of price and allocation $\left\{p_t\left(s^t\right), \left\{c_t^i\left(s^t\right)\right\}_{i=A,B}\right\}_{t=0}^{\infty}$ for $s_t = H, L$, such that

(1) Consumer i solves the problem

$$\max\ E_0 \sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\left(s^t\right)\right)$$

$s.t.$

$$\sum_{t=0}^{\infty}\sum_{s^t} p_t\left(s^t\right) c_t^i\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} p_t\left(s^t\right) w_t^i\left(s_t\right)$$

(2) Market clear, i.e.

$$c_t^A\left(s^t\right) + c_t^B\left(s^t\right) = w_t^A\left(s_t\right) + w_t^B\left(s_t\right)$$

(b) Determine the competitive equilibrium allocation in terms of primitives.

The first order condition for consumer's problem is

$$\frac{\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right)}{\pi\left(H\right) u'\left(c_0^i\left(H\right)\right)} = \frac{p_t\left(s^t\right)}{p_0\left(H\right)}$$

$$\Rightarrow \quad \frac{\beta^t \pi\left(s^t\right) c_0^i\left(H\right)}{\pi c_t^i\left(s^t\right)} = \frac{p_t\left(s^t\right)}{p_0\left(H\right)}$$

where $\pi\left(s^t\right)$ represents the unconditional probability of s^t. The necessary and sufficient conditions for the competitive equilibrium with date-0 trading are the F.O.C. for each consumer, budget constraint, and market clearing condition, i.e.

$$\frac{\beta^t \pi\left(s^t\right) c_0^A\left(H\right)}{\pi c_t^A\left(s^t\right)} = \frac{p_t\left(s^t\right)}{p_0\left(H\right)} \tag{1}$$

$$\frac{c_0^A\left(H\right)}{c_t^A\left(s^t\right)} = \frac{c_0^B\left(H\right)}{c_t^B\left(s^t\right)} \tag{2}$$

$$\sum_{t=0}^{\infty}\sum_{s^t} p_t\left(s^t\right) c_t^A\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} p_t\left(s^t\right) w_t^A\left(s_t\right) \tag{3}$$

$$c_t^A\left(s^t\right) + c_t^B\left(s^t\right) = w_t^A\left(s_t\right) + w_t^B\left(s_t\right) \tag{4}$$

Here we can exclude person B's budget constraint because of Walras law (i.e. we have one redundancy in our system of equations).

To solve for this equation explicitly, we make some simplifications. Since consumption in complete market does not depend on history, and only depends on aggregate endowment, we know that

$$c_t^A\left(s^t\right) = c_t^A\left(s_t\right) = \begin{cases} c^A\left(H\right) & \text{if } s_t = H \\ c^A\left(L\right) & \text{if } s_t = L \end{cases} \tag{5}$$

$$c_t^B\left(s^t\right) = c_t^B\left(s_t\right) = \begin{cases} c^B\left(H\right) & \text{if } s_t = H \\ c^B\left(L\right) & \text{if } s_t = L \end{cases} \tag{6}$$

Also we normalize the price as

$$p_0\left(H\right) = 1$$

Plug equation (5) and (6) into equation (2), we have

$$\begin{aligned}
& \frac{c_0^A\left(H\right)}{c_t^A\left(s^t\right)} = \frac{c_0^B\left(H\right)}{c_t^B\left(s^t\right)} \\
\Rightarrow\ & \frac{c^A\left(H\right)}{c^A\left(s_t\right)} = \frac{c^B\left(H\right)}{c^B\left(s_t\right)} \qquad (from\ \ (5),(6)) \\
\Rightarrow\ & \frac{c^A\left(H\right)}{c^A\left(L\right)} = \frac{c^B\left(H\right)}{c^B\left(L\right)} \quad (for\ s_t = L) \\
\Rightarrow\ & \frac{c^A\left(H\right)}{c^A\left(L\right)} = \frac{w^A\left(H\right) + w^B\left(H\right) - c^A\left(H\right)}{w^A\left(L\right) + w^B\left(L\right) - c^A\left(L\right)} \\
& (plug\ in\ \ (4)) \\
\Rightarrow\ & \frac{c^A\left(H\right)}{c^A\left(L\right)} = \frac{3 - c^A\left(H\right)}{1 - c^A\left(L\right)} \\
\Rightarrow\ & \frac{c^A\left(H\right)}{c^A\left(L\right)} = \frac{c^B\left(H\right)}{c^B\left(L\right)} = 3
\end{aligned} \tag{7}$$

Plug equation (5) and (6) into equation (1), we have

$$\begin{aligned}
& \frac{\beta^t \pi\left(s^t\right) c_0^A\left(H\right)}{\pi c_t^A\left(s^t\right)} = \frac{p_t\left(s^t\right)}{p_0\left(H\right)} \\
\Rightarrow\ & p_t\left(s^t\right) = \frac{\beta^t \pi\left(s^t\right) c^A\left(H\right)}{\pi c^A\left(s_t\right)} \\
\Rightarrow\ & p_t\left(s^t\right) = \begin{cases} \beta^t \frac{\pi\left(s^{t-1},H\right)}{\pi} & for\ s_t = H \\ \beta^t \frac{\pi\left(s^{t-1},L\right) c^A(H)}{\pi c^A(L)} & for\ s_t = L \end{cases} \\
\Rightarrow\ & p_t\left(s^t\right) = \begin{cases} \beta^t \frac{\pi\left(s^{t-1},H\right)}{\pi} & for\ s_t = H \\ 3\beta^t \frac{\pi\left(s^{t-1},L\right)}{\pi} & for\ s_t = L \end{cases} \qquad (plug\ in\ 7)
\end{aligned} \tag{8}$$

Now plug (7) and (8) into (3), we have

$$\begin{aligned}
& \sum_{t=0}^{\infty}\sum_{s^t} p_t\left(s^t\right) c_t^A\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} p_t\left(s^t\right) w_t^A\left(s_t\right) \\
\Rightarrow\ & \sum_{t=0}^{\infty}\left[\sum_{s^{t-1}} \beta^t \frac{\pi\left(s^{t-1},H\right)}{\pi} c^A(H) + 3\beta^t \frac{\pi\left(s^{t-1},L\right)}{\pi} c^A(L)\right] \\
=\ & \sum_{t=0}^{\infty}\left[\sum_{s^{t-1}} \beta^t \frac{\pi\left(s^{t-1},H\right)}{\pi} w_t^A(H) + 3\beta^t \frac{\pi\left(s^{t-1},L\right)}{\pi} w_t^A(L)\right] \\
& \text{(here we plug in } (5),(6),(8)) \\
\Rightarrow\ & \sum_{t=0}^{\infty}\left[\sum_{s^{t-1}} \beta^t \frac{\pi\left(s^{t-1}\right)}{\pi} \pi(H) c^A(H) + 3\beta^t \frac{\pi\left(s^{t-1}\right)}{\pi} \pi(L) c^A(L)\right] \\
=\ & \sum_{t=0}^{\infty}\left[\sum_{s^{t-1}} \beta^t \frac{\pi\left(s^{t-1}\right)}{\pi} \pi(H) w_t^A(H) + 3\beta^t \frac{\pi\left(s^{t-1}\right)}{\pi} \pi(L) w_t^A(L)\right] \\
& \text{(here we use the assumption of independence)} \\
\Rightarrow\ & \sum_{t=0}^{\infty}\left[\beta^t c^A(H) + 3\beta^t \frac{1-\pi}{\pi} c^A(L)\right] \\
=\ & \sum_{t=0}^{\infty}\left[\beta^t w_t^A(H) + 3\beta^t \frac{1-\pi}{\pi} w_t^A(L)\right] \quad \left(\text{since } \sum_{s^{t-1}} \pi\left(s^{t-1}\right) = 1\right) \\
\Rightarrow\ & \sum_{t=0}^{\infty}\left[\beta^t \frac{1}{\pi} c^A(H)\right] = \sum_{t=0}^{\infty}\left[2\beta^t + 3\beta^t \frac{1-\pi}{\pi}\right] \\
& \text{(here we plug in } (7), w_t^A(H) \text{ and } w_t^A(L)) \\
\Rightarrow\ & c^A(H) = 3-\pi \qquad (9)
\end{aligned}$$

Plug back into (7) and (4), we have

$$\begin{aligned}
c^A(H) &= 3-\pi \\
c^A(L) &= 1-\frac{1}{3}\pi \\
c^B(H) &= \pi \\
c^B(L) &= \frac{1}{3}\pi
\end{aligned}$$

which is equilibrium allocation.

(c) Determine the prices of the Arrow securities in terms of primitives.

Recall from part (b) that the Arrow-Debreu price is

$$p_t\left(s^t\right) = \begin{cases} \beta^t \frac{\pi\left(s^{t-1},H\right)}{\pi} & for\ s_t = H \\ 3\beta^t \frac{\pi\left(s^{t-1},L\right)}{\pi} & for\ s_t = L \end{cases}$$

Therefore, the price of Arrow securities are

$$
\begin{aligned}
q_t(H,H) &= \frac{p_{t+1}(s^{t-1},H,H)}{p_t(s^{t-1},H)} = \frac{\beta^{t+1}\frac{\pi\left(s^{t-1},H,H\right)}{\pi}}{\beta^{t}\frac{\pi\left(s^{t-1},H\right)}{\pi}} = \frac{\beta\pi\left(s^{t-1}\right)\pi(H)\pi(H)}{\pi\left(s^{t-1}\right)\pi(H)} = \beta\pi \\
q_t(H,L) &= \frac{p_{t}(s^{t-1},H,L)}{p_t(s^{t-1},H)} = \frac{3\beta^{t+1}\frac{\pi\left(s^{t-1},H,L\right)}{\pi}}{\beta^{t}\frac{\pi\left(s^{t-1},H\right)}{\pi}} = \frac{3\beta\pi\left(s^{t-1}\right)\pi(H)\pi(L)}{\pi\left(s^{t-1}\right)\pi(H)} = 3\beta(1-\pi) \\
q_t(L,H) &= \frac{p_{t+1}(s^{t-1},L,H)}{p_t(s^{t-1},L)} = \frac{\beta^{t+1}\frac{\pi\left(s^{t-1},L,H\right)}{\pi}}{3\beta^{t}\frac{\pi\left(s^{t-1},L\right)}{\pi}} = \frac{\beta\pi\left(s^{t-1}\right)\pi(L)\pi(H)}{3\pi\left(s^{t-1}\right)\pi(L)} = \frac{\beta}{3}\pi \\
q_t(L,L) &= \frac{p_{t+1}(s^{t-1},L,L)}{p_t(s^{t-1},L)} = \frac{3\beta^{t+1}\frac{\pi\left(s^{t-1},L,L\right)}{\pi}}{3\beta^{t}\frac{\pi\left(s^{t-1},L\right)}{\pi}} = \frac{\beta\pi\left(s^{t-1}\right)\pi(L)\pi(L)}{\pi\left(s^{t-1}\right)\pi(L)} = \beta(1-\pi)
\end{aligned}
$$

(d) Use your answer from part (c) to determine the average rate of return on a (one-period) riskless bond in this economy.

The price of a risk-free asset is

$$
\begin{aligned}
q_t^{rf}(H) &= q_t(H,H) + q_t(H,L) = \beta\pi + 3\beta(1-\pi) = \beta(3-2\pi) \\
q_t^{rf}(L) &= q_t(L,H) + q_t(L,L) = \frac{\beta}{3}\pi + \beta(1-\pi) = \frac{\beta}{3}(3-2\pi)
\end{aligned}
$$

Therefore, the average rate of return on a (one-period) riskless bond in this economy is

$$
\begin{aligned}
E\left(R^{rf}\right) &= \pi(H)\frac{1}{q_t^{rf}(H)} + \pi(L)\frac{1}{q_t^{rf}(L)} \\
&= \pi \times \frac{1}{\beta(3-2\pi)} + (1-\pi) \times \frac{1}{\frac{\beta}{3}(3-2\pi)} \\
&= \frac{1}{\beta}
\end{aligned}
$$