Econ 510a (second half)
Yale University
Fall 2004
Prof. Tony Smith

HOMEWORK #1

This homework assignment is due at 5PM on Friday, October 29 in Ted Papageorgiou's mailbox.

1. Consider a growth model with two sectors, one producing consumption goods and one producing investment goods. Consumption is given by $c_t = F(k_{ct}, L_{ct})$ and investment by $i_t = G(k_{it}, L_{it})$, where k_{ct} is the amount of capital used in time period t in the consumption-goods sector and k_{it} is the amount of capital used in time period t in the investment-goods sector. Similarly, L_{ct} and L_{it} are the amounts of labor used in the two sectors. Total capital evolves according to: $k_{t+1} = (1-\delta)k_t + i_t$. At each point in time, capital and labor must be allocated across the two sectors: $k_t = k_{ct} + k_{it}$ and $L = L_{ct} + L_{it}$, where the total amount of labor L is fixed. Finally, suppose that F and G have constant returns to scale. (Recall that this means, for example, that $F(K, L) = Lf(K/L)$, where $f(K/L) \equiv F(K/L, 1)$.)

As in the Solow growth model, we will make ad hoc behavioral assumptions, in this case about how to allocate k_t and L at a point in time. In particular, assume that, for all t, $k_{ct}/L_{ct} = k_{it}/L_{it}$, i.e., the capital-to-labor ratios are the same in both sectors. In addition, assume that a constant fraction θ of total labor is allocated to the consumption-goods sector: $L_{ct} = \theta L$, where $\theta \in (0, 1)$.

State and prove a "global convergence" result. That is, provide assumptions on the primitives $(F, G, L, \delta, \theta)$ and show that, given these assumptions, there is global convergence to a steady state (i.e., the dynamic system converges from any initial condition).

2. Consider a growth model with capital accumulation equation $k_{t+1} = f(k_t)$ if t is even and $k_{t+1} = g(k_t)$ if t is odd. Assume that:

(i) $f(0) = g(0) = 0$.

(ii) $f'(0)g'(0) > 1$.

(iii) $\lim_{k\to\infty} f'(g(k))g'(k) < 1$ and $\lim_{k\to\infty} g'(f(k))f'(k) < 1$.

(iv) f and g are strictly increasing and strictly concave.

Show that there is global convergence to a "two-cycle" in which k_t oscillates between two values. How are these values determined?

Suppose instead that $f(k_t) = ak_t$ and $g(k_t) = bk_t$, where a and b are positive constants. Completely characterize the dynamics of k_t as a function of a and b.

3. For the neoclassical growth model that we developed in lecture on October 20, we derived the following Euler equation:

$$u'(f(k) - k') = \beta u'(f(k') - k'')f'(k').$$

Recognizing that $k' = g(k)$ and $k'' = g(k') = g(g(k))$, where g is the optimal decision rule, one can state a "functional" version of the Euler equation:

$$u'(f(k) - g(k)) = \beta u'(f(g(k)) - g(g(k)))f'(g(k)).$$

As described in Section 4.2.3 in Chapter 4 in the lecture notes, the slope of g at the steady state k^* can be found by differentiating the functional Euler equation, evaluating it at k^*, solving a quadratic equation in $g'(k^*)$, and then discarding the "unstable" root.

Make the following assumptions about functional forms:

(i) u has constant elasticity of intertemporal substitution σ^{-1}:

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},$$

where $\sigma > 0$ and $u(c) = \log(c)$ if $\sigma = 1$.

(ii) $f(k) = Ak^\alpha + (1 - \delta)k$, where $A > 0$, $\alpha \in (0, 1)$, and $\delta \in [0, 1]$.

Discuss how A, α, β, δ, and σ affect the speed of convergence to the steady state. (As discussed in Section 4.2 in Chapter 4 of the lecture notes, the speed of convergence near the steady state is inversely related to the slope of the decision rule at the steady state.)

4. This problem studies a neoclassical growth model with "time-to-build", as in the famous article by Kydland and Prescott (*Econometrica*, 1982) recently awarded the Nobel Prize in Economics. Specifically, suppose that it takes two time periods to build and install new capital (rather than one period as in the standard growth model). Let s_{2t} denote new investment projects initiated in period t; s_{2t} is a choice variable at all points in time. The stock of partially completed investment projects in period t (i.e., investment projects one period from completion in period t) is denoted s_{1t}. The stocks of partially completed and new investment projects are related by $s_{1,t+1} = s_{2t}$. The capital accumulation equation reads: $k_{t+1} = (1 - \delta)k_t + s_{1t}$, where k_t is the stock of completed projects.

The resource constraint is: $c_t + i_t = F(k_t)$, where investment $i_t = (1 - \phi)s_{1t} + \phi s_{2t}$ and $\phi \in [0, 1]$. In other words, starting a new investment project of size s is a commitment to invest resources ϕs in the first stage (or period) of its construction and resources $(1-\phi)s$ in the second stage of its construction. Finally, assume that F has the standard properties.

(a) Formulate the Bellman equation for the social planning problem in this economy, assuming that a typical consumer's preferences are given by: $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where u has the standard properties. Be clear about what the state variables are and what the choice variables are.

(b) Derive the first-order and envelope conditions for this problem.

(c) Under the assumptions that $F(k) = k^{\alpha}$ and $\phi = 1$, derive an expression for the steady-state capital stock in terms of the structural parameters.