Econ 510a (second half)
Yale University
Fall 2005
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HOMEWORK #2

This homework assignment is due at the beginning of class on Wednesday, November 2.

1. This problem introduces various forms of heterogeneity into a two-period model with endogenous capital accumulation similar to the one that you studied in the third problem on Homework #1.

(a) Suppose that there are two types of consumers distinguished by their initial endowments of capital. In particular, type-1 consumers (who comprise fraction θ of the population) are richer than type-2 consumers (who comprise fraction $1-\theta$ of the population): type-1 consumers are endowed with k_0^1 units of capital and type-2 consumers are endowed with k_0^2 units of capital, where $k_0^1 > k_0^2$. The two types of consumers are identical in all other respects. Each consumer takes prices as given (i.e., each consumer takes the aggregate, or total, capital stock in period 1 as given) when making savings decisions in period 0. The equilibrium (or consistency) condition is that the total savings of the two types of consumers must equal the aggregate capital stock that consumers take as given when deciding how much to save in period 0.

Using the same functional forms as in the third problem on Homework #1 (but set $z = 1$), derive the equilibrium aggregate capital stock in period 1 as a function of primitives. Use your answer to show that changes in k_0^1 and k_0^2 that keep aggregate capital in period 0 (i.e., $\theta k_0^1 + (1-\theta)k_0^2$) constant have no effect either on equilibrium aggregate savings or on equilibrium prices. This is a version of an *aggregation theorem* for this economy: holding the total amount of capital in period 0 constant, the behavior of the aggregates in this economy does not depend on the distribution of capital in period 0.

(b) Suppose that the period utility function (or the felicity function) takes the form: $u(c) = (1-\sigma)^{-1}\left(c^{1-\sigma} - 1\right)$, where $\sigma > 0$. Does an aggregation theorem like the one described in part (a) hold for this economy? Explain why or why not.

(c) Suppose instead that the two types of consumers have the same initial endowments of capital, but instead have different endowments of effective labor. Although each type of consumer is endowed with one unit of time, type-1 consumers are more productive than type-2 consumers: each type-1 consumer earns $\epsilon_1 w$, where

w is the (equilibrium) wage, for each unit of time that he works, whereas each type-2 consumer earns $\epsilon_2 w$ for each unit of time that he works, where $\epsilon_1 > \epsilon_2$. Because consumers supply labor inelastically, total labor supply in each period is $\bar{\epsilon} \equiv \theta\epsilon_1 + (1-\theta)\epsilon_2$ and the firm's total wage bill is $\bar{\epsilon}\, w$. Derive the equilibrium aggregate capital stock in period 1 as a function of primitives (assume again that consumers have logarithmic felicity functions).

(d) Now suppose that consumers have identical endowments of capital and labor but differ in their patience: the discount factor of type-1 consumers is β_1 and the discount factor of type-2 consumers is β_2, where $\beta_1 > \beta_2$. Derive the equilibrium aggregate capital stock in period 1 as a function of primitives (assume again that consumers have logarithmic felicity functions).

2. Consider a two-consumer, two-period exchange economy with date-0 trading like the one that we discussed in lecture on Wednesday, October 26. Type-1 consumers are endowed with 2 units of the consumption good in period 0 and 0 units of the consumption good in period 1; type-2 consumers are endowed with 0 units of the consumption good in period 0 and 2 units of the consumption good in period 1. The consumption good is not storable. Both types of consumers have the same preferences: $\log(c_0^i) + \beta \log(c_1^i)$, where c_t^i is the consumption of a type-i consumer in period t. Let the measures (fractions) of the two types of consumers both be equal to one-half.

(a) Find the equilibrium consumption allocation and the equilibrium (relative) price of period-1 consumption goods. (Adopt the normalization that the period-0 consumption good is the numeraire: that is, the price of the period-0 consumption good is 1.) Express the equilibrium allocation and the equilibrium price in terms of β. Explain intuitively why type-1 consumers have higher consumption than type-2 consumers in equilibrium.

(b) Find the allocation of consumption chosen by a social planner who maximizes:

$$\alpha[\log(c_0^1) + \beta \log(c_1^1)] + (1-\alpha)[\log(c_0^2) + \beta \log(c_1^2)]$$

subject to the resource constraints: $c_t^1 + c_t^2 = \omega_t^1 + \omega_t^2$, for $t = 0, 1$. The weight $\alpha \in (0, 1)$. Your answer should depend on α and β.

(c) An allocation in this economy is Pareto efficient if, one, it is feasible (i.e., it satisfies the resource constraints); and, two, it has the property that any reallocation that makes one (type of) consumer better off makes the other (type of) consumer worse off. Show that the set of solutions to the planning problem (as the weight α varies) coincides with the set of Pareto efficient allocations in this economy. (Hint: In this economy, the set of Pareto efficient allocations is the set of feasible allocations for which the intertemporal marginal rates of substitution of the two types of consumers are equated.)

(d) Find the value of α for which the allocation chosen by the planner is identical to the competitive equilibrium allocation that you computed in part (a). Your answer should depend only on β. Explain intuitively why the social planner puts more weight on type-1 consumers than on type-2 consumers.

(e) Show that the value of α that you computed in part (d) is equal to:

$$\frac{1/\lambda^1}{1/\lambda^1 + 1/\lambda^2},$$

where λ^i is the marginal utility of period-0 consumption for a type-i consumer in competitive equilibrium. Give an intuitive explanation for this result.

3. Consider a consumer with the following sequence of budget constraints:

$$c_t + a_{t+1} = Ra_t + w, \quad t = 0, 1, 2, \ldots,$$

where w is the consumer's (constant) wage income in each period. The consumer's consumption cannot be negative. The gross interest rate R is greater than 1 (so that the net interest rate is strictly positive).

(a) Use the no-Ponzi-game (nPg) restriction that $\lim_{t\to\infty} a_{t+1}/R^t \geq 0$ to derive a consolidated, or intertemporal, budget constraint for this consumer. Interpret your answer.

(b) Show that the nPg restriction is equivalent to imposing a constraint that the consumer's asset holdings never fall below a fixed amount B, where B is allowed to be negative. In other words, show that there is a borrowing limit B such that the set of feasible consumption levels defined by the sequential budget constraints and the nPg restriction is identical to the set of feasible consumption levels defined by the sequential budget constraints and the borrowing constraint $a_t \geq B$ for $t \geq 0$. Express B in terms of R and w. (Hint: Imagine a consumer who has zero consumption and whose asset holdings do not change over time.)