Econ 510a (second half)
Yale University
Fall 2007
Prof. Tony Smith

HOMEWORK #2

This homework assignment is due at noon on Friday, November 9. Please put your assignment in Evrim Aydin's mailbox.

1. Consider a neoclassical growth model with two sectors, one producing consumption goods and one producing investment goods. Consumption is given by $C_t = F(K_{Ct}, L_{Ct})$ and investment is given by $I_t = G(K_{It}, L_{It})$, where K_{jt} is the amount of capital in sector j at the beginning of period t and L_{jt} is the amount of labor used in sector j in period t. The total amount of labor in each period is equal to L (leisure is not valued). Labor can be freely allocated in each period between the two sectors: $L = L_{Ct} + L_{It}$. Capital, by contrast, is sector-specific: once it is installed in a given sector, it cannot be moved to the other sector. Investment goods, however, can be used to augment the capital stock in either sector. In particular, the capital stocks in the two sectors evolve according to:
$$K_{j,t+1} = (1-\delta)K_{jt} + I_{jt}, \quad j = C, I,$$
where $I_t = I_{Ct} + I_{It}$.

   The social planner seeks to maximize $\sum_{t=0}^{\infty} \beta^t u(C_t)$, given K_{C0} and K_{I0}, subject to the constraints on technology. Note that although leisure is not valued (i.e., the total amount of labor supply L does not appear in the planner's objective), the planner must nonetheless decide in each period how to allocate L across the two sectors.

   (a) Formulate the planner's optimization problem as a dynamic programming problem. What are the state variable(s)? What are the choice variable(s)?

   (b) Find a set of first-order conditions and envelope conditions that an optimal solution to the planning problem must satisfy.

   (c) Use your answer to part (b) to find a set of equations that determine the steady-state values of capital and labor (in each sector) in this economy.

   (d) *[This part is optional because the algebra is intense. Even if you don't work out the answer, you should make sure that you understand what you are being asked to do.]* Suppose that $F(K_{Ct}, L_{Ct}) = K_{Ct}^{\alpha} L_{Ct}^{1-\alpha}$ and $G(K_{It}, L_{It}) = K_{It}^{\gamma} L_{It}^{1-\gamma}$. Use your answer from part (c) to express the steady state as a function of the structural parameters (i.e., β, δ, α, and γ).

2. Consider a consumer with the following optimization problem:

$$\max_{\{c_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \text{given } b_0$$

subject to:

$$c_t + qb_{t+1} = b_t + w, \quad t = 0, 1, 2, \ldots,$$

and the no-Ponzi-game (nPg) condition:

$$\lim_{t \to \infty} q^t b_t = 0.$$

Assume that the gross interest rate q^{-1} is greater than one, that the wage w is positive, and that the discount factor $\beta \in (0, 1)$. The felicity function u is strictly increasing, strictly concave, and twice continuously differentiable.

(a) Derive the consumer's consolidated (or lifetime) budget constraint. (We derived the consolidated budget constraint in lecture by repeatedly substituting using the period-by-period budget constraints and then imposing the nPg condition; see p. 21 of the lecture notes by Per Krusell to refresh your memory.)

(b) Find the transversality condition for this problem. Show that the nPg condition is met if the transversality condition and the Euler equation are both satisfied.

(c) Modify the proof of Proposition 3.4 in the lecture notes by Per Krusell (pp. 23–24) to prove that a sequence $\{b_t^*\}_{t=0}^{\infty}$ that satisfies the transversality condition and the Euler equation maximizes the consumer's objective, subject to the sequence of budget constraints and the nPg condition.

3. Consider an exchange economy with one consumption good per period and two types of consumers, type-A and type-B. Normalize the size of the total population to two (2) and let each type of consumer comprise one-half of the population (so that each type of consumer has measure one). The two types of consumers differ both in their preferences and their endowments. Type-i consumers ($i = A, B$) value consumption streams according to $\sum_{t=0}^{\infty} \beta^t u_i(c_t)$, where u_i has constant elasticity of intertemporal substitution equal to σ_i^{-1}. Consumer i's endowment of consumption goods is $\{\omega_t^i\}_{t=0}^{\infty}$, $i = A, B$. Consumption goods are perishable (i.e., they cannot be stored and used for consumption in future periods).

(a) Carefully define a competitive equilibrium with date-0 trading for this economy.

(b) Suppose that $\omega_t^A = 2$ for all t and $\omega_t^B = 1$ for all t. Find the competitive equilibrium allocations and prices. In addition, find each type of consumer's marginal utility of wealth in equilibrium (i.e., using the notation from lecture, find λ^i, $i = A, B$).

(c) Suppose now that the endowments fluctuate deterministically: the endowment stream of a type-A consumer is $\{2, 1, 2, 1, 2, 1, \dots\}$ and the endowment stream of a type-B consumer is $\{1, 2, 1, 2, 1, 2, \dots\}$. Find the competitive equilibrium allocations and prices as well as the λ^is of the two types of consumers.

(d) Assume now that $\sigma_A = \sigma_B = \sigma$, but do not restrict either σ or the endowment streams. Show that the competitive equilibrium allocation takes the form

$$c_t^i = \theta^i \bar{\omega}_t, \qquad i = A, B,$$

where $\bar{\omega}_t$ is the aggregate endowment in period t and θ^i is a (time-invariant) share of the aggregate endowment that depends on primitives (i.e., preferences and endowments).

(e) Suppose that the endowment stream of a type-A consumer is the same as in part (b), but that the endowment stream of a type-B consumer is the same as in part (c). Use your answer from part (d) to determine the competitive equilibrium allocations and prices when $\sigma = 1$ (the logarithmic case). In addition, find each of the λ^is.