Econ 510a (second half)
Yale University
Fall 2005
Prof. Tony Smith

HOMEWORK #3

This homework assignment is due at the beginning of class on Wednesday, November 9.

1. Consider a consumer with the following optimization problem:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \text{given } a_0 > 0,$$

subject to:

$$c_t + a_{t+1} = Ra_t + w, \quad t = 0, 1, 2, \ldots,$$

and the no-Ponzi-game (nPg) restriction that

$$\lim_{t \to \infty} \frac{a_{t+1}}{R^t} \geq 0.$$

Assume that the gross interest rate R is greater than one, that the wage w is positive, and that the discount factor $\beta \in (0,1)$. The felicity function u is strictly increasing, strictly concave, twice continuously differentiable, and satisfies an Inada condition: $\lim_{c \to 0} u'(c) = \infty$.

(a) Find the transversality condition for this problem. Show that the nPg restriction is met if the transversality condition and the Euler equation are both satisfied.

(b) Modify the proof that we discussed in lecture (see also pp. 23–24 in the lecture notes) to prove that a sequence $\{a_t^*\}_{t=0}^{\infty}$ that satisfies the transversality condition and the Euler equation maximizes the consumer's objective, subject to the sequence of budget constraints and the nPg restriction.

2. This problem considers a two-period economy similar to the one that you studied in the third problem on Homework #1, but under the assumption that firms, rather than consumers, control the capital stock. Firms, in turn, are owned by consumers. In particular, consumers own shares in the firm; each share entitles its owner to a fraction of the firm's profits. Normalize the total number of shares to one, so that each of the identical consumers (whose total measure is also normalized to one) owns one share. In addition, in period 0 consumers can trade riskfree bonds (i.e., claims to the consumption good in the next period) in a competitive market.

The (representative) consumer takes prices as given and seeks to maximize his lifetime utility of consumption:

$$\max_{c_0,\, c_1,\, b_1} \quad u(c_0) + \beta u(c_1)$$

subject to:

$$c_0 + q_0 b_1 = w_0 + s_0 \pi_0 + b_0$$

and

$$c_1 = w_1 + b_1 + s_1 \pi_1$$

where q_0 is the price of a bond in period 0, π_t is profits per share in period t, b_t is the amount of bonds owned by the consumer in period t, and s_t is the amount of shares owned by the consumer in period t. Assume that, initially, each consumer holds no bonds: $b_0 = 0$. As discussed above, each consumer owns one share in the firm in each time period: $s_0 = s_1 = 1$. Consumers are also endowed with unit of time in each time period, which they supply inelastically in a competitive labor market.

The (representative) firm takes prices as given and seeks to maximize the net present value of profits:

$$\max_{n_0,\, n_1,\, k_1} \quad y_0 - w_0 n_0 - k_1 + q_0 \left(y_1 - w_1 n_1 \right),$$

where $y_t = f(k_t, n_t)$ is the firm's output in period t, k_t is the amount of capital owned by the firm at the beginning of period t, n_t is labor demand in period t, and w_t is the wage in period t. The firm discounts profits in period 1 using the price q_0 of period-1 consumption goods (relative to the price of period-0 consumption goods). Assume that each firm is endowed with the same amount of capital in period 0 and that the production function f exhibits constant returns to scale.

In equilibrium, the markets for labor, bonds, and goods must clear: in particular, $n_t = 1$, $b_1 = 0$, and total output in period t is either consumed or invested in period t (note that investment in period 1 is zero because the economy ends).

(a) Show that the competitive equilibrium allocation for this economy is identical to the one that obtains when consumers control the capital stock directly (as in the third problem on Homework #1). As part of your answer, determine the price q_0 in terms of the primitives u, f, and k_0. (You do not have to explicitly solve for q_0, but you have to explain how to compute it.)

(b) Suppose now that a market for shares in the firm is introduced into the economy. In particular, a typical consumer's budget constraints now read:

$$c_0 + q_0 b_1 + p_0 s_1 = w_0 + b_0 + (p_0 + \pi_0) s_0$$

and

$$c_1 = w_1 + b_1 + s_1 \pi_1,$$

where p_0 is the price of a share in period 0. Each consumer is endowed with one share in period 0 (i.e., $s_0 = 1$), but he is free to make any choice for s_1. In equilibrium, the market for shares must clear: $s_1 = 1$. Show that the introduction of this additional market does not change the allocation from the one in part (a). As part of your answer, determine the price p_0 in terms of the primitives u, f, and k_0. Interpret.

(c) Suppose now that the government taxes income from profits at a proportional rate τ and returns the proceeds to consumers in a lump-sum fashion (so that its budget balances in each period). Is the competitive equilibrium allocation in this economy Pareto optimal? Explain why or why not.

3. Consider an exchange economy with two (types of) consumers. Each type of consumer has measure one, so that the total population of consumers equals two. The two (types of) consumers have identical preferences: they each value consumption streams according to $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where u has constant elasticity of substitution σ^{-1}, i.e., $u(c) = (1-\sigma)^{-1}\left(c^{1-\sigma} - 1\right)$. A consumer of type-$i$, where $i = 1, 2$, is endowed with a sequence of (perishable) consumption goods denoted by $\{\omega_t^i\}_{t=0}^{\infty}$.

(a) Carefully define a competitive equilibrium with date-0 trading for this economy.

(b) Suppose that $\omega_t^1 = 2$ for all t and $\omega_t^2 = 1$ for all t. Find the competitive equilibrium allocations and prices. (Hint: Your answers to all of the questions in this problem should depend only on β and σ. Before you consider the general case, it might be helpful to consider the special case $\sigma = 1$, i.e., logarithmic utility.)

(c) Suppose now that the consumers' endowments fluctuate deterministically: the endowment stream of a type-1 consumer is $\{2, 1, 2, 1, 2, 1, \ldots\}$ and the endowment stream of a type-2 consumer is $\{1, 2, 1, 2, 1, 2, \ldots\}$. Find the competitive equilibrium allocations and prices.

(d) In parts (b) and (c) there is no variation in the *aggregate* endowment across time. Suppose that, as in part (b), the endowment of a type-1 consumer is 2 in every period but that the endowment stream of a type-2 consumer fluctuates deterministically: it is given by $\{0, 1, 0, 1, 0, 1, \ldots\}$. Find the competitive equilibrium allocations and prices.

(e) State the social planning problem for this economy. For each of the pairs of endowment streams in parts (b), (c), and (d), find the Pareto weights (i.e., the weights on each of the two types of consumers) that deliver the competitive equilibrium allocation as the solution to the planning problem. Interpret the (relative) sizes of the weights.

(f) Carefully define a competitive equilibrium with sequential trading for this economy. Use your results from parts (b), (c), and (d) to determine the sequence of equilibrium interest rates for each pair of endowment streams. In addition, for each case determine how each consumer's asset holdings (i.e., claims to consumption) vary over time (assume that each consumer starts with zero assets in period 0).