Econ 510a (second half)
Yale University
Fall 2006
Prof. Tony Smith

HOMEWORK #3

This homework assignment is due at NOON *on Friday, November 17 in Marnix Amand's mailbox.*

1. This problem introduces wealth inequality into a two-period economy with production like the one discussed in lecture on Friday, November 10.

(a) Suppose that there are two types of consumers distinguished by their initial endowments of capital. In particular, type-1 consumers (who comprise fraction θ of the population) are richer than type-2 consumers (who comprise fraction $1-\theta$ of the population): type-1 consumers are endowed with k_0^1 units of capital and type-2 consumers are endowed with k_0^2 units of capital, where $k_0^1 > k_0^2$. The two types of consumers are identical in all other respects. Each consumer takes prices as given (in particular, each consumer takes the aggregate, or total, capital stock in period 1 as given) when making savings decisions in period 0. The equilibrium (or consistency) condition is that the total savings of the two types of consumers in period 0 must equal the aggregate capital stock that consumers take as given when deciding how much to save.

Assume that each consumer's utility function takes the form $u(c_0) + \beta u(c_1)$, with $u(c) = \log(c)$. The production technology available to firms is: $y = k^\alpha n^{1-\alpha}$, with $0 < \alpha < 1$, where y is the firm's output and k and n are the services of capital and labor, respectively. (Consumers do not value leisure; if each consumer's endowment of time is normalized to one, then $n = 1$ in each time period.)

Derive the equilibrium aggregate capital stock in period 1 as a function of primitives (i.e., the parameters α, β, and θ and the initial capital stocks k_0^1 and k_0^2).

(b) Use your answer to part (a) to show that changes in k_0^1 and k_0^2 that keep aggregate capital in period 0 (i.e., $\theta k_0^1 + (1-\theta)k_0^2$) *constant* have no effect either on equilibrium aggregate savings or on equilibrium prices. This is a version of an *aggregation theorem* for this economy: holding the total amount of capital in period 0 constant, the behavior of the aggregates in this economy does not depend on the distribution of capital in period 0.

(c) Suppose that the felicity function takes the form: $u(c) = (1-\sigma)^{-1}\left(c^{1-\sigma} - 1\right)$, where $\sigma > 0$. Does an aggregation theorem like the one described in part (b) hold for this economy? Explain why or why not.

2. Consider a neoclassical growth model in which consumers have time-separable preferences given by: $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Let the aggregate production (or resource) function take the form:

$$f(\bar{k}, n) = A\bar{k}^{\alpha} n^{1-\alpha} + (1-\delta)\bar{k},$$

where δ is the rate of depreciation of capital. The parameters satisfy: $0 < \beta < 1$, $A > 0$, $0 < \alpha < 1$, and $0 < \delta \leq 1$. Consumers are endowed with one unit of time in each period but do not value leisure (so that $n = 1$). In this problem, you will solve explicitly for the recursive competitive equilibrium of this economy under the assumptions that $u(c) = \log(c)$ and $\delta = 1$. (Assume too that the economy is decentralized in the manner that we have discussed in class.)

(a) Suppose that aggregate capital evolves according to $\bar{k}' = G(\bar{k}) = sf(\bar{k}, 1)$. (You will verify the validity of this conjecture below.) Find explicit formulas for the value function $v(k, \bar{k})$ and the decision rule $k' = g(k, \bar{k})$ of a "small" (or typical) consumer who takes the law of motion for aggregate capital as given. The functions v and g depend on s as well as on primitives of technology and preferences. (Hint: Guess that $v(k, \bar{k}) = a + b\log(k + d\bar{k}) + e\log(\bar{k})$ and then find expressions for the unknown coefficients a, b, d, and e in terms of the structural parameters α and β and the behavioral parameter s.)

(b) Find the competitive equilibrium value of s by imposing the consistency condition $G(\bar{k}) = g(\bar{k}, \bar{k})$. Verify that the resulting law of motion for aggregate capital solves the planning problem for this economy. Display v and g for the equilibrium value of s.

(c) How does an increase in aggregate capital affect the savings behavior and the (indirect) utility of a typical consumer (holding fixed the consumer's own holdings of capital)?

3. This problem studies a neoclassical growth model with an externality in production. Leisure is not valued and the (representative) consumer has time-separable preferences with discount factor $\beta \in (0, 1)$. Consumers, who own the factors of production, are endowed with k_0 units of capital in period 0 and with one unit of time in each period. There is a large number of identical profit-maximizing firms each of which has the following production technology:

$$f(k, n, \bar{k}) = Ak^{\alpha} n^{1-\alpha} \bar{k}^{\gamma} + (1-\delta)k,$$

where k is the amount of capital rented by the firm, n is the amount of labor hired by the firm, $\bar{k}$ is the aggregate capital stock, δ is the rate of depreciation of capital. The parameters satisfy: $0 < \gamma < 1 - \alpha$, $0 < \alpha < 1$, and $0 < \delta \leq 1$. Thus there is a productive externality from the rest of the economy: a higher aggregate capital stock increases the productivity of each firm. A typical (small) firm takes the aggregate capital stock as given when choosing its inputs.

(a) Carefully define a sequential competitive equilibrium for this economy.

(b) Carefully define a recursive competitive equilibrium for this economy.

(c) Find a second-order difference equation that governs the evolution of the aggregate capital stock in competitive equilibrium. (Hint: Find a typical consumer's Euler equation and then impose equilibrium conditions.) Use this equation to find an expression for the steady-state aggregate capital stock in competitive equilibrium.

(d) Display the Bellman equation for the social planning problem in this economy. The planner internalizes the externality in production: his production technology is

$$h(\bar{k}, n) \equiv f(\bar{k}, n, \bar{k}) = A\bar{k}^{\alpha+\gamma}n^{1-\alpha} + (1-\delta)\bar{k}.$$

Is the competitive equilibrium allocation Pareto optimal? (Hint: Compare the planner's Euler equation to the second-order difference equation that you found in part (c).)

(e) Now introduce a government that subsidizes savings at a proportional rate τ and finances these subsidies by means of a lump-sum tax on consumers. The investment subsidy is constant across time but the lump-sum tax varies over time so as to balance the government's budget in every period. Define a recursive competitive equilibrium for this economy.

(f) For what subsidy rate τ is the competitive equilibrium steady-state aggregate capital stock equal to the steady-state aggregate capital stock in the planning problem?