Econ 510a (second half)
Yale University
Fall 2004
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HOMEWORK #4

This homework assignment is due at the beginning of the help session on Thursday, November 18.

1. Consider an infinite-horizon one-sector growth model with an externality in production. Leisure is not valued and the (representative) consumer has time-separable preferences with discount factor $\beta \in (0,1)$. Consumers own the factors of production. Capital depreciates at rate δ. There is a large number of identical firms each of which has the following production technology:

$$F(k, \ell, \bar{k}) = Ak^{\alpha}\ell^{1-\alpha}\bar{k}^{\gamma},$$

where k is the capital rented by the firm, $\bar{k}$ is the aggregate capital stock, and the parameters α and γ satisfy $0 < \gamma < 1 - \alpha$ and $\alpha \in (0,1)$. Thus there is a productive externality from the rest of the economy: a higher aggregate capital stock increases the productivity of each firm. A typical (small) firm takes the aggregate capital stock as given when choosing its inputs.

(a) Define a recursive competitive equilibrium for this economy. Be clear about which variables consumers and firms take as given when they solve their optimization problems. Find a second-order difference equation that governs the evolution of the economy's aggregates. (Hint: Find a typical consumer's Euler equation and then impose equilibrium conditions.)

(b) Write the planning problem for this economy in recursive form. The planner internalizes the externality in production: his production technology is

$$G(\bar{k}, \ell) \equiv F(\bar{k}, \ell, \bar{k}) = A\bar{k}^{\alpha+\gamma}\ell^{1-\alpha}.$$

Is the competitive equilibrium allocation Pareto optimal? (Hint: Compare the planner's Euler equation to the second-order difference equation that you found in part (a).)

(c) Now introduce a government that subsidizes savings at a proportional rate τ and finances these subsidies by means of a lump-sum tax on consumers. The investment subsidy is constant across time but the lump-sum tax varies over time so as to balance the government's budget in every period. Define a recursive competitive equilibrium for this economy.

(d) For what subsidy rate τ is the competitive equilibrium steady-state aggregate capital stock equal to the steady-state aggregate capital stock in the planning problem?

2. Consider a neoclassical growth model with logarithmic felicity function, Cobb-Douglas production function $F(\bar{k}, \ell) = A\bar{k}^{\alpha}\ell^{1-\alpha}$, full depreciation of the capital stock in one period (the rate of depreciation is equal to 1), and inelastic labor supply (leisure is not valued). In this problem, you will solve explicitly for the recursive competitive equilibrium of this economy (assuming that the economy is decentralized in the manner that we have discussed in class).

(a) Suppose that aggregate capital evolves according to $\overline{k}' = G(\overline{k}) = sF(\overline{k}, 1)$. (You will verify the validity of this conjecture below.) Find explicit formulas for the value function $v(k, \overline{k})$ and the decision rule $k' = g(k, \overline{k})$ of a "small" (or typical) consumer who takes the law of motion for aggregate capital as given. The functions v and g depend on s as well as on primitives of technology and preferences. (Hint: Guess that $v(k, \overline{k}) = a + b\log(k + d\overline{k}) + e\log(\overline{k})$ and then find expressions for the unknown coefficients a, b, d, and e in terms of the structural parameters α and β and the behavioral parameter s.)

(b) Find the competitive equilibrium value of s by imposing the consistency condition $G(\overline{k}) = g(\overline{k}, \overline{k})$. Verify that the resulting law of motion for aggregate capital solves the planning problem for this economy. Display v and g for the equilibrium value of s.

(c) How does an increase in aggregate capital affect the savings behavior and the (indirect) utility of a typical consumer (holding fixed the consumer's own holdings of capital)?

(d) How does the equilibrium utility of a typical consumer vary with aggregate capital (taking into account that the consumer's own holdings of capital equal aggregate capital in equilibrium)?

3. Consider the planning problem for a neoclassical growth model with logarithmic utility, full depreciation of the capital stock in one period, and a production function of the form $y = zk^{\alpha}$, where z is a random shock to productivity. The shock z is observed before making the current-period savings decision. Assume that the capital stock can take on only two values: i.e., k is restricted to the set $\{\bar{k}_1, \bar{k}_2\}$. In addition, assume that z takes on values in the set $\{\bar{z}_1, \bar{z}_2\}$ and that z follows a Markov chain with transition probabilities $p_{ij} = P(z' = \bar{z}_j | z = \bar{z}_i)$.

(a) Let $\bar{z}_1 = 0.9$, $\bar{z}_2 = 1.1$, $p_{11} = 0.95$, and $p_{22} = 0.9$. Find the invariant distribution associated with the Markov chain for z. Use the invariant distribution to compute the long-run (or unconditional) expected value of z; that is, compute $E(z) = \pi_1 \bar{z}_1 + \pi_2 \bar{z}_2$, where π_1 and π_2 determine the invariant distribution.

(b) Let $\beta = 0.9$, $\alpha = 0.36$, $\bar{k}_1 = 0.95 k_{ss}$, and $\bar{k}_2 = 1.05 k_{ss}$, where k_{ss} is the steady-state capital stock in a version of this model without shocks and with no restrictions on capital (i.e., $k_{ss} = (\alpha\beta)^{\frac{1}{(1-\alpha)}}$). Let $g(k, z)$ denote the planner's optimal decision rule. Prove that $g(k_i, z_j) = k_j$.

(c) The decision rule from part (b) and the law of motion for z jointly determine an invariant distribution over (k, z)-pairs. Find this distribution. (That is, find probabilities $\pi_{ij} = P(k = k_i, z = z_j)$ that "reproduce" themselves: if π_{ij} is the unconditional probability that the economy is in state (k_i, z_j) today, then it is also the unconditional probability that the economy is in this state tomorrow. For a more complete discussion of this concept, see pp. 78 and 79 in the lecture notes by Per Krusell.) Use your answer to compute the long-run (or unconditional) expected values of the capital stock and of output.

(d) In Matlab, use the optimal decision rule, the law of motion for z, and a random number generator to create a simulated time series $\{k_t, y_t\}_{t=0}^{T}$, given an initial condition (k_0, z_0). Compute $T^{-1} \sum_{t=1}^{T} k_t$ and $T^{-1} \sum_{t=1}^{T} y_t$ for a suitably large value of T and confirm that these sample means are close to the corresponding population means that you computed in part (c). (You may find useful the Matlab code by Ljunqvist and Sargent for simulating a Markov chain that I have posted on the course web site.)