Econ 510a (second half)
Yale University
Fall 2005
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HOMEWORK #4

This homework assignment is due at the beginning of class on Wednesday, November 16.

1. Consider a neoclassical growth model in which consumers have time-separable preferences given by: $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Let the aggregate production (or resource) function take the form:

$$f(\bar{k}, n) = A\bar{k}^{\alpha} n^{1-\alpha} + (1-\delta)\bar{k},$$

where δ is the rate of depreciation of capital. The parameters satisfy: $0 < \beta < 1$, $A > 0$, $0 < \alpha < 1$, and $0 < \delta \leq 1$. Consumers are endowed with one unit of time in each period but do not value leisure (so that $n = 1$). In this problem, you will solve explicitly for the recursive competitive equilibrium of this economy under the assumptions that $u(c) = \log(c)$ and $\delta = 1$. (Assume too that the economy is decentralized in the manner that we have discussed in class.)

(a) Suppose that aggregate capital evolves according to $\bar{k}' = G(\bar{k}) = sf(\bar{k}, 1)$. (You will verify the validity of this conjecture below.) Find explicit formulas for the value function $v(k, \bar{k})$ and the decision rule $k' = g(k, \bar{k})$ of a "small" (or typical) consumer who takes the law of motion for aggregate capital as given. The functions v and g depend on s as well as on primitives of technology and preferences. (Hint: Guess that $v(k, \bar{k}) = a + b\log(k + d\bar{k}) + e\log(\bar{k})$ and then find expressions for the unknown coefficients a, b, d, and e in terms of the structural parameters α and β and the behavioral parameter s.)

(b) Find the competitive equilibrium value of s by imposing the consistency condition $G(\bar{k}) = g(\bar{k}, \bar{k})$. Verify that the resulting law of motion for aggregate capital solves the planning problem for this economy. Display v and g for the equilibrium value of s.

(c) How does an increase in aggregate capital affect the savings behavior and the (indirect) utility of a typical consumer (holding fixed the consumer's own holdings of capital)?

2. This problem studies a neoclassical growth model with an externality in production. Leisure is not valued and the (representative) consumer has time-separable preferences with discount factor $\beta \in (0, 1)$. Consumers, who own the factors of production, are endowed with k_0 units of capital in period 0 and with one unit of time in each period. There is a large number of identical profit-maximizing firms each of which has the following production technology:

$$f(k, n, \bar{k}) = Ak^{\alpha} n^{1-\alpha} \bar{k}^{\gamma} + (1 - \delta)k,$$

where k is the amount of capital rented by the firm, n is the amount of labor hired by the firm, $\bar{k}$ is the aggregate capital stock, δ is the rate of depreciation of capital. The parameters satisfy: $0 < \gamma < 1 - \alpha$, $0 < \alpha < 1$, and $0 < \delta \leq 1$. Thus there is a productive externality from the rest of the economy: a higher aggregate capital stock increases the productivity of each firm. A typical (small) firm takes the aggregate capital stock as given when choosing its inputs.

(a) Carefully define a sequential competitive equilibrium for this economy.

(b) Carefully define a recursive competitive equilibrium for this economy.

(c) Find a second-order difference equation that governs the evolution of the economy's aggregates in competitive equilibrium. (Hint: Find a typical consumer's Euler equation and then impose equilibrium conditions.)

(d) Display the Bellman equation for the social planning problem in this economy. The planner internalizes the externality in production: his production technology is

$$h(\bar{k}, n) \equiv f(\bar{k}, n, \bar{k}) = A\bar{k}^{\alpha+\gamma} n^{1-\alpha} + (1 - \delta)\bar{k}.$$

Is the competitive equilibrium allocation Pareto optimal? (Hint: Compare the planner's Euler equation to the second-order difference equation that you found in part (c).)

(e) Now introduce a government that subsidizes savings at a proportional rate τ and finances these subsidies by means of a lump-sum tax on consumers. The investment subsidy is constant across time but the lump-sum tax varies over time so as to balance the government's budget in every period. Define a recursive competitive equilibrium for this economy.

(f) For what subsidy rate τ is the competitive equilibrium steady-state aggregate capital stock equal to the steady-state aggregate capital stock in the planning problem?

3. Consider a neoclassical growth model in which consumers value leisure: a typical consumer's preferences over sequences of consumption and leisure are described by $\sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t)$, where c_t is consumption in period t and ℓ_t is leisure in period t. The felicity function u is strictly increasing in both arguments, strictly concave, twice continuously differentiable, and satisfies $\lim_{c \to 0} u_c(c, \ell) = \infty$. In all other respects, this economy is identical to the one that we have discussed in class.

(a) Carefully define a sequential competitive equilibrium for this economy.

(b) Carefully define a recursive competitive equilibrium for this economy. (Hint: You need *two* functions to describe the behavior of the aggregate economy.)

(c) Show that the competitive equilibrium allocation for this economy solves a social planning problem. (Hint: Compare first-order conditions.)

(d) Let the aggregate production function take the form $f(\bar{k}, \bar{n}) = \bar{k}^{\alpha}\bar{n}^{1-\alpha} + (1-\delta)\bar{k}$, where $\bar{n}$ is aggregate labor supply, and let $u(c, \ell) = \lambda \log(c) + (1-\lambda)\log(\ell)$. The parameters satisfy: $0 < \alpha < 1$, $0 < \delta \leq 1$, and $0 < \lambda < 1$. Solve explicitly (in terms of parameters) for the steady-state values of aggregate capital and aggregate leisure in competitive equilibrium.