HOMEWORK #4
This homework assignment is due at the beginning of class on Wednesday, November 16.

1. Consider a neoclassical growth model in which consumers have time separable preferences given by: $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Let the aggregate production (or resource) function take the form:

$$f(\bar{k}, n) = A\bar{k}^\alpha n^{1-\alpha} + (1-\delta)\bar{k},$$

where $\delta$ is the rate of depreciation of capital. The parameters satisfy: $0 < \beta < 1$, $A > 0$, $0 < \alpha < 1$, and $0 < \delta \leq 1$. Consumers are endowed with one unit of time in each period but do not value leisure (so that $n = 1$). In this problem, you will solve explicitly for the recursive competitive equilibrium of this economy under the assumptions that $u(c) = \log(c)$ and $\delta = 1$. (Assume too that the economy is decentralized in the manner that we have discussed in class.)

(a) Suppose that aggregate capital evolves according to $\bar{k}' = G(\bar{k}) = sf(\bar{k}, 1)$. (You will verify the validity of this conjecture below.) Find explicit formulas for the value function $v(k, \bar{k})$ and the decision rule $k' = g(k, \bar{k})$ of a “small” (or typical) consumer who takes the law of motion for aggregate capital as given. The functions $v$ and $g$ depend on $s$ as well as on primitives of technology and preferences. (Hint: Guess that $v(k, \bar{k}) = a + b \log(k + d\bar{k}) + e \log(\bar{k})$ and then find expressions for the unknown coefficients $a, b, d,$ and $e$ in terms of the structural parameters $\alpha$ and $\beta$ and the behavioral parameter $s$.)

(b) Find the competitive equilibrium value of $s$ by imposing the consistency condition $G(\bar{k}) = g(\bar{k}, \bar{k})$. Verify that the resulting law of motion for aggregate capital solves the planning problem for this economy. Display $v$ and $g$ for the equilibrium value of $s$.

(c) How does an increase in aggregate capital affect the savings behavior and the (indirect) utility of a typical consumer (holding fixed the consumer’s own holdings of capital)?

2. This problem studies a neoclassical growth model with an externality in production. Leisure is not valued and the (representative) consumer has time separable preferences with discount factor $\beta \in (0, 1)$. Consumers, who own the factors of production, are
endowed with \( k_0 \) units of capital in period 0 and with one unit of time in each period. There is a large number of identical profit-maximizing firms each of which has the following production technology:

\[
f(k, n, \bar{k}) = A k^\alpha n^{1 - \alpha} \bar{k}^\gamma + (1 - \delta)k,
\]

where \( k \) is the amount of capital rented by the firm, \( n \) is the amount of labor hired by the firm, \( \bar{k} \) is the aggregate capital stock, \( \delta \) is the rate of depreciation of capital. The parameters satisfy: \( 0 < \gamma < 1 - \alpha \), \( 0 < \alpha < 1 \), and \( 0 < \delta \leq 1 \). Thus there is a productive externality from the rest of the economy: a higher aggregate capital stock increases the productivity of each firm. A typical (small) firm takes the aggregate capital stock as given when choosing its inputs.

(a) Carefully define a sequential competitive equilibrium for this economy.

(b) Carefully define a recursive competitive equilibrium for this economy.

(c) Find a second-order difference equation that governs the evolution of the economy’s aggregates in competitive equilibrium. (Hint: Find a typical consumer’s Euler equation and then impose equilibrium conditions.)

(d) Display the Bellman equation for the social planning problem in this economy. The planner internalizes the externality in production: his production technology is

\[
h(\bar{k}, n) \equiv f(\bar{k}, n, \bar{k}) = A \bar{k}^{\alpha + \gamma} n^{1 - \alpha} + (1 - \delta)\bar{k}.
\]

Is the competitive equilibrium allocation Pareto optimal? (Hint: Compare the planner’s Euler equation to the second-order difference equation that you found in part (c).)

(e) Now introduce a government that subsidizes savings at a proportional rate \( \tau \) and finances these subsidies by means of a lump-sum tax on consumers. The investment subsidy is constant across time but the lump-sum tax varies over time so as to balance the government’s budget in every period. Define a recursive competitive equilibrium for this economy.

(f) For what subsidy rate \( \tau \) is the competitive equilibrium steady-state aggregate capital stock equal to the steady-state aggregate capital stock in the planning problem?

3. Consider a neoclassical growth model in which consumers value leisure: a typical consumer’s preferences over sequences of consumption and leisure are described by

\[
\sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t),
\]

where \( c_t \) is consumption in period \( t \) and \( \ell_t \) is leisure in period \( t \). The felicity function \( u \) is strictly increasing in both arguments, strictly concave, twice continuously differentiable, and satisfies \( \lim_{\epsilon \to 0} u(c, \ell) = \infty \). In all other respects, this economy is identical to the one that we have discussed in class.
(a) Carefully define a sequential competitive equilibrium for this economy.

(b) Carefully define a recursive competitive equilibrium for this economy. (Hint: You need two functions to describe the behavior of the aggregate economy.)

(c) Show that the competitive equilibrium allocation for this economy solves a social planning problem. (Hint: Compare first-order conditions.)

(d) Let the aggregate production function take the form \( f(\bar{k}, \bar{n}) = \bar{k}^\alpha \bar{n}^{1-\alpha} + (1-\delta)\bar{k}, \) where \( \bar{n} \) is aggregate labor supply, and let \( u(c, \ell) = \lambda \log(c) + (1 - \lambda) \log(\ell). \) The parameters satisfy: 0 < \( \alpha < 1, \) 0 < \( \delta \leq 1, \) and 0 < \( \lambda < 1. \) Solve explicitly (in terms of parameters) for the steady-state values of aggregate capital and aggregate leisure in competitive equilibrium.