Econ 510a (second half)
Yale University
Fall 2007
Prof. Tony Smith

HOMEWORK #5

This homework assignment is due at 4PM on Friday, December 7. Please put your assignment in Evrim Aydin's mailbox.

1. Consider an exchange economy with two types of infinitely-lived consumers. Each type of consumer has measure one. Both types of consumers have identical preferences over consumption streams given by: $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$, where u exhibits constant relative risk aversion equal to $\gamma > 0$. Let $\bar{\omega}$ denote the aggregate endowment of the consumption good in any given period, and assume that $\bar{\omega}$ is a random variable that equals 2 with probability η and 1 with probability $1-\eta$. In any given period, type-1 consumers receive fraction θ of the aggregate endowment (so that type-2 consumers receive fraction $1-\theta$ of the aggregate endowment), where θ is a random variable that equals 3/4 with probability α and 1/2 with probability $1-\alpha$. Assume that the probabilities η and α are time-invariant and that the aggregate endowment $\bar{\omega}$ and the fraction θ are independent random variables. Endowments are non-storable.

   (a) This economy has four states of the world in any given period. Determine the probability of each state and the endowment that each type of consumer receives in each state.

   (b) Carefully define a competitive equilibrium with date-0 trading for this economy.

   (c) Show that, in equilibrium, relative prices depend on the aggregate endowment but not on how the aggregate endowment is distributed across the two types of consumers. In other words, show that the ratio of any two prices of date- and state-contingent consumption goods does not depend on the specific values that the random variable θ takes on.

   (d) Determine the competitive equilibrium allocation of consumption under the assumption that $\gamma = 1$ (i.e., assume that u is logarithmic).

   (e) Determine the prices of the Arrow securities in this economy (how many of them are there?) and use them to find the price of a riskfree bond, expressed in terms of today's consumption goods, in each state. (Recall that a riskfree bond is a sure claim to one unit of the consumption good tomorrow.)

2. Consider the planning problem for a neoclassical growth model with logarithmic utility, full depreciation of the capital stock in one period, and a production function of the form $y = zk^\alpha$, where z is a random shock to productivity. The shock z is observed before making the current-period savings decision. Assume that the capital stock can take on only two values: i.e., k is restricted to the set $\{\bar{k}_1, \bar{k}_2\}$. In addition, assume that z takes on values in the set $\{\bar{z}_1, \bar{z}_2\}$ and that z follows a Markov chain with transition probabilities $p_{ij} = P(z' = \bar{z}_j | z = \bar{z}_i)$.

   (a) Let $\bar{z}_1 = 0.9$, $\bar{z}_2 = 1.1$, $p_{11} = 0.9$, and $p_{22} = 0.8$. Find the invariant distribution associated with the Markov chain for z. Use the invariant distribution to compute the long-run (or unconditional) expected value of z; that is, compute $E(z) = \pi_1 \bar{z}_1 + \pi_2 \bar{z}_2$, where π_1 and π_2 characterize the invariant distribution.

   (b) Let $\beta = 0.9$, $\alpha = 0.36$, $\bar{k}_1 = 0.95k_{ss}$, and $\bar{k}_2 = 1.05k_{ss}$, where k_{ss} is the steady-state capital stock in a version of this model without shocks and with no restrictions on capital (i.e., $k_{ss} = (\alpha\beta)^{\frac{1}{(1-\alpha)}}$). Let $g(k, z)$ denote the planner's optimal decision rule. Prove that $g(k_i, z_j) = k_j$ for all i and j.

   (c) The decision rule from part (b) and the law of motion for z jointly determine an invariant distribution over (k, z)-pairs. Find this distribution. (That is, find probabilities $\pi_{ij} = P(k = k_i, z = z_j)$ that "reproduce" themselves: if π_{ij} is the unconditional probability that the economy is in state (k_i, z_j) today, then it is also the unconditional probability that the economy is in this state tomorrow. For a more complete discussion of this concept, see pp. 78 and 79 in the lecture notes by Per Krusell.) Use your answer to compute the long-run (or unconditional) expected values of the capital stock and of output.

3. In the Mehra-Prescott model that we discussed in lecture on December 3 (or will discuss, if you are reading this before then!), for what value of γ (the degree of risk aversion) does the model generate an average (annual) equity premium of 6%? (Set $\beta = 0.96$ and fix the other parameters at the values that we used in lecture.) For this degree of risk aversion, what is the average (annual) risk-free rate? Is it close to the historical average of 1%?

4. *(You do not need to submit an answer for this problem, but you are responsible for the material that it covers.)* This problem studies a neoclassical growth model with an externality in production. Leisure is not valued and the (representative) consumer has time-separable preferences with discount factor $\beta \in (0, 1)$. Consumers, who own the factors of production, are endowed with k_0 units of capital in period 0 and with one unit of time in each period. There is a large number of identical profit-maximizing firms each of which has the following production technology:

$$f(k, n, \bar{k}) = Ak^\alpha n^{1-\alpha}\bar{k}^\gamma + (1 - \delta)k,$$

where k is the amount of capital rented by the firm, n is the amount of labor hired by the firm, $\bar{k}$ is the aggregate capital stock, δ is the rate of depreciation of capital. The parameters satisfy: $0 < \gamma < 1 - \alpha$, $0 < \alpha < 1$, and $0 < \delta \leq 1$. Thus there is a productive externality from the rest of the economy: a higher aggregate capital stock increases the productivity of each firm. A typical (small) firm takes the aggregate capital stock as given when choosing its inputs.

(a) Carefully define a recursive competitive equilibrium for this economy.

(b) Find a second-order difference equation that governs the evolution of the aggregate capital stock in competitive equilibrium. (Hint: Find a typical consumer's Euler equation and then impose equilibrium conditions.) Use this equation to find an expression for the steady-state aggregate capital stock in competitive equilibrium.

(c) Display the Bellman equation for the social planning problem in this economy. The planner internalizes the externality in production: his production technology is

$$h(\bar{k}, n) \equiv f(\bar{k}, n, \bar{k}) = A\bar{k}^{\alpha+\gamma}n^{1-\alpha} + (1-\delta)\bar{k}.$$

Is the competitive equilibrium allocation Pareto optimal? (Hint: Compare the planner's Euler equation to the second-order difference equation that you found in part (b).)

(d) Now introduce a government that subsidizes savings at a proportional rate τ and finances these subsidies by means of a lump-sum tax on consumers. The investment subsidy is constant across time but the lump-sum tax varies over time so as to balance the government's budget in every period. Define a recursive competitive equilibrium for this economy.

(e) For what subsidy rate τ is the competitive equilibrium steady-state aggregate capital stock equal to the steady-state aggregate capital stock in the planning problem?