Econ 511b (Part I)
Yale University
Spring 2004
Prof. Tony Smith

HOMEWORK #5

This homework assignment should be handed in by 5PM on Friday, February 13 to Jinhui Bai's mailbox in the basement of 28 Hillhouse.

1. Consider the planning problem for a neoclassical growth model with logarithmic utility, full depreciation of the capital stock in one period, and a production function of the form $y = zk^\alpha$, where z is a random shock to productivity. The shock z is observed before making the current-period savings decision. Assume that the capital stock can take on only two values: i.e., k is restricted to the set $\{\bar{k}_1, \bar{k}_2\}$. In addition, assume that z takes on values in the set $\{\bar{z}_1, \bar{z}_2\}$ and that z follows a Markov chain with transition probabilities $p_{ij} = P(z' = \bar{z}_j | z = \bar{z}_i)$.

(a) Let $\bar{z}_1 = 0.9$, $\bar{z}_2 = 1.1$, $p_{11} = 0.95$, and $p_{22} = 0.9$. Find the invariant distribution associated with the Markov chain for z. Use the invariant distribution to compute the long-run (or unconditional) expected value of z.

(b) Let $\beta = 0.9$, $\alpha = 0.36$, $\bar{k}_1 = 0.95k_{ss}$, and $\bar{k}_2 = 1.05k_{ss}$, where k_{ss} is the steady-state capital stock in a version of this model without shocks and with no restrictions on capital (i.e., $k_{ss} = (\alpha\beta)^{\frac{1}{(1-\alpha)}}$). Using Matlab (if you need to), find the optimal decision rule for capital, i.e., a function mapping pairs of the form (k, z) into the optimal choice for capital.

(c) The decision rule from part (b) and the law of motion for z jointly determine an invariant distribution over (k, z)-pairs. Find this distribution. Use your answer to compute the long-run (or unconditional) expected values of the capital stock and of output.

(d) In Matlab, use the optimal decision rule, the law of motion for z, and a random number generator to create a simulated time series $\{k_t, y_t\}_{t=0}^T$, given an initial condition (k_0, z_0). Compute $T^{-1}\sum_{t=1}^T k_t$ and $T^{-1}\sum_{t=1}^T y_t$ for a suitably large value of T and confirm that these sample means are close to the corresponding population means that you computed in part (c).

2. Consider a two-period exchange economy with two (types of) consumers labelled A and B. The two types of consumers have identical preferences given by $u(c_0) + \beta E u(c_1)$, where u is strictly increasing and strictly concave. Each consumer is endowed with one consumption good in period 0. In period 1, each type A consumer is endowed with θy consumption goods and each type B consumer is endowed with $(1-\theta)y$ consumption goods. The random variable θ can be interpreted as the consumer's share of the aggregate endowment y. Let θ equal $1/2+z$ with probability p and equal $1/2-z$ with probability $1-p$, where $0 < z < 1/2$. The aggregate endowment y is also random: it equals $1+x$ with probability one-half and equals $1-x$ with probability one-half. The random variables y and θ are statistically independent.

(a) Assume that markets are complete: in period 0, consumers can trade a full set of Arrow securities. Express the competitive equilibrium allocations and prices in terms of primitives as explicitly as you can. You might want to start with the case where $p = 1/2$ (so that the consumers are identical in all respects) and then consider the more general case $p \neq 1/2$.

(b) Use the prices of the Arrow securities from part (a) to find the equilibrium period-0 price of a risk-free bond (i.e., an asset that pays one unit of the consumption in all states of the world in period 1). If you are unable to solve for the Arrow prices explicitly, then show how you would use these prices to compute the price of a risk-free bond.

(c) Now suppose that markets are incomplete: in period 0, the only asset that consumers are allowed to trade is a risk-free bond. The net supply of bonds is zero (since the economy is closed). Find the competitive equilibrium allocations and the equilibrium price of the bond as explicitly as you can. Compare your answers to those in parts (a) and (b). Show that eliminating complete markets makes consumers worse off. Does eliminating complete markets increase or decrease the risk-free rate of return (i.e., the inverse of the bond price)? Why? (Again, you might want to start with the case $p = 1/2$ before considering the case $p \neq 1/2$.)

(d) Introduce a second asset into the economy you studied in part (c). Specifically, in addition to the endowments described above (which can be viewed as "labor income"), let each consumer be endowed with one "Lucas tree" in period 0. Each tree yields a "dividend" of d consumption goods in period 1, where d equals d_H if y equals $1+x$ and equals $d_L < d_H$ if y equals $1-x$. Trees, as well as risk-free bonds, can be bought and sold in competitive markets in period 0. Without doing any explicit calculations, show how you would go about solving for the equilibrium prices of the two assets in this economy.

(e) Determine (as completely as you can) the prices of a risk-free bond and of a Lucas tree under the assumption that consumers can trade a full set of Arrow securities in the economy that you studied in part (d). Compare (if possible) these prices to the corresponding prices in part (d).

3. Consider a complete-markets exchange economy populated by identical consumers whose preferences exhibit "habit persistence":

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma},$$

where $\sigma > 0$, $\beta \in (0,1)$, and λ is positive and bounded. Each consumer has the same endowment ω_t in period t. Assume, for simplicity, that ω_t grows deterministically according to: $\omega_{t+1} = g\,\omega_t$, with $g > 1$.

(a) Formulate the consumer's consumption-savings problem as a dynamic programming problem: there is one asset, a one-period riskless bond whose price is q.

(b) Derive the Euler equation for the consumer's problem.

(c) Find the equilibrium bond price in this model as a function of the structural parameters.

(d) Suppose now that the endowment grows stochastically: $\omega_{t+1} = g_{t+1}\omega_t$, where the growth rate g_{t+1} is independent across time and takes on the two values $\lambda_1 > 1$ and $\lambda_2 < 1$ with equal probability. Find the prices of the Arrow securities and use them to compute the (long-run) average rate of return on a riskless bond in this model. If you cannot find explicit solutions for the prices of the Arrow securities, then show what conditions they must satisfy (i.e., find a set of equations that determine these prices) and explain how you would use them to compute the average rate of return on a riskless bond in this model.