1 Uncertainty

Our program of study will comprise the following four topics:

1. Basic Concepts
2. Examples of common stochastic processes in macroeconomics
3. Maximization under uncertainty
4. Competitive equilibrium under uncertainty

The first two are closely related to time series analysis. The last two are a generalization of the tools we have already introduced to the case where the decision makers face uncertainty.

1.1 Basic concepts

We will introduce the basic elements with which uncertain events are modeled. The main mathematical notion underlying the concept of uncertainty is that of a probability space.

Definition 1 *A probability space is a mathematical object consisting of three elements: 1) a set Ω of possible outcomes ϖ; 2) a collection $\mathcal{F}$ of subsets of Ω that constitute the "events" to which probability is assigned (a σ-algebra); and 3) a set function P that assigns probability values to those events. A probability space is denoted by:*

$$(\Omega, \mathcal{F}, P)$$

Definition 2 *A σ-algebra ($\mathcal{F}$) is a special kind of family of subsets of a space Ω that verify three properties: 1) $\Omega \in \mathcal{F}$; 2) $\mathcal{F}$ is closed under complementation: $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$; 3) $\mathcal{F}$ is closed under countable union: if $\{E_i\}_{i=1}^{\infty}$ is a sequence of sets such that $E_i \in \mathcal{F}$ $\forall i$, then $(\cup_{i=1}^{\infty} E_i) \in \mathcal{F}$.*

Definition 3 *A random variable is a function whose domain is the set of events Ω and whose image is the real numbers (or a subset thereof):*

$$x : \Omega \to \Re$$

For any real number α, define the set

$$E_\alpha = \{\varpi : x(\varpi) < \alpha\}$$

Definition 4 *A function x is said to be measurable with respect to the σ-algebra $\mathcal{F}$ (or $\mathcal{F}$-measurable) if the following property is satisfied:*

$$\forall \alpha \in \Re : E_\alpha \in \mathcal{F}$$

Conceptually, if x is $\mathcal{F}$-measurable then we can assign probability to the event $x < \alpha$ for any real number α. [We may equivalently have used $>$, $\leq$ or $\geq$ for the definition of measurability, but that is beyond the scope of this course. You only need to know that if x is $\mathcal{F}$-measurable, then we can sensibly talk about the probability of x taking values in virtually any subset of the real line you can think of (the *Borel* sets).]

Now define a sequence of σ-algebras:

$$\{\mathcal{F}_t\}_{t=1}^{\infty} : \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}$$

Conceptually, each σ-algebra $\mathcal{F}_t$ "refines" $\mathcal{F}_{t-1}$, in the sense that distinguishes (in a probabilistic sense) between "more" events than the previous one.

Finally, let a sequence of random variables x_t be $\mathcal{F}_t$-measurable for each t: this models a stochastic process. Consider an $\varpi \in \Omega$, and choose an $\alpha \in \Re$. Then for each t, the set $E_{\alpha t} \equiv \{\varpi : x_t(\varpi) < \alpha\}$ will be a set included in the collection (the σ-algebra) $\mathcal{F}_t$. Since $\mathcal{F}_t \subseteq \mathcal{F}$ for all t, $E_{\alpha t}$ also belongs to $\mathcal{F}$. Hence, we can assign probability to $E_{\alpha t}$ using the set function P: $P[E_{\alpha t}]$ is well defined

The following example may clarify:

Consider the probability space $(\Omega, \mathcal{F}, P)$, where:

$\Omega = [0, 1]$

$\mathcal{F} = \mathcal{B}$ (the Borel sets restricted to $[0, 1]$)

$P = \lambda$ - the length of an interval: $\lambda([a, b]) = b - a$

Consider the following collections of sets:

$$A_t = \left\{ \left\{ \left[\frac{j}{2^t}, \frac{j+1}{2^t} \right) \right\}_{j=0}^{2^t-2}, \left[\frac{2^t-1}{2^t}, 1 \right] \right\}$$

For every t, let $\mathcal{F}_t$ be the minimum σ-algebra containing A_t. Denote by $\sigma(A_t)$ the collection of all possible unions of the sets in A_t (notice that $\Omega \in \sigma(A_t)$). Then $\mathcal{F}_t = \{\emptyset, A_t, \sigma(A_t)\}$ (you should check that this is a σ-algebra).

For example,

$$A_1 = \left\{ [0, 1], \emptyset, \left[0, \tfrac{1}{2}\right), \left[\tfrac{1}{2}, 1\right] \right\}$$

$$\Rightarrow \mathcal{F}_1 = \left\{ [0, 1], \emptyset, \left[0, \tfrac{1}{2}\right), \left[\tfrac{1}{2}, 1\right] \right\}$$

$$A_2 = \left\{ \left[0, \tfrac{1}{4}\right), \left[\tfrac{1}{4}, \tfrac{1}{2}\right), \left[\tfrac{1}{2}, \tfrac{3}{4}\right), \left[\tfrac{3}{4}, 1\right] \right\}$$

$$\Rightarrow \sigma(A_2) = \left\{ \left[0, \tfrac{1}{2}\right), \left[0, \tfrac{3}{4}\right), \left[\tfrac{1}{4}, \tfrac{3}{4}\right), \left[\tfrac{1}{4}, 1\right], \left[\tfrac{1}{2}, 1\right], \left[0, \tfrac{1}{4}\right) \cup \left[\tfrac{1}{2}, \tfrac{3}{4}\right) \right\} \cup$$

$$\cup\left\{\left[0,\,\tfrac{1}{4}\right)\cup\left[\tfrac{1}{2},\,1\right],\,\left[0,\,\tfrac{1}{4}\right)\cup\left[\tfrac{3}{4},\,1\right],\,\left[0,\,\tfrac{1}{2}\right)\cup\left[\tfrac{3}{4},\,1\right],\,\left[\tfrac{1}{4},\,\tfrac{1}{2}\right)\cup\left[\tfrac{3}{4},\,1\right],\,[0,\,1]\right\}$$

Now consider the experiment of repeated fair coin flips: $c_t \in \{0,\,1\}$. The infinite sequence $\{c_t\}_{t=0}^{\infty}$ is a stochastic process that can be modeled with the probability space and associated sequence of σ-algebras that we have defined above. Each sequence $\{c_t\}_{t=0}^{\infty}$ is an "outcome", represented by a number $\varpi \in \Omega$.

For every t let $y_t = \{c_j\}_{j=1}^{t}$ (this will be a t-dimensional vector of zeros and ones), and to each possible configuration of y_t (there are 2^t possible ones), associate a distinct interval in A_t. For example, for $t = 1$ and $t = 2$, let

$$\begin{aligned} I_1\left[(0)\right] &= \left[0,\,\tfrac{1}{2}\right) \\ I_1\left[(1)\right] &= \left[\tfrac{1}{2},\,1\right] \\ I_2\left[(0,\,0)\right] &= \left[0,\,\tfrac{1}{4}\right) \\ I_2\left[(0,\,1)\right] &= \left[\tfrac{1}{4},\,\tfrac{1}{2}\right) \\ I_2\left[(1,\,0)\right] &= \left[\tfrac{1}{2},\,\tfrac{3}{4}\right) \\ I_2\left[(1,\,0)\right] &= \left[\tfrac{3}{4},\,1\right] \end{aligned}$$

For $t = 3$, we will have a three-coordinate vector, and we will have the following restrictions on I_3:

$$\begin{aligned} I_3\left[(0,\,0,\,\cdot)\right] &\subset \left[0,\,\tfrac{1}{4}\right) \\ I_3\left[(0,\,1,\,\cdot)\right] &\subset \left[\tfrac{1}{4},\,\tfrac{1}{2}\right) \\ I_3\left[(1,\,0,\,\cdot)\right] &\subset \left[\tfrac{1}{2},\,\tfrac{3}{4}\right) \\ I_3\left[(1,\,1,\,\cdot)\right] &\subset \left[\tfrac{3}{4},\,1\right] \end{aligned}$$

and so on for the following t.

Then a number $\varpi \in \Omega$ implies a sequence of intervals $\{I_t\}_{t=0}^{\infty}$ that represents, for ever t, the "partial" outcome realized that far.

Finally, the stochastic process will be modeled by a function x_t that, for each t and for each $\varpi \in \Omega$, associates a real number; such that x_t is $\mathcal{F}_t$-measurable. For example, take $\varpi' = .7$ and $\varpi'' = .8$, then $I_1\left[y_1'\right] = I_1\left[y_1''\right] = \left[\tfrac{1}{2},\,1\right]$ - that is, the first element of the respective sequences c_t', c_t'' is a 1 (say "Heads").

Then we must have $x_1\left(\varpi'\right) = x_1\left(\varpi''\right) \equiv b$. We are ready now to answer the following question: What is the probability that the first toss in the experiment is Heads? Or, in our model, what is the probability that $x_1\left(\varpi\right) = b$? To answer this question, we look at measure of the set of ϖ that will produce the value $x_1\left(\varpi\right) = b$:

$$E = \left\{\varpi : x_1\left(\varpi\right) = b\right\} = \left[\tfrac{1}{2},\,1\right] \quad \left(\in \mathcal{F}_1\right)$$

And the probability of the event $\left[\frac{1}{2},\, 1\right]$ is calculated using $P\left(\left[\frac{1}{2},\, 1\right]\right) = \lambda\left(\left[\frac{1}{2},\, 1\right]\right) = \frac{1}{2}$.

That is, the probability that the event $\{c_t\}_{t=1}^{\infty}$ to be drawn produces a Head as its first toss is $\frac{1}{2}$. ■

Definition 5 *Let $B \in \mathcal{F}$. Then the joint probability of the events $(x_{t+1},\, ...,\, x_{t+n}) \in B$ is given by*

$$P_{t+1,\, ...,\, t+n}(B) = P\left[\varpi \in \Omega : \left[x_{t+1}\left(\varpi\right),\, ...,\, x_{t+n}\left(\varpi\right)\right] \in B\right]$$

Definition 6 *A stochastic process is stationary if $P_{t+1,\, ...,\, t+n}(B)$ is independent of t, $\forall t$, $\forall n$, $\forall B$.*

Conceptually, if a stochastic process is stationary, then the joint probability distribution for any $(x_{t+1},\, ...,\, x_{t+n})$ is independent of time.

Given an observed realization of the sequence $\{x_j\}_{j=1}^{\infty}$ in the last s periods: $(x_{t-s},\, ...,\, x_t) = (a_{t-s},\, ...,\, a_t)$, the *conditional* probability of the event $(x_{t+1},\, ...,\, x_{t+n}) \in B$ is denoted by

$$P_{t+1,\, ...,\, t+n}\left[B \,|x_{t-s} = a_{t-s},\, ...,\, x_t = a_t\right]$$

Definition 7 *A first order Markov Process is a stochastic process with the property that:*

$$P_{t+1,\, ...,\, t+n}\left[B \,|x_{t-s} = a_{t-s},\, ...,\, x_t = a_t\right] = P_{t+1,\, ...,\, t+n}\left[B \,|x_t = a_t\right]$$

Definition 8 *A stochastic process is weakly stationary (or covariance stationary) if the first two moments of the joint distribution of $(x_{t+1},\, ...,\, x_{t+n})$'s are independent of time.*

A usual assumption in macroeconomics is that the exogenous randomness affecting the economy can be modeled as a (weakly) stationary stochastic process. The task then is to look for stochastic processes for the endogenous variables (capital, output, etc.) that are stationary. This stochastic stationarity is the analogue to the steady state in deterministic models.

For example, suppose that productivity is subject to a two-state shock:

$$\begin{aligned} y &= z \cdot F(k) \\ z &\in \{z_L,\, z_H\} \end{aligned}$$

Imagine for example that the z_t's are iid, with $\Pr\left[z_t = z_H\right] = \frac{1}{2} = \Pr\left[z_t = z_L\right]$ $\forall t$. Then the policy function will now be a function of both the initial capital stock K and the realization of the shock z: $g\left(k,\, z\right) \in \left\{g\left(k,\, z_L\right),\, g\left(k,\, z_H\right)\right\}$ $\forall K$. We need to find the functions $g\left(k,\, \cdot\right)$. Notice that they will determine a stochastic process for capital - the trajectory of capital in this economy will be subject to a random shock. The following chart shows an example of such a trajectory:

The interval $\left(\underline{k}^*, \overline{k}^*\right)$ is the *ergodic set*: once the level of capital enters this set, it will not leave it again. The capital stock will follow a stationary stochastic process within the limits of the ergodic set.

1.2 Examples of common stochastic processes in macroeconomics

The two main types of modeling techniques that macroeconomists make use of are:

1. Markov chains
2. Linear stochastic difference equations

Markov chains

Let $x_t \in X$, where $X = \{\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\}$ is a finite set of values. A stationary Markov Chain is a stochastic process $\{x_t\}_{t=0}^{\infty}$ defined by an X, a transition matrix $\underset{n\times n}{P}$, and an initial probability distribution $\underset{1\times n}{\pi_0}$ for x_0 (the first element in the stochastic process).

The elements of $\underset{n\times n}{P}$ represent the following probabilities:

$$P_{ij} = \Pr\left[x_{t+1} = \overline{x}_j \,|x_t = \overline{x}_i\,\right]$$

Notice that these probabilities are independent of time. We also have that the probability two periods ahead is given by:

$$\begin{aligned}\Pr\left[x_{t+2} = \overline{x}_j \,|x_t = \overline{x}_i\right] &= \sum_{k=1}^{n} P_{ik} \cdot P_{kj} \\ &\equiv \left[P^2\right]_{i,j}\end{aligned}$$

where $\left[P^2\right]_{i,j}$ denotes the $(i,\, j)^{\text{th}}$ entry of the matrix P^2.

Given π_0, π_1 will be the probability distribution of x_1, as at time $t = 0$, and will be given by:

$$\pi_1 = \pi_0 \cdot P$$

Analogously,

$$\begin{aligned}\pi_2 &= \pi_0 \cdot P^2 \\ \vdots &= \vdots \\ \pi_t &= \pi_0 \cdot P^t\end{aligned}$$

and also:

$$\pi_{t+1} = \pi_t \cdot P$$

Definition 9 *A stationary (or invariant) distribution for P is a probability vector π such that*

$$\pi = \pi \cdot P$$

A stationary distribution then verifies

$$1 \cdot \pi = \pi \cdot P$$

and

$$\begin{aligned}\pi - \pi \cdot P &= 0 \\ \pi \cdot [I - P] &= 0\end{aligned}$$

That is, π is an eigenvector of P, associated with the eigenvalue $\lambda = 1$.

Example 10 $P = \begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix} \Rightarrow \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \cdot \begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix}$

You should verify that $\pi = \begin{pmatrix} \frac{4}{7} & \frac{3}{7} \end{pmatrix}$.

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \pi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$P = \begin{pmatrix} 1 & 0 \\ .1 & .9 \end{pmatrix} \Rightarrow \pi = \begin{pmatrix} 1 & 0 \end{pmatrix}$ *$\rightarrow$ 1 is said to be an "absorbing" state*

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \pi = \begin{pmatrix} a & 1-a \end{pmatrix}, \ a \in [0, 1]$$

In the last case, there is a continuum of invariant distributions

Question: Does π_t converge, in some sense, to a number π_∞ as $t \to \infty$? (This would mean that $\pi_\infty = \pi_\infty \cdot P$.) If so, does π_∞ depend on the initial condition π_0?

If the answers to these two questions are "Yes" and "No", respectively, then the stochastic process is said to be "asymptotically stationary", with a unique invariant distribution. Fortunately, we can borrow on the following result for sufficient conditions for asymptotic stationarity:

Theorem 11 *P has a unique invariant distribution (and is asymptotically stationary) if $P_{ij} > 0 \ \forall i, \forall j$.*

Linear stochastic difference equations [AR(1), ARIMA(4,7,1), ...]

Let $x_t \in \Re^n, \ w_t \in \Re^m$,

$$x_{t+1} = \underset{n \times n}{A} \cdot x_t + \underset{n \times n}{C} \cdot w_{t+1}$$

We normally assume:

$$\begin{aligned} E_t\left[w_{t+1}\right] &= E_t\left[w_{t+1} \left| w_t, \, w_{t-1}, \, ...\right.\right] = 0 \\ E_t\left[w_{t+1} \cdot w_{t+1}'\right] &= I \end{aligned}$$

Example 12 (AR(1) Process)

$$y_{t+1} = \rho \cdot y_t + \varepsilon_{t+1} + b$$

$$\begin{aligned} E_t\left[\varepsilon_{t+1}\right] &= 0 \\ E_t\left[\varepsilon_{t+1}^2\right] &= \sigma^2 \\ E_t\left[\varepsilon_{t+k} \cdot \varepsilon_{t+k+1}\right] &= 0 \end{aligned}$$

Even if y_0 is known, the $\{y_t\}_{t=0}^{\infty}$ process will not be stationary in general. However, the process may become stationary as $t \to \infty$. By repeated substitution, we have that

$$E_0\left[y_t\right] = \rho^t \cdot y_0 + \frac{b}{1-\rho} \cdot \left(1 - \rho^t\right)$$

$$|\rho| < 1 \Rightarrow \lim_{t \to \infty} E_0 [y_t] = \frac{b}{1-\rho}$$

then the process will be stationary if $|\rho| < 1$. *Similarly, the autocorrelation function is given by*

$$\gamma(t,\, k) \equiv E_0 [(y_t - E[y_t]) \cdot (y_{t-k} - E[y_{t-k}])] = \sigma^2 \cdot \rho^k \cdot \frac{1-\rho^{t-k}}{1-\rho^2}$$

$$|\rho| < 1 \Rightarrow \lim_{t \to \infty} \gamma(t,\, k) = \frac{\sigma^2}{1-\rho^2} \cdot \rho^k$$

Then if $|\rho| < 1$, *the process is asymptotically weakly stationary.*

We can also regard x_0 (or y_0, in the case of an AR(1) process) as drawn from a distribution with mean μ_0 and covariance $E\left[(x_0 - \mu_0) \cdot (x_0 - \mu_0)'\right] \equiv \Gamma_0$. Then the following are sufficient conditions for $\{x_t\}_{t=0}^{\infty}$ to be weakly stationary process:

(i) μ_0 is the eigenvector associated to the eigenvalue $\lambda_1 = 1$ of A:

$$\mu_0' = \mu_0' \cdot A$$

(ii) All other eigenvalues of A are smaller than 1 in absolute value:

$$|\lambda_i| < 1 \quad i = 2, ..., n$$

To see this, notice that condition (i) implies that

$$x_{t+1} - \mu_0 = A \cdot (x_t - \mu_0) + C \cdot w_{t+1}$$

then

$$\Gamma_0 = \Gamma(0) \equiv E\left[(x_t - \mu_0) \cdot (x_t - \mu_0)'\right] = A \cdot \Gamma(0) \cdot A' + CC'$$

and

$$\Gamma(k) \equiv E\left[(x_{t+k} - \mu_0) \cdot (x_t - \mu_0)'\right] = A^k \cdot \Gamma(0)$$

This is the matrix version of the autocovariance function $\gamma(t,\, k)$ presented above. Notice we drop t as a variable in this function.

For example, let $x_t = y_t \in \Re$, $A = \rho$, $C = \sigma^2$, $w_t = \dfrac{\varepsilon_t}{\sigma}$ - we are accommodating the AR(1) process seen before to this notation. We can do the following change of variables:

$$\begin{aligned} \widehat{y}_t &= \begin{pmatrix} y_t \\ 1 \end{pmatrix} \\ \widehat{y}_{t+1} &= \underbrace{\begin{pmatrix} \rho & b \\ 0 & 1 \end{pmatrix}}_{\widehat{A}} \cdot \widehat{y}_t + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \cdot w_{t+1} \end{aligned}$$

Then using the previous results:

$$\begin{aligned} \Gamma(0) &= \rho^2 \cdot \Gamma(0) + \sigma^2 \\ \Rightarrow \quad \Gamma(0) &= \frac{\sigma^2}{1-\rho^2} \end{aligned}$$

(ignoring the constant).

1.3 Maximization under uncertainty

We will approach this topic by illustrating with examples. Let us begin with a simple 2-period model, where an agent faces a decision problem in which he needs to make the following choices:

1. Consume and save in period 1.
2. Consume and work in period 2.

The uncertainty arises in period 2's income: the wage is stochastic. We will assume that there are n possible states of the world in this second period:

$$\varpi^2 \in \{\varpi_1, ..., \varpi_n\}$$

where $\pi_i \equiv \Pr\left[\varpi^2 = \varpi_i\right]$, for $i = 1, ..., n$.

The consumer's utility function has the von Neumann - Morgenstern type - he is an expected utility maximizer. Leisure in the second period is valued:

$$U = \sum_{i=1}^{n} \pi_i \cdot u\left(c_0, c_{1i}, n_i\right) \equiv E\left[u\left(c_0, c_{1i}, n_i\right)\right]$$

Specifically, the utility function is assumed to have the form:

$$U = u\left(c_0\right) + \beta \cdot \sum_{i=1}^{n} \left[u\left(c_{1i}\right) + v\left(n_i\right)\right]$$

(with $v'\left(n_i\right) < 0$.)

Market Structure: We will assume that there is a "risk free" asset denoted by a, and priced q, such that every unit of a purchased in period 0 pays 1 unit in period 1, whatever the state of the world. The consumer faces the following budget restriction in the first period:

$$c_0 + a \cdot q = I$$

And, at each realization of the random state of the world, his budget is given by:

$$c_{1i} = a + w_i \cdot n_i \quad i = 1, ..., n$$

Therefore, the consumer's problem is:

$$\max_{c_0,\, a,\, \{c_{1i},\, n_{1i}\}_{i=1}^{n}} u\left(c_0\right) + \beta \cdot \sum_{i=1}^{n} \left[u\left(c_{1i}\right) + v\left(n_i\right)\right]$$

$$\begin{aligned} s.t.\ c_0 + a \cdot q &= I \\ c_{1i} &= a + w_i \cdot n_i \quad i = 1, ..., n \end{aligned}$$

First order conditions:

$$u'(c_0) = \lambda = \sum_{i=1}^{n} \lambda_i \cdot R$$

where $R \equiv \frac{1}{q}$.

$$\begin{aligned} \beta \cdot \pi_i \cdot u'(c_{1i}) &= \lambda_i \\ -\beta \cdot \pi_i \cdot v'(n_{1i}) &= \lambda_i \cdot w_i \\ \Rightarrow -u'(c_{1i}) \cdot w_i &= v'(n_{1i}) \end{aligned}$$

$$\begin{aligned} u'(c_0) &= \beta \cdot \sum_{i=1}^{n} \pi_i \cdot u'(c_{1i}) \cdot R \\ &\equiv \beta \cdot E[u'(c_{1i}) \cdot R] \end{aligned}$$

Example 13 *Let $u(c)$ belong to the CES class; that is $u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$. This is a common assumption in the literature. Recall that σ is the coefficient of relative risk aversion (the higher σ, the less variability in consumption across states is the consumer willing to suffer). In particular, let $\sigma = 1$, then $u(c) = \log(c)$. Assume also that $v(n) = \log(1 - n)$. Replacing in the first order conditions, these assumptions yield:*

$$c_{1i} = w_i \cdot (1 - n_i)$$

and, using the budget constraint at i,

$$c_{1i} = \frac{a \cdot R + w_i}{2}$$

Therefore,

$$\frac{1}{I - a} = \beta \cdot \sum_{i=1}^{n} \pi_i \cdot R \cdot \frac{2}{a \cdot R + w_i}$$

This is one equation in a - we get a unique solution, even if not explicit, for the amount of savings given the price q. Finally, notice that there is incomplete insurance in this model (why?).

We will now modify the market structure in the previous example. Instead of a risk free asset yielding the same payout in each state, we will allow "Arrow securities" (state-contingent claims): n assets are traded in period 0, and each unit of asset i purchased pays off 1 unit if the realized state is i, and 0 otherwise. The new budget constraint in period 0 reads:

$$c_0 + \sum_{i=1}^{n} q_i \cdot a_i = I$$

And in the second period, if the realized state is i then the consumer must abide by:

$$c_{1i} = a_i + n_i \cdot w_i$$

Notice that a risk free asset can be constructed by purchasing one unit of each a_i. Assume that the total price paid for such a portfolio is the same as before, that is:

$$q = \sum_{i=1}^{n} q_i$$

Then the question is whether the consumer will be better or worse off with this market structure than before. Intuitively, we can see that the structure of wealth transfer across periods that was available before (namely, the risk free asset) is also available now at the same cost. Therefore, the agent could not be worse off. Moreover, the market structure now allows the wealth transfer across periods to be state-specific: not only can the consumer reallocate his income between periods 0 and 1; now he is also able to move his wealth across states of the world. Conceptually, this ability to move income across states will lead to a welfare improvement if the w_i's are truly random, and if preferences show risk aversion (i.e., if the utility index $u(\cdot)$ is strictly concave).

Solving for a_i in the period-1 budget constraints, and replacing in the period-0 constraint, we can rewrite:

$$c_0 + \sum_{i=1}^{n} q_i \cdot c_{1i} = I + \sum_{i=1}^{n} q_i \cdot w_i \cdot n_i$$

We can interpret this expression in the following way. q_i is the price, in terms of c_0, of consumption goods in period 1 if the realized state is i. And $q_i \cdot w_i$ is the remuneration to labor if the realized state is i, measured in term of c_0 as well (remember that budget consolidation only makes sense if all expenditures and income are measured in the same unit of account - in this case, monetary units, where the price of c_0 has been normalized to 1, and q_i is the resulting level of relative prices).

Notice that we have thus reduced the $n+1$ constraints to 1, whereas in the previous problem we could only eliminate one and reduce them to n.

First order conditions:

$$\begin{aligned} c_0 &= \frac{1}{1+2\cdot\beta} \cdot \left(I + \sum_{i=1}^{n} q_i \cdot w_i \right) \\ c_{1i} &= \beta \cdot c_0 \cdot \frac{\pi_i}{q_i} \\ n_i &= 1 - \frac{c_{1i}}{w_i} \end{aligned}$$

The second condition says that consumption in each period is proportional to consumption in c_0. And this proportionality is a function of the cost of insurance: the higher q_i is in relation to π_i, the lower the wealth transfer into state i.

Under this equilibrium allocation, we have that the marginal rates of substitution between consumption in period 0 and consumption in period 1, for any realization of the state of the world, is given by

$$MRS\left(c_0,\, c_{1j}\right) = q_i$$

And the marginal rates of substitution across states are:

$$MRS\left(c_{1i},\, c_{1j}\right) = \frac{q_i}{q_j}$$

If $\sum_{i=1}^{n} q_i = q$, then the consumer is better off with this market structure. To see this, notice that by purchasing a risk free portfolio, then:

$$u'(c_0) = \beta \cdot \sum_{i=1}^{n} \pi_i \cdot u'(c_{1i}) \cdot R$$

where $R \equiv \dfrac{1}{q} = \dfrac{1}{\sum_{i=1}^{n} q_i}$.

1.3.1 Stochastic neoclassical growth model

Notation We introduce uncertainty into the neoclassical growth model through a stochastic shock affecting factor productivity. A very usual assumption is that of a *neutral* shock, affecting total factor productivity ("TFP"). Under certain assumptions (for example, Cobb-Douglass $y = AK^{\alpha}n^{1-\alpha}$ production technology), a productivity shock is always *neutral*, even if it is modeled as affecting a specific component (capital K, labor n, technology A).

Specifically, a neoclassical (constant returns to scale) aggregate production function subject to a TFP shock has the form:

$$F_t\left(k_t,\, 1\right) = z_t \cdot f\left(k_t\right)$$

z is a stochastic process, and the realizations z_t are drawn from a set Z: $z_t \in Z$, $\forall t$. Let Z^t denote a t-times Cartesian product of Z. We will assume throughout that Z is a countable set (a generalization of this assumption only

requires to generalize the summations into integration - however this brings in additional technical complexities which are beyond the scope of this course).

Let z^t denote a *history* of realizations: a t-component vector keeping track of the previous values taken by the z_j for all periods j from 0 to t:

$$z^t = (z_t,\, z_{t-1},\, ...,\, z_0)$$

Notice that $z^0 = z_0$, and we can write $z^t = \left(z_t,\, z^{t-1}\right)$.

Let $\pi\left(z^t\right)$ denote the probability of occurrence of the event $(z_t,\, z_{t-1},\, ...,\, z_0)$. Under this notation, a first order Markov process has

$$\pi\left[\left(z_{t+1},\, z^t\right)\left|z^t\right.\right] = \pi\left[\left(z_{t+1},\, z_t\right)\left|z_t\right.\right]$$

(care must be taken on the objects on which probability is assigned).

Sequential Formulation The planning problem in sequential form in this economy requires to maximize the function

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \cdot \pi\left(z^t\right) \cdot u\left[c_t\left(z^t\right)\right] \equiv E\left[\sum_{t=0}^{\infty} \beta^t \cdot u\left[c_t\right]\right]$$

Notice that as t increases, the dimension of the space of events Z^t increases. The choice variables in this problem are the consumption and investment amounts at each date and for each possible realization of the sequence of shocks as of that date. The consumer has to choose a stochastic process for c_t and another one for k_{t+1} :

$$\begin{array}{ll} c_t\left(z^t\right) & \forall z^t,\ \forall t \\ k_{t+1}\left(z^t\right) & \forall z^t,\ \forall t \end{array}$$

Notice that now there is only one kind of asset (k_{t+1}) available at each date.

Let $(t,\, z^t)$ denote a realization of the sequence of shocks z^t as of date t. The budget constraint in this problem requires that the consumer chooses a consumption and investment amount that is feasible at each $(t,\, z^t)$:

$$c_t\left(z^t\right) + k_{t+1}\left(z^t\right) \leq z_t \cdot f\left[k_t\left(z^{t-1}\right)\right] + (1-\delta) \cdot k_t\left(z^{t-1}\right)$$

You may observe that this restriction is consistent with the fact that at the moment of choosing, the agent's information is z^t.

Assuming that the utility index $u(\cdot)$ is strictly increasing, we may as well write the restriction in terms of equality. Then the consumer solves:

$$\max_{\{c_t(z^t),\, k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \cdot \pi\left(z^t\right) \cdot u\left[c_t\left(z^t\right)\right] \tag{SP}$$

$$s.t.\, c_t\left(z^t\right) + k_{t+1}\left(z^t\right) = z_t \cdot f\left[k_t\left(z^{t-1}\right)\right] + (1-\delta) \cdot k_t\left(z^{t-1}\right) \quad \forall\left(t,\, z^t\right)$$

$$k_0 \text{ given}$$

First order conditions:

$$k_{t+1}\left(z^t\right) : -\pi\left(z^t\right) \cdot u'\left[c_t\left(z^t\right)\right] + \sum_{z^{t+1} \in Z^{t+1}} \beta \cdot \pi\left(z_{t+1},\, z^t\right) \cdot u'\left[c_{t+1}\left(z_{t+1},\, z^t\right)\right] \cdot \left[z_{t+1} \cdot f'\left[k_{t+1}\left(z^t\right)\right] + 1 - \delta\right] = 0$$

Or, if we denote $\pi\left[\left(z_{t+1},\, z^t\right) | z^t\right] \equiv \dfrac{\pi\left(z_{t+1},\, z^t\right)}{\pi\left(z^t\right)}$, then we can rewrite:

$$\begin{aligned} u'\left[c_t\left(z^t\right)\right] &= \sum_{z^{t+1} \in Z^{t+1}} \beta \cdot \pi\left[\left(z_{t+1},\, z^t\right) | z^t\right] \cdot u'\left[c_{t+1}\left(z_{t+1},\, z^t\right)\right] \cdot \left[z_{t+1} \cdot f'\left[k_{t+1}\left(z^t\right)\right] + 1 - \delta\right] \\ &\equiv E_{z^t}\left[u'\left[c_{t+1}\left(z_{t+1},\, z^t\right)\right] \cdot R_{t+1}\right] \end{aligned} \tag{SEE}$$

where $R_{t+1} \equiv z_{t+1} \cdot f'\left[k_{t+1}\left(z^t\right)\right] + 1 - \delta$ is the marginal return on capital realized for each z_{t+1}.

(SEE) is a nonlinear, stochastic difference equation. In general, we will not be able to solve it analytically, so numerical methods or linearization techniques will be necessary.

Recursive Formulation The planner's problem in recursive version is:

$$V(k,\, z) = \max_{k'} \left\{ u\left[z \cdot f(k) - k' + (1-\delta) \cdot k\right] + \beta \cdot \sum_{z' \in Z} \pi\left(z' | z\right) \cdot V\left(k',\, z'\right) \right\} \tag{FE}$$

where yes, we have sneaked in a first order Markov assumption on the process $\{z_t\}_{t=0}^{\infty}$. The solution to this problem involves the policy rule

$$k' = g(k,\, z)$$

If we additionally assume that Z is not only countable but finite:

$$Z = \{z_1,\, ...,\, z_n\}$$

then the problem can also be written:

$$V_i(k) = \max_{k'} \left\{ u\left[z_i \cdot f(k) - k' + (1-\delta)\cdot k\right] + \beta \cdot \sum_{j=1}^{n} \pi_{ij} \cdot V_j(k') \right\}$$

where π_{ij} denotes the probability of moving from state i into state j:

$$\pi_{ij} \equiv \pi\left[z_{t+1} = z_j \,|z_t = z_i\right]$$

Solving the model: Linearization of the Euler equation Both the recursive and the sequential formulation lead to the Stochastic Euler Equation:

$$u'(c_t) = \beta \cdot E_{z_t}\left[u'(c_{t+1}) \cdot \left[z_{t+1} \cdot f'(k_{t+1}) + 1 - \delta\right]\right] \qquad \text{(SEE)}$$

Our strategy to solve this equation will be to use a linear approximation of it around the deterministic steady state. We will guess a linear policy function, and replace the choice variables with it. Finally, we will solve for the coefficients of this linear guess.

We rewrite (SEE) in terms of capital and using dynamic programming notation:

$$u'\left[z \cdot f(k) + (1-\delta)\cdot k - k'\right] = \beta \cdot E_z\left[u'\left[z' \cdot f(k') + (1-\delta)\cdot k' - k''\right] \cdot \left[z' \cdot f'(k') + 1 - \delta\right]\right] \qquad \text{(SEE)}$$

Denote

$$\begin{aligned} LHS &\equiv u'\left[z \cdot f(k) + (1-\delta)\cdot k - k'\right] \\ RHS &\equiv \beta \cdot E_z\left[u'\left[z' \cdot f(k') + (1-\delta)\cdot k' - k''\right] \cdot \left[z' \cdot f'(k') + 1 - \delta\right]\right] \end{aligned}$$

Let $\overline{k}$ be the steady state associated to the realization $\{z_t\}_{t=0}^{\infty}$ that has $z_t = \overline{z}$ for all but a infinite number of periods t. That is, $\overline{z}$ is the long run value of z.

Example 14 *Suppose that $\{z_t\}_{t=0}^{\infty}$ follows an AR(1) process:*

$$z_{t+1} = \rho \cdot z_t + (1-\rho)\cdot \overline{z} + \varepsilon_t$$

where $|\rho| < 1$. Then if $E[\varepsilon] = 0$, $E\left[\varepsilon^2\right] = \sigma^2 < \infty$, $E\left[\varepsilon_t \cdot \varepsilon_{t+j}\right] = 0 \;\; \forall j \geq 1$, by the Law of Large Numbers we have:

$$plim\ z_t = \overline{z}$$

Having the long run value of z, the associated steady state level of capital $\overline{k}$ is solved from the usual deterministic Euler equation:

$$\begin{aligned} u'(\overline{c}) &= \beta \cdot u'(\overline{c}) \cdot \left[\overline{z} \cdot f(\overline{k}) + 1 - \delta\right] \\ &\Rightarrow \quad \overline{z} \cdot f(\overline{k}) + 1 - \delta = \frac{1}{\beta} \\ &\Rightarrow \quad \overline{k} = f^{-1}\left(\frac{\beta^{-1} - (1-\delta)}{\overline{z}}\right) \\ &\Rightarrow \quad \overline{c} = \overline{z} \cdot f(\overline{k}) - \delta \cdot \overline{k} \end{aligned}$$

Let

$$\begin{aligned} \widehat{k} &\equiv k - \overline{k} \\ \widehat{z} &\equiv z - \overline{z} \end{aligned}$$

denote the variables expressed as deviations from their steady state values, and using this notation we write down a first order Taylor expansion of (SEE) around the long run values:

$$\begin{aligned} LHS &\approx LLHS \equiv a_L \cdot \widehat{z} + b_L \cdot \widehat{k} + c_L \cdot \widehat{k}' + d_L \\ RHS &\approx LRHS \equiv E_z\left[a_R \cdot \widehat{z}' + b_R \cdot \widehat{k}' + c_R \cdot \widehat{k}''\right] + d_R \end{aligned}$$

where the coefficients a_L, a_R, etcetera are the derivatives of the expressions LHS and RHS with respect to the corresponding variables, evaluated at the steady state (for example, $a_L = u''(\overline{c}) \cdot f(\overline{k})$ - you should derive the remaining ones). In addition, $LLHS = LRHS$ needs to hold for $\widehat{z} = \widehat{z}' = \widehat{k} = \widehat{k}' = \widehat{k}'' = 0$ (the steady state), so $d_L = d_R$.

Next we introduce our linear policy function guess in terms of deviations with respect to the steady state:

$$\widehat{k}' = g_k \cdot \widehat{k} + g_z \cdot \widehat{z}$$

The coefficients g_k, g_z are our unknowns. We substitute this guess into the linearized stochastic euler equation:

$$LLHS = a_L \cdot \widehat{z} + b_L \cdot \widehat{k} + c_L \cdot g_k \cdot \widehat{k} + c_L \cdot g_z \cdot \widehat{z} + d_L$$

$$\begin{aligned} LRHS &= E_z\left[a_R \cdot \widehat{z}' + b_R \cdot g_k \cdot \widehat{k} + b_R \cdot g_z \cdot \widehat{z} + c_R \cdot g_k \cdot \widehat{k}' + c_R \cdot g_z \cdot \widehat{z}'\right] + d_R \\ &= E_z\left[a_R \cdot \widehat{z}' + b_R \cdot g_k \cdot \widehat{k} + b_R \cdot g_z \cdot \widehat{z} + c_R \cdot g_k^2 \cdot \widehat{k} + c_R \cdot g_k \cdot g_z \cdot \widehat{z} + c_R \cdot g_z \cdot \widehat{z}'\right] + d_R \\ &= a_R \cdot E_z[\widehat{z}'] + b_R \cdot g_k \cdot \widehat{k} + b_R \cdot g_z \cdot \widehat{z} + c_R \cdot g_k^2 \cdot \widehat{k} + c_R \cdot g_k \cdot g_z \cdot \widehat{z} + c_R \cdot g_z \cdot E_z[\widehat{z}'] + d_R \end{aligned}$$

and our equation is:

$$LLHS = LRHS \qquad \text{(LS)}$$

(notice that d_L, d_R will simplify away). Using the assumed form of the stochastic process $\{z_t\}_{t=0}^{\infty}$, we can replace $E_z\left[\widehat{z}'\right]$.

The system (LS) needs to hold for all values of $\widehat{k}$ and $\widehat{z}$. Given the values of the coefficients a_i, b_i, c_i (for $i = L, R$), the task is to find the values of g_k, g_z that solve the system. Rearranging, (LS) can be written:

$$\widehat{z} \cdot A + E_z\left[\widehat{z}'\right] \cdot B + \widehat{k} \cdot C = 0$$

where

$$\begin{aligned} A &= a_L + c_L \cdot g_z - b_R \cdot g_z - c_R \cdot g_k \cdot g_z \\ B &= -a_R - c_R \cdot g_z \\ C &= b_L + c_L \cdot g_k - b_R \cdot g_k - c_R \cdot g_k^2 \end{aligned}$$

C is a second order polynomial in g_k; therefore the solution will involve two roots. We know that the eigenvalue smaller than one in absolute value will be the stable solution to the system.

Example 15 *Let $\{z_t\}_{t=0}^{\infty}$ follow an AR(1) process, as in the previous example:*

$$z_{t+1} = \rho \cdot z_t + (1 - \rho) \cdot \overline{z} + \varepsilon_t$$

Then

$$\begin{aligned} \widehat{z}' &\equiv z' - \overline{z} \\ &= \rho \cdot z + (1 - \rho) \cdot \overline{z} + \varepsilon - \overline{z} \\ &= \rho \cdot (z - \overline{z}) + \varepsilon \end{aligned}$$

So

$$E_z\left[\widehat{z}'\right] = \rho \cdot \widehat{z}$$

Replacing,

$$LRHS = a_R \cdot \rho \cdot \widehat{z} + b_R \cdot g_k \cdot \widehat{k} + b_R \cdot g_z \cdot \widehat{z} + c_R \cdot g_k^2 \cdot \widehat{k} + c_R \cdot g_k \cdot g_z \cdot \widehat{z} + c_R \cdot g_z \cdot \rho \cdot \widehat{z} + d_R$$

We can rearrange (LS) to:

$$\widehat{z} \cdot A + \widehat{k} \cdot B = 0$$

where

$$\begin{aligned} A &= a_L + c_L \cdot g_z - a_R \cdot \rho - b_R \cdot g_z - c_R \cdot g_k \cdot g_z - c_R \cdot g_z \cdot \rho \\ B &= b_L + c_L \cdot g_k - b_R \cdot g_k - c_R \cdot g_k^2 \end{aligned}$$

The solution to (LS) requires

$$\begin{aligned} A &= 0 \\ B &= 0 \end{aligned}$$

Therefore, the procedure is to solve first for g_k from B (picking the eigenvalue λ_1 that has $|\lambda_1| < 1$); and then use this value to solve for g_z from A.

Simulation Once we have solved for the coefficients g_k, g_z, we can *simulate* the model, by drawing values of $\{\widehat{z}_t\}_{t=0}^{T}$ from the assumed distribution, and an arbitrary $\widehat{k}_0$. This will yield a stochastic path for capital from the policy rule

$$\widehat{k}_{t+1} = g_k \cdot \widehat{k}_t + g_z \cdot \widehat{z}_t$$

Impulse response

We may also be interested in observing the effect on the capital accumulation path in an economy if there is a one-time productivity shock $\widehat{z}$. The usual procedure for this analysis is to set $\widehat{k}_0 = 0$ (that is, we begin from the steady state capital stock associated to the long run value $\overline{z}$), and $\widehat{z}_0$ to some arbitrary number. The values of $\widehat{z}_t$ for $t > 0$ are then derived by eliminating the stochastic component in the $\{\widehat{z}_t\}_{t=0}^{T}$ process.

For example, let $\{z_t\}_{t=0}^{\infty}$ be an AR(1) process as in the previous examples, then:

$$\widehat{z}_{t+1} = \rho \cdot \widehat{z}_t + \varepsilon_t$$

Let $\widehat{z}_0 = \Delta$, and set $\varepsilon_t = 0$ for all t. Using the policy function, we obtain the following path for capital:

$$\begin{aligned}
\widehat{k}_0 &= 0 \\
\widehat{k}_1 &= g_z \cdot \Delta \\
\widehat{k}_2 &= g_k \cdot g_z \cdot \Delta + g_z \cdot \rho \cdot \Delta = (g_k \cdot g_z + g_z \cdot \rho) \cdot \Delta \\
\widehat{k}_3 &= \left(g_k^2 \cdot g_z + g_k \cdot g_z + g_z \cdot \rho^2\right) \cdot \Delta \\
\vdots &= \vdots \\
\widehat{k}_t &= \left(g_k^{t-1} + g_k^{t-1} \cdot \rho + \ ... \ + g_k \cdot \rho^{t-2} + \rho^{t-1}\right) \cdot g_z \cdot \Delta
\end{aligned}$$

and

$$|g_k| < 1 \ \& \ |\rho| < 1 \ \Rightarrow \lim_{t \to \infty} \widehat{k}_t = 0$$

The capital stock returns to its steady state value if $|g_k| < 1$ and $|\rho| < 1$.

An example *impulse response* plot, using $g_z = .8$, $g_k = .9$, $\rho = -.75$

References and comments on the linear-quadratic setup You can find most of the material we have discussed on the neoclassical growth model in King, Plosser and Rebelo, 1988. Hansen and Sargent discuss the model in a Linear-Quadratic environment: assuming that the production technology is linear in z and k, and u is quadratic:

$$\begin{aligned} y(z,\, k) &= a_y \cdot z + b_y \cdot k \\ u(c) &= -a_u \cdot c^2 + b_u \end{aligned}$$

This set up leads to a linear Euler equation, therefore the linear policy function guess is exact. In addition, the linear-quadratic model has a property called "certainty equivalence": g_k and g_z do not depend on second or higher order moments of the shock ε. This implies that it is possible to solve the problem, at all t, by replacing z_{t+k} with $E_t\left[z_{t+k}\right]$ and thus transform it into a deterministic problem.

This approach provides an alternative to linearizing the stochastic euler equation. We can solve the problem by replacing the return function with a quadratic approximation, and the (technological) constraint by a linear function. Then we solve the resulting linear-quadratic problem:

$$\sum_{t=0}^{\infty} \beta^t \cdot \underbrace{u\left[F\left(k_t\right) + (1-\delta) \cdot k_t - k_{t+1}\right]}_{\text{Return function}}$$

The approximation of the return function can be done by taking a second order Taylor series expansion around the steady state. This will yield the same results as the linearization.

Finally, the following shortfalls of the linear-quadratic setup must be kept in mind:

- The quadratic return function leads to satiation: there will be consumption amounts with zero marginal utility.

- Non-negativity constraints may cause problems. In practice, the method requires such constraints to not bind. Otherwise, the Euler equation will involve Lagrange multipliers, for a significant increase in the complexity of the solution.

- A linear production function implies a constant returns to scale technology, which may not be consistent with economic intuition.

2 Competitive equilibrium under uncertainty

The welfare properties of competitive equilibrium are affected by the introduction of uncertainty through the *market structure*. The relevant distinction is whether such structure involves *complete* or *incomplete markets*. Intuitively, a *complete markets* structure allows trading in each single commodity. Recall our previous discussion of the neoclassical growth model under uncertainty where commodities are defined as consumption goods indexed by time and state of the world. For example, if z_1^t and z_2^t denote two different realizations of the random sequence $\{z_j\}_{j=0}^t$, then a unit of the physical good c consumed in period t, if the state of the world is z_1^t - denoted by $c_t(z_1^t)$ - is a commodity different than $c_t(z_2^t)$. A complete markets structure will allow contracts between parties to specify the delivery of physical good c in different amounts at (t, z_1^t) than at (t, z_2^t), and for a different price.

In an incomplete markets structure, such a contract might be impossible to enforce - i.e. parties might be unable to sign a "legal" contract that makes the delivery amount contingent on the realization of the random shock. A usual incomplete markets structure is one where agents may only agree to the delivery of goods on a date basis, regardless of the shock. A contract specifying $c_t(z_1^t) \neq c_t(z_2^t)$ is not enforceable in such an economy.

You may notice that the structure of markets is an assumption of an *institutional* nature - nothing should prevent, in theory, the market structure to be complete. However, markets are incomplete in the real world, and this seems to play a key role in the economy (for example in the distribution of wealth, in the business cycle, perhaps even in the *equity premium* puzzle that we will discuss in due time).

Before embarking on the study of the subject, it is worth mentioning that the structure of markets need not be explicit. For example, the accumulation of capital may supply the role of transferring wealth across states of the world (not just across time). But allowing for the transfer of wealth across states is one of the functions specific to markets; therefore if these are incomplete then capital accumulation can (to some extent) perform this missing function. An extreme example is the deterministic model. In that case, there is only one state of the world, and only transfers of wealth across time are relevant. In such a case, the possibility of accumulating capital is enough to ensure that markets are complete - allowing agents also to engage in trade of dated commodities is redundant. Another example shows up in real business cycle models, which we shall analyze later on in this course. A usual result in the real business cycle literature (consistent with actual economic data) is that agents choose to accumulate more capital whenever there is a "good" realization of the productivity shock. An intuitive interpretation is that savings play role of a "buffer" used to smooth out the consumption path - a function that markets could perform.

Hence, you may correctly suspect that whenever we talk about market completeness or incompleteness, in fact we are referring not to the actual, explicit contracts that agents are allowed to sign, but to the degree to which they are able to transfer wealth across states of the world. This ability will depend on the institutional framework assumed for the economy.

2.0.2 The neoclassical growth model with complete markets

We will begin by analyzing the neoclassical growth model in an uncertain environment. We assume that given a stochastic process $\{z_t\}_{t=0}^{\infty}$, there is a market for each consumption commodity $c_t(z^t)$, as well as for capital and labor services at each date and state of the world. There are two alternative setups: Arrow-Debreu date-0 trading, or sequential trade.

Arrow-Debreu date-0 trading The consumer's budget constraint reads:

$$\sum_{t=0}^{\infty}\sum_{Z^t} p_t\left(z^t\right)\cdot\left[c_t\left(z^t\right)+K_{t+1}\left(z^t\right)\right] \leq \sum_{t=0}^{\infty}\sum_{Z^t} p_t\left(z^t\right)\cdot\left[\left(r_t\left(z^t\right)+1-\delta\right)\cdot K_t\left(z^{t-1}\right)+w_t\left(z^t\right)\cdot n_t\left(z^t\right)\right]$$

You should check this: Specify the objective of the consumer, derive first order conditions, the stochastic Euler equation, and show that it is identical to the Euler equation in the planner's problem.

Sequential trade In order to allow wealth transfers across dates, agents must be able to borrow and lend. It suffices to have one-period assets, even with an infinite time horizon. We will assume the existence of these one-period assets, and, for simplicity, that Z is a finite set with n possible shock values:

Assume that there are q assets, with asset j paying off r_{ij} consumption units in $t+1$ if the realized state is z_i. The following matrix shows the payoff of each asset for every realization of z_{t+1}:

$$\begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_q \end{array} \\ \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1q} \\ r_{21} & r_{22} & \cdots & r_{2q} \\ r_{31} & r_{32} & \cdots & r_{3q} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nq} \end{pmatrix} \end{array} \equiv R$$

Then portfolio $a = (a_1, a_2, ..., a_q)$ is pays off (in terms of consumption goods at $t+1$):

$$\underset{n\times 1}{p} = \underset{n\times q}{R} \cdot \underset{q\times 1}{a}$$

Each component $p_i = \sum_{j=1}^{q} R_{ij} \cdot a_j$ is the amount of consumption goods obtained in state i from holding portfolio a.

Matrix algebra has the answer to the following important question: What restrictions must we impose on R so that any arbitrary payoff combination $p \in \Re^n$ can be generated (by the appropriate portfolio choice)? The answer is that we must have

1. $q \geq n$
2. $rank(R) = n$

If R satisfies condition number (2) (which requires the first one, of course), then the market structure is *complete*. The whole space $\Re^n$ is *spanned* by R - we say that there is *spanning*.

Arrow securities: Recall that these were mentioned before. Arrow security i pays off 1 unit if the realized state is i, and 0 otherwise. If there are $q < n$ different Arrow securities, then the payoff matrix reads:

$$\begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_q \\ \vdots \\ z_n \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_q \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

2.0.3 General equilibrium under uncertainty: Multiple agents

Motivation: If we compare the outcome of the neoclassical growth model with uncertainty and one representative agent with the two different market structures:

1 Only (sequential) trade in capital. There is no *spanning* in this setup: only one asset for n states.

2 *Spanning* (either with Arrow-Debreu date-0, or sequential trading).

Will equilibria look different with these structures? The answer is no, and the reason is that there is a single agent. Let us turn to the case where the economy is populated by more than one agent to analyze the validity of such a result.

Illustration: 2-period model, 2 agents. We will compare the equilibrium allocation of this economy under the market structures (1) and (2) mentioned above.

Assumptions

Random Shock: We assume there are n states of the world, corresponding to n different values of the shock to technology to be described below:

$$\begin{aligned} z &\in \{z_1, z_2, ..., z_n\} \\ \pi_j &= \Pr[z = z_j] \end{aligned}$$

Let $\overline{z}$ denote the expected value of z:

$$\overline{z} = \sum_{j=1}^{n} \pi_j \cdot z_j$$

Tastes: Agents derive utility from consumption only (not from leisure). Preferences satisfy the axioms of expected utility, with utility index $u(\cdot)$. Specifically, we assume that:

$$U_i = u_i\left(c_0^i\right) + \beta \cdot \sum_{j=1}^{n} \pi_j \cdot u_i(c_j^i) \qquad i = 1, 2$$

where $u_1(x) = x$, and $u_2(x)$ is strictly concave: $u_2' > 0$, $u_2'' < 0$. We also assume that $\lim_{x \to 0} u_2'(x) = \infty$.

Thus, agents' preferences exhibit different attitudes towards risk: Agent 1 is *risk neutral*, and Agent 2 is *risk averse*.

Endowments: Each agent is endowed with ϖ_0 consumption goods in period 0, and with 1 unit of labor in period 1 (which will be supplied inelastically since leisure is not valued).

Technology: Consumption goods are produced in period 1 with a constant returns to scale technology represented by the Cobb Douglass production function

$$y_j = z_j \cdot K^{\alpha} \cdot \left(\frac{n}{2}\right)^{1-\alpha}$$

where K, n denote the aggregate supply of capital and labor services in period 1, respectively. We know that $n = 2$, so

$$y_j = z_j \cdot K^{\alpha}$$

Therefore, the remunerations to factors in period 1, if state j is realized, are given by:

$$\begin{aligned} r_j &= z_j \cdot \alpha \cdot K^{\alpha-1} \\ w_j &= z_j \cdot \frac{(1-\alpha)}{2} \cdot K^{\alpha} \end{aligned}$$

Structure 1

Trading in only 1 asset (capital) is allowed in this setup. With K denoting the aggregate capital stock, a_i denotes the capital stock held by agent i, then asset market clearing requires that

$$a_1 + a_2 = K$$

The budget constraints for each agent is given by:

$$\begin{aligned} c_0^i + a_i &= \varpi_0 \\ c_j^i &= a_i \cdot r_j + w_j \end{aligned}$$

To solve this problem, we proceed to maximize each consumer's utility subject to his budget constraint. We take first order conditions:

Agent 1:

$$\begin{aligned} c_0^1 &: 1 = \lambda \\ c_j^1 &: \beta \cdot \pi_j = \lambda_j \\ a_1 &: \lambda = \sum_{j=1}^{n} r_j \cdot \lambda_j \end{aligned}$$

The Euler equation is, replacing for r_{1j}, is

$$\begin{aligned} 1 &= \beta \cdot \sum_{j=1}^{n} \pi_j \cdot \alpha \cdot z_j \cdot K^{\alpha-1} \qquad \text{(EE1)} \\ 1 &= \alpha \cdot \beta \cdot K^{\alpha-1} \cdot \sum_{j=1}^{n} \pi_j \cdot z_j \end{aligned}$$

Therefore, the optimal choice of K from Agent 1's preferences is given by:

$$K^* = (\overline{z} \cdot \alpha \cdot \beta)^{\frac{1}{1-\alpha}}$$

Notice that only the average value of the random shock matters for Agent 1, consistently with the fact of this agent being *risk neutral*.

Agent 2:

$$u_2'(\varpi_0 - a_2) = \beta \cdot \sum_{j=1}^{n} \pi_j \cdot u_2'\left(a_2 \cdot r_j^* + w_j^*\right) \cdot r_j^* \qquad \text{(EE2)}$$

Given K^* from Agent 1's problem, we have the value of r_j^* and w_j^* for each realization j. Therefore, Agent 2's Euler equation (EE2) is one equation in one unknown: a_2. Since $\lim_{x \to 0} u_2'(x) = \infty$, there exists a unique solution.

Let a_2^* be the solution to (EE2), then the values of the remaining choice variables follow:

$$\begin{aligned} a_1^* &= K^* - a_2^* \\ c_0^i &= \varpi_0 - a_i^* \end{aligned}$$

More importantly, Agent 2 will face a stochastic consumption prospect for period 1:

$$c_j^2 = a_2^* \cdot r_j^* + w_j^*$$

With r_j^* and w_j^* are stochastic. This implies that Agent 1 has <u>not</u> provided *full insurance* to Agent 2.

Structure 2

Trading in n different Arrow securities is allowed in this setup. In this case, these securities are (contingent) claims on the total remuneration to capital (you could think of them as rights to collect future dividends in a company, according to the realized state of the world). Notice this implies *spanning*; that is, markets are complete. Let a_j denote the Arrow security paying off 1 unit if the realized state is z_j, zero otherwise, and let q_j denote the price of a_j.

In this economy, agents *save* by accumulating contingent claims (they save by buying future dividends in a company). Total savings are thus given by

$$S \equiv \sum_{j=1}^{n} q_j \cdot (a_{1j} + a_{2j})$$

Investment is the accumulation of physical capital, K. Then clearing of the savings-investment market requires that:

$$\sum_{j=1}^{n} q_j \cdot (a_{1j} + a_{2j}) = K \qquad (S = I)$$

Constant returns to scale implies that the total remuneration to capital services in state j will be given by $K \cdot r_j$ (Euler Theorem). Therefore, the contingent claims that get activated when this state is realized must exactly match this amount (each unit of "dividends" that the company will pay out must have an owner, but the total claims can not exceed the actual amount of dividends to be paid out).

In other words, clearing of (all of) the Arrow security markets requires that

$$a_{1j} + a_{2j} = K \cdot r_j \quad j = 1, ..., n \qquad \text{(ASMC)}$$

If we multiply both sides of (ASMC) by q_j, for each j, and then sum up over j's, we get:

$$\sum_{j=1}^{n} q_j \cdot (a_{1j} + a_{2j}) = K \cdot \sum_{j=1}^{n} q_j \cdot r_j$$

But, using $(S = I)$ to replace total savings by total investment,

$$K = K \cdot \sum_{j=1}^{n} q_j \cdot r_j$$

Therefore the equilibrium condition is that

$$\sum_{j=1}^{n} q_j \cdot r_j = 1 \qquad \text{(EC)}$$

(EC) can be interpreted as a "no arbitrage" condition, in the following way. The left hand side $\sum_{j=1}^{n} q_j \cdot r_j$ is the total price (in terms of foregone consumption units) of the marginal unit of a portfolio yielding the same (expected) marginal return as physical capital investment. And the right hand side is the price (also in consumption units) of a marginal unit of capital investment.

Then suppose that $\sum_{j=1}^{n} q_j \cdot r_j > 1$. An agent could in principle make unbounded profits by selling an infinite amount of units of such a portfolio,

and using the proceeds from this sale to finance an unbounded physical capital investment. In fact, since no agent would be willing to be on the buy side of such a deal, no trade would actually occur. But there would be an infinite *desired* supply of such a portfolio, and an infinite *desired* demand of physical capital units. In other words, asset markets would not be in equilibrium. A similar reasoning would lead to the conclusion that $\sum_{j=1}^{n} q_j \cdot r_j < 1$ could not be an equilibrium either.

With the equilibrium conditions at hand, we are able to solve the model. With this market structure, the budget constraint of each Agent i reads:

$$\begin{aligned} c_0^i + \sum_{j=1}^{n} q_j \cdot a_{ij} &= \varpi_0 \\ c_j^i &= a_j + w_j \end{aligned}$$

First order conditions on Agent 1's problem leads to the equilibrium prices:

$$q_j = \beta \cdot \pi_j$$

You should also check that

$$K^* = (\overline{z} \cdot \alpha \cdot \beta)^{\frac{1}{1-\alpha}}$$

(as in the previous problem). Therefore, Agent 1 is as well off with the current market structure as in the previous setup.

Agent 2's problem yields the Euler equation

$$u_2'\left(c_0^2\right) = \lambda = q_j^{-1} \cdot \beta \cdot \pi_j \cdot u_2'\left(c_j^2\right)$$

Replacing for the equilibrium prices derived from Agent 1's problem, this simplifies to

$$u_2'\left(c_0^2\right) = u_2'\left(c_j^2\right) \qquad j = 1, ..., n$$

Therefore, with the new market structure, Agent 2 is able to obtain *full insurance* from Agent 1. From the First Welfare Theorem (that requires completeness of markets) we know that the allocation prevailing under market *Structure* 2 is a Pareto optimal allocation. It is your task to determine whether the allocation resulting from *Structure* 1 was optimal as well, or not.