Econ 525a
Fall 2014
Yale University
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PROBLEM SET #5

This problem set is due on October 17.

1. Consider a two-period growth model with idiosyncratic shocks to labor productivity. There are two types of consumers, each of whom has measure one-half. Both types of consumers have the same initial wealth ω in period 1. Consumers do not value leisure and work one unit of time in period 2. Each consumer's labor productivity in period 2 is random. There are two states of the world in period 2. With probability one-half, type-1 consumers have productivity e^1 and type-2 consumers have productivity e^2; with probability one-half, type-1 consumers have productivity e^2 and type-2 consumers have productivity e^1. Total (or aggregate) labor supply in period 2 is then $L_2 \equiv (e^1 + e^2)/2$ in both states of the world.

   Output in period 2 is produced by a profit-maximizing competitive firm with (aggregate) production function $F(K_2, L_2)$, where K_2 is the total amount of capital in period 2. The production function exhibits constant-returns-to-scale and is strictly increasing and strictly concave in both of its arguments.

   Consumers can save in the form of capital and they can purchase state-contingent claims to consumption. Assuming a complete set of state-contingent claims, each consumer's period-1 budget constraint reads: $c_1 = \omega - k_2 - q^1 a^1 - q^2 a^2$ and each consumer's period-2 budget constraint reads: $c_2^i = r(K_2, L_2)k_2 + w(K_2, L_2)e^i + a^i$, where c_1 is consumption in period 1, c_2^i is consumption in state i in period 2, k_2 is savings in capital, and a^i is the number of Arrow securities paying one unit of the consumption good in state i (and zero units in state $j \neq i$). (Note that consumers do not save in the second period and instead consume all their resources.) The rental rate of capital r and the wage rate w are the appropriate marginal products of the production function; q^i is the price of an Arrow security that pays off in state i. Consumers choose k_2, a^1, and a^2 to maximize $u(c_1) + \beta E[u(c_2)]$ and they face a no-borrowing constraint on capital: $k_2 \geq 0$.

   The equilibrium conditions are that $K_2 = k_2$ (i.e., total savings in period 2 is the sum of the savings of both types of consumers—which are the same for both consumers since they face symmetric decision problems) and that the markets for the two Arrow securities clear.

(a) Prove that the competitive equilibrium prices and allocations for this economy are the same as those that obtain in an economy without uncertainty (i.e., one in which $e_1 = e_2$).

(b) Now consider an economy with incomplete markets. In particular, assume that consumers cannot trade Arrow securities, but that they can save in the form of capital. Find an equation that determines aggregate savings K_2 in period 2. Is aggregate savings higher or lower than in part (a)? (In the absence of a proof, try a numerical example with logarithmic utility.)

(c) In the economy with incomplete markets, suppose that the government introduces a proportional tax on investment and returns the proceeds to consumers in a lump-sum fashion. The consumer's budget constraints now read:

$$c_1 = \omega - (1 + \tau)k_2 + \tau K_2$$

and

$$c_2^i = r(K_2, L_2)k_2 + w(K_2, L_2)e^i.$$

A typical consumer takes the lump-sum subsidy τK_2 as given when making his choices. Prove that increasing τ from 0 reduces aggregate savings. (Hint: Find an equation that determines K_2 implicitly as a function of τ, totally differentiate it with respect to τ, and then evaluate the resulting expression at $\tau = 0$.)

(d) Prove that increasing τ from 0 increases a typical consumer's ex ante welfare. Try to provide intuition for this result. (Hint: You may need to use the fact that if F is homogeneous of degree 1, then $KF_{KK} + LF_{KL} = 0$.)

2. In part (d) of the second problem on Problem Set #4, you generated a long simulation for the bond holdings of a typical agent. Fix a grid of, say, 20 points for bond holdings and then use your simulated data to calculate, using the notation from lecture on October 10, $\hat{G}(b_i, \epsilon_j)$, i.e., the fraction of time in the simulation that the agent has bond holdings $\leq b_i$ and labor productivity $\epsilon = \epsilon_j$. For each value of ϵ, connect these fractions using a cubic spline. Now, using $\hat{G}$ as an initial guess, iterate as described in lecture to compute functions $G(b, \epsilon_h)$ and $G(b, \epsilon_l)$ satisfying the following pair of functional equations:

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\begin{aligned}
G(b, \epsilon_h) &= G(g_h^{-1}(b), \epsilon_h)\pi_{hh} + G(g_l^{-1}(b), \epsilon_l)\pi_{lh} \\
G(b, \epsilon_l) &= G(g_h^{-1}(b), \epsilon_h)\pi_{hl} + G(g_l^{-1}(b), \epsilon_l)\pi_{ll},
\end{aligned}
$$

where $b' = g_j(b)$ is the decision rule for bond holdings when $\epsilon = \epsilon_j$. (Note: to compute $g_j^{-1}(b)$, you can use a root-finding algorithm to find the value of x satisfying $b = g_j(x)$. Recall that in part (c) of the second problem on Problem Set #4 you used linear interpolation to obtain optimal decisions off the grid; you can do so again here.) These two functions characterize the invariant distribution in the Huggett model (given a bond price q). How close are they to $\hat{G}$?