SUPPLEMENT TO "TEMPTATION AND TAXATION"

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GENERAL FRAMEWORK FOR THE PROOFS OF PROPOSITIONS 1–3

Letting $\bar{u}(c_1, c_2) = u(c_1, c_2) + v(c_1, c_2)$, the first-order conditions for the competitive consumer's maximization problem are given by

$$(1+\tau_i)\bar{u}_1(c_1, c_2) = r_2\bar{u}_2(c_1, c_2) \quad \text{and}$$
$$(1+\tau_i)v_1(\tilde{c}_1, \tilde{c}_2) = r_2 v_2(\tilde{c}_1, \tilde{c}_2),$$

where

$$c_1 = r_1 k_1 + w_1 + s - (1+\tau_i)k_2, \quad c_2 = r_2 k_2 + w_2,$$
$$\tilde{c}_1 = r_1 k_1 + w_1 + s - (1+\tau_i)\tilde{k}_2, \quad \text{and}$$
$$\tilde{c}_2 = r_2\tilde{k}_2 + w_2.$$

Using the first-order conditions of the consumer, it is easy to show that $\bar{k}_2 > \bar{\tilde{k}}_2$ and $\bar{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2) = u_1(c_1, c_2) + v_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2) > 0$. We will use these below. The value function of the representative agent is given by

$$\begin{aligned} U(\bar{k}_1, P, \tau_i) &= \bar{u}(r_1\bar{k}_1 + w_1 - \bar{k}_2, r_2 k_2 + w_2) \\ &\quad - v(r_1\bar{k}_1 + w_1 + \tau_i(\bar{k}_2 - \bar{\tilde{k}}_2) - \bar{\tilde{k}}_2, r_2\bar{\tilde{k}}_2 + w_2). \end{aligned}$$

Differentiating the value function with respect to τ_i and using the consumer's first-order conditions, we obtain

$$\begin{aligned} \frac{dU}{d\tau_i} &= \bar{u}_1(c_1, c_2)\tau_i\frac{d\bar{k}_2}{d\tau_i} + \bar{u}_2(c_1, c_2)\left(\frac{dr_2}{d\tau_i}\bar{k}_2 + \frac{dw_2}{d\tau_i}\right) \\ &\quad - v_1(\tilde{c}_1, \tilde{c}_2)\left\{\bar{k}_2 - \bar{\tilde{k}}_2 + \tau_i\frac{d\bar{k}_2}{d\tau_i}\right\} - v_2(\tilde{c}_1, \tilde{c}_2)\left(\frac{dr_2}{d\tau_i}\bar{\tilde{k}}_2 + \frac{dw_2}{d\tau_i}\right). \end{aligned}$$

PROOF OF PROPOSITION 1: In partial equilibrium, $\frac{dr_2}{d\tau_i} = 0$ and $\frac{dw_2}{d\tau_i} = 0$. Therefore, we obtain

$$\frac{dU}{d\tau_i} = (\bar{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2))\tau_i\frac{d\bar{k}_2}{d\tau_i} - v_1(\tilde{c}_1, \tilde{c}_2)\{\bar{k}_2 - \bar{\tilde{k}}_2\}.$$

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Since $\bar{k}_2 > \bar{\bar{k}}_2$ and $\bar{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2) > 0$, then $\frac{dU}{d\tau_i} < 0$ for all $\tau_i \geq 0$. Therefore, the optimal tax rate has to be negative. *Q.E.D.*

PROOF OF PROPOSITION 3: In this case, $\frac{dr_2}{d\tau_i}\bar{k}_2 + \frac{dw_2}{d\tau_i} = w'(\bar{k}_2) + r'(\bar{k}_2)\bar{k}_2 = 0$ and $\frac{dr_2}{d\tau_i}\bar{\bar{k}}_2 + \frac{dw_2}{d\tau_i} = r'(\bar{k}_2)(\bar{\bar{k}}_2 - \bar{k}_2)\frac{d\bar{k}_2}{d\tau_i}$. Using these relations,

$$\begin{aligned}
\frac{dU}{d\tau_i} &= (\bar{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2))\tau_i \frac{d\bar{k}_2}{d\tau_i} \\
&\quad - v_1(\tilde{c}_1, \tilde{c}_2)\{\bar{k}_2 - \bar{\bar{k}}_2\} + v_2(\tilde{c}_1, \tilde{c}_2)(\bar{k}_2 - \bar{\bar{k}}_2)\frac{dr(\bar{k}_2)}{d\tau_i} \\
&= (\bar{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2))\tau_i \frac{d\bar{k}_2}{d\tau_i} \\
&\quad + v_1(\tilde{c}_1, \tilde{c}_2)\{\bar{k}_2 - \bar{\bar{k}}_2\}\left\{\widetilde{\mathrm{MRS}}\frac{dr(\bar{k}_2)}{d\tau_i} - 1\right\},
\end{aligned}$$

where $\widetilde{\mathrm{MRS}} = \frac{v_2(\tilde{c}_1, \tilde{c}_2)}{v_1(\tilde{c}_1, \tilde{c}_2)}$, where MRS denotes "marginal rate of substitution." Taking the derivative of the first-order condition for the actual choice with respect to τ_i, we can show that $\frac{d\bar{k}_2}{d\tau_i} < 0$. We will show that $1 - \widetilde{\mathrm{MRS}}\frac{dr(\bar{k}_2)}{d\tau_i} > 0$. This implies that $\frac{dU}{d\tau_i} < 0$ for all $\tau_i \geq 0$. Thus, the optimal tax is negative, that is, $\tau_i < 0$. To show this, note that in equilibrium, $r(\bar{k}_2) \times \mathrm{MRS} = r(\bar{k}_2) \times \widetilde{\mathrm{MRS}} = 1 + \tau_i$, where $\mathrm{MRS} = \frac{\bar{u}_2(c_1, c_2)}{\bar{u}_1(c_1, c_2)}$. Therefore, it is enough to show that $1 - \mathrm{MRS}\frac{dr(\bar{k}_2)}{d\tau_i} > 0$. Taking the derivative of $r(\bar{k}_2) \times \mathrm{MRS} = 1 + \tau_i$ with respect to τ_i, we obtain $1 - \mathrm{MRS}\frac{dr(\bar{k}_2)}{d\tau_i} = \frac{dMRS}{d\tau_i}r(\bar{k}_2)$. Given that $\mathrm{MRS} = \frac{\bar{u}_2(c_1, c_2)}{\bar{u}_1(c_1, c_2)}$ and $\frac{d\bar{k}_2}{d\tau_i} < 0$, it is then clear that $\frac{dMRS}{d\tau_i} > 0$. *Q.E.D.*

PROOF OF PROPOSITION 2: In this case, $\frac{dw_2}{d\tau_i} = 0$, $\bar{k}_1 = 0$, $\bar{k}_2 = 0$, $c_1 = w_1$, and $c_2 = w_2$. Given these, we obtain

$$\begin{aligned}
\frac{dU}{d\tau_i} &= \bar{u}_1(c_1, c_2)\tau_i \frac{d\bar{k}_2}{d\tau_i} - v_1(\tilde{c}_1, \tilde{c}_2)\left\{-\bar{\bar{k}}_2 + \tau_i \frac{d\bar{k}_2}{d\tau_i}\right\} - v_2(\tilde{c}_1, \tilde{c}_2)\frac{dr_2}{d\tau_i}\bar{\bar{k}}_2 \\
&= (\bar{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2))\tau_i \frac{d\bar{k}_2}{d\tau_i} \\
&\quad + v_1(\tilde{c}_1, \tilde{c}_2)\bar{\bar{k}}_2\left(1 - \widetilde{\mathrm{MRS}}\frac{dr_2}{d\tau_i}\right).
\end{aligned}$$

The key difference between the previous case and this one is that the consumer consumes his endowment, that is, $\mathrm{MRS} = \frac{\bar{u}_2(w_1, w_2)}{\bar{u}_1(w_1, w_2)}$. Therefore, $\frac{d\mathrm{MRS}}{d\tau_i} = 0$, which implies that $1-\widetilde{\text{MRS}}\frac{dr_2}{d\tau_i}=0$. Second, $\frac{d\bar{k}_2}{d\tau_i}=0$. Thus, we obtain that $\frac{dU}{d\tau_i}=0$ independent of τ_i, which implies that the consumer is indifferent to any τ_i. *Q.E.D.*

For the proof of Proposition 4, see the proof to Proposition 8, which studies a T-period economy with logarithmic utility.

PROOF OF PROPOSITION 5: The problem of the consumer can be written as

$$U(k_1,\bar{k}_1,\tau_i)=\max_{c_1,c_2}(1+\gamma)\frac{c_1^{1-\sigma}}{1-\sigma}+\delta(1+\beta\gamma)\frac{c_2^{1-\sigma}}{1-\sigma}$$
$$-\gamma\left[\max_{\tilde{c}_1,\tilde{c}_2}\frac{\tilde{c}_1^{1-\sigma}}{1-\sigma}+\delta\beta\frac{\tilde{c}_2^{1-\sigma}}{1-\sigma}\right]$$

s.t.

$$c_1+\frac{c_2}{r(\bar{k}_2)}(1+\tau_i)=r(\bar{k}_1)k_1+w(\bar{k}_1)+s+\frac{w(\bar{k}_2)}{r(\bar{k}_2)}(1+\tau_i)=Y.$$

The first-order conditions are

$$c_1^{-\sigma}=\frac{\delta(1+\beta\gamma)}{1+\gamma}\underbrace{\frac{r(\bar{k}_2)}{1+\tau_i}}_{m(\bar{k}_2,\tau_i)}c_2^{-\sigma}\quad\text{and}\quad\tilde{c}_1^{-\sigma}=\delta\beta\frac{r(\bar{k}_2)}{1+\tau_i}\tilde{c}_2^{-\sigma}.$$

This implies

$$c_1=\frac{Y}{1+\left[\frac{\delta(1+\beta\gamma)}{1+\gamma}\right]^{1/\sigma}[m(\bar{k}_2,\tau_i)]^{(1-\sigma)/\sigma}}\quad\text{and}$$
$$c_2=\left[\frac{\delta(1+\beta\gamma)}{1+\gamma}m(\bar{k}_2,\tau_i)\right]^{1/\sigma}c_1,$$
$$\tilde{c}_1=\frac{Y}{1+[\delta\beta]^{1/\sigma}[m(\bar{k}_2,\tau_i)]^{(1-\sigma)/\sigma}}\quad\text{and}\quad\tilde{c}_2=[\delta\beta m(\bar{k}_2,\tau_i)]^{1/\sigma}\tilde{c}_1.$$

From these expressions we obtain

$$\frac{\tilde{c}_1}{c_1}=\frac{1+\left[\frac{\delta(1+\beta\gamma)}{1+\gamma}\right]^{1/\sigma}[m(\bar{k}_2,\tau_i)]^{(1-\sigma)/\sigma}}{1+[\delta\beta]^{1/\sigma}[m(\bar{k}_2,\tau_i)]^{(1-\sigma)/\sigma}}=x_1$$

and

$$\frac{\tilde{c}_2}{c_2} = \left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma} \frac{\tilde{c}_1}{c_1} = x_2 = \left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma} x_1.$$

Then we can write the objective function of the government, inserting the expressions above, as

$$\begin{aligned}
U(\bar{k}_1, \bar{k}_1, \tau_i) &= (1+\gamma)\frac{c_1^{1-\sigma}}{1-\sigma} + \delta(1+\beta\gamma)\frac{c_2^{1-\sigma}}{1-\sigma} \\
&\quad - \gamma\left[\frac{\tilde{c}_1^{1-\sigma}}{1-\sigma} + \delta(1+\beta\gamma)\frac{\tilde{c}_2^{1-\sigma}}{1-\sigma}\right] \\
&= \frac{c_1^{1-\sigma}}{1-\sigma} + \delta\frac{c_2^{1-\sigma}}{1-\sigma} \\
&\quad + \gamma\left[(1-x_1^{1-\sigma})\frac{c_1^{1-\sigma}}{1-\sigma} + (1-x_2^{1-\sigma})\delta\beta\frac{c_2^{1-\sigma}}{1-\sigma}\right],
\end{aligned}$$

where

$$c_1 = (1-d)\bar{k}_1 + f(\bar{k}_1) - \bar{k}_2 \quad \text{and} \quad c_2 = (1-d)\bar{k}_2 + f(\bar{k}_2).$$

Taking the derivative of the objective function with respect to τ_i and inserting $\frac{dx_2}{d\tau_i} = [\frac{\beta(1+\gamma)}{1+\beta\gamma}]^{1/\sigma}\frac{dx_1}{d\tau_i}$, we obtain

$$\begin{aligned}
&\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i) \\
&\quad = [-c_1^{-\sigma} + \delta r(\bar{k}_2)c_2^{-\sigma}]\frac{d\bar{k}_2}{d\tau_i} \\
&\qquad + \gamma[-(1-x_1^{1-\sigma})c_1^{-\sigma} + (1-x_2^{1-\sigma})\delta\beta r(\bar{k}_2)c_2^{-\sigma}]\frac{d\bar{k}_2}{d\tau_i} \\
&\qquad - \gamma\left[x_1^{-\sigma}c_1^{1-\sigma} + \delta\beta\left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma} x_2^{-\sigma}c_2^{1-\sigma}\right]\frac{dx_1}{d\tau_i}.
\end{aligned}$$

Let τ_i^* be the tax rate that maximizes the commitment utility. Then τ_i^* will generate the condition

$$c_1^{-\sigma} = \delta r(\bar{k}_2)c_2^{-\sigma}.$$

Using the first-order condition $c_1^{-\sigma} = \frac{\delta(1+\beta\gamma)}{1+\gamma}m(\bar{k}_2, \tau_i)c_2^{-\sigma}$, this implies

$$\frac{(1+\beta\gamma)}{1+\gamma}m(\bar{k}_2, \tau_i^*) = r(\bar{k}_2).$$

It is easy to see that $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*) = 0$ at $\sigma = 1$. Thus the subsidy that maximizes utility under logarithmic utility is the same as the subsidy that maximizes the commitment utility.

We now characterize the condition under which

$$\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*) < 0 \quad \text{for} \quad \sigma > 1$$

holds, so that for $\sigma > 1$, the optimal subsidy is larger than the optimal subsidy that maximizes commitment utility. To do this, we take the derivative of $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*)$ with respect to σ and evaluate at $\sigma = 1$. If the derivative is negative at $\sigma = 1$, then $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*) < 0$ for σ marginally above $\sigma = 1$. If the derivative is positive at $\sigma = 1$, then $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*) > 0$ for σ marginally above $\sigma = 1$.

First, for later use, we compute the objects

$$\begin{aligned}
\frac{dx_1}{d\tau_i} &= \frac{1-\sigma}{\sigma}[m(\bar{k}_2, \tau_i)]^{(1-2\sigma)/\sigma} \\
&\quad \times \frac{\left[\dfrac{\delta(1+\beta\gamma)}{1+\gamma}\right]^{1/\sigma} - [\delta\beta]^{1/\sigma}}{[1 + [\delta\beta]^{1/\sigma}[m(\bar{k}_2, \tau_i)]^{(1-\sigma)/\sigma}]^2} \frac{dm(\bar{k}_2, \tau_i)}{d\tau_i} \\
&= \frac{1-\sigma}{\sigma} H_1 \frac{dm(\bar{k}_2, \tau_i)}{d\tau_i}, \\
\frac{dx_2}{d\tau_i} &= \left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma} \frac{dx_1}{d\tau_i},
\end{aligned}$$

and

$$H_1(\sigma = 1) = [m(\bar{k}_2, \tau_i)]^{-1} \frac{\delta(1-\beta)}{(1+\gamma)[1+\delta\beta]^2}.$$

Second, to find $\frac{d\bar{k}_2}{d\tau_i}$, take the derivative of the expression $c_1^{-\sigma} = \frac{\delta(1+\beta\gamma)}{1+\gamma} \times m(\bar{k}_2, \tau_i)c_2^{-\sigma}$ with respect to τ_i to obtain

$$\begin{aligned}
\frac{d\bar{k}_2}{d\tau_i} &= \frac{\dfrac{\delta(1+\beta\gamma)}{1+\gamma} c_2^{-\sigma}}{\left[\sigma c_1^{-\sigma-1} + \sigma c_2^{-\sigma-1} \dfrac{\delta(1+\beta\gamma)}{1+\gamma} m(\bar{k}_2, \tau_i) r(\bar{k}_2)\right]} \frac{dm(\bar{k}_2, \tau_i)}{d\tau_i} \\
&= H_2 \frac{dm(\bar{k}_2, \tau_i)}{d\tau_i}.
\end{aligned}$$

We know that $\frac{d\bar{k}_2}{d\tau_i} < 0$ and thus $\frac{dm(\bar{k}_2,\tau_i)}{d\tau_i} < 0$ too. Moreover, $H_2(\sigma = 1) = \frac{(1+\beta\gamma)}{1+\gamma}\frac{c_1}{\sigma r(\bar{k}_2)[1+1/\delta]}$.

At $\sigma = 1$, we have that

$$x_1 = \frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\gamma)(1+\delta\beta)} \quad \text{and} \quad x_2 = \frac{\beta}{1+\beta\gamma}\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\delta\beta)}.$$

Using the expressions above, we can write $\frac{d}{d\tau_i}U(\bar{k}_1,\bar{k}_1,\tau_i^*)$ as

$$\begin{aligned}
&\frac{d}{d\tau_i}U(\bar{k}_1,\bar{k}_1,\tau_i^*)\\
&\quad = \gamma\underbrace{[-(1-x_1^{1-\sigma})+(1-x_2^{1-\sigma})\beta]}_{K_{11}}\underbrace{H_2 c_1^{-\sigma}\frac{dm(\bar{k}_2,\tau_i)}{d\tau_i}}_{K_{12}}\\
&\qquad -\gamma\underbrace{\frac{1-\sigma}{\sigma}}_{K_{21}}\\
&\qquad \times\underbrace{\left[x_1^{-\sigma}+\delta\beta\left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma}x_2^{-\sigma}[\delta r(\bar{k}_2)]^{(1-\sigma)/\sigma}\right]H_1 c_1^{1-\sigma}\frac{dm(\bar{k}_2,\tau_i)}{d\tau_i}}_{K_{22}}.
\end{aligned}$$

Take the derivative of $\frac{d}{d\tau_i}U(\bar{k}_1,\bar{k}_1,\tau_i^*)$ with respect to σ to obtain

$$\begin{aligned}
&\frac{d}{d\sigma}\left[\frac{d}{d\tau_i}U(\bar{k}_1,\bar{k}_1,\tau_i^*)\right]\frac{1}{\gamma}\\
&\quad = K_{11}\frac{dK_{12}}{d\sigma}+K_{12}\frac{dK_{11}}{d\sigma}-K_{21}\frac{dK_{22}}{d\sigma}-K_{22}\frac{dK_{21}}{d\sigma}.
\end{aligned}$$

If we evaluate this expression at $\sigma = 1$, we obtain

$$\begin{aligned}
&\frac{d}{d\sigma}\left[\frac{d}{d\tau_i}U(\bar{k}_1,\bar{k}_1,\tau_i^*)\right]\frac{1}{\gamma}\\
&\quad = K_{12}\frac{dK_{11}}{d\sigma}-K_{22}\frac{dK_{21}}{d\sigma}\\
&\quad = \frac{(1+\beta\gamma)}{1+\gamma}\frac{\delta}{\sigma r(\bar{k}_2)[1+\delta]}\frac{dm(\bar{k}_2,\tau_i)}{d\tau_i}\frac{dK_{11}}{d\sigma}\\
&\qquad -\frac{\delta(1-\beta)}{1+\gamma+\delta(1+\beta\gamma)}[m(\bar{k}_2,\tau_i)]^{-1}\frac{dm(\bar{k}_2,\tau_i)}{d\tau_i}\frac{dK_{21}}{d\sigma},
\end{aligned}$$

where $\frac{dK_{21}}{d\sigma} = -\frac{1}{\sigma^2}$ and $\frac{dK_{11}}{d\sigma} = \beta \log(x_2) - \log(x_1)$. Evaluating at $\sigma = 1$ and inserting $m(\bar{k}_2, \tau_i) = \frac{1+\gamma}{1+\beta\gamma} r(\bar{k}_2)$, we obtain

$$\begin{aligned}
&\frac{d}{d\sigma}\left[\frac{d}{d\tau_i} U(\bar{k}_1, \bar{k}_1, \tau_i^*)\right]\frac{1}{\gamma} \\
&\quad = \frac{dm(\bar{k}_2, \tau_i)}{d\tau_i} \frac{\delta(1+\beta\gamma)}{(1+\gamma) r(\bar{k}_2)} \\
&\qquad \times \left[\left(\beta \log\left[\frac{\beta}{1+\beta\gamma} \frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\delta\beta)}\right]\right.\right. \\
&\qquad \left.\left. - \log\left[\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\gamma)(1+\delta\beta)}\right]\right) \frac{1}{[1+\delta]} + \frac{(1-\beta)}{1+\gamma+\delta(1+\beta\gamma)}\right].
\end{aligned}$$

Since $\frac{dm(\bar{k}_2, \tau_i)}{d\tau_i} < 0$, if

$$\begin{aligned}
&\left(\beta \log\left[\frac{\beta}{1+\beta\gamma} \frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\delta\beta)}\right]\right. \\
&\quad \left. - \log\left[\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\gamma)(1+\delta\beta)}\right]\right) \frac{1}{[1+\delta]} + \frac{(1-\beta)}{1+\gamma+\delta(1+\beta\gamma)} > 0,
\end{aligned}$$

then $\frac{d}{d\sigma}[\frac{d}{d\tau_i} U(\bar{k}_1, \bar{k}_1, \tau_i^*)]\frac{1}{\gamma} < 0$ at $\sigma = 1$. Therefore, it is optimal to increase the subsidy for $\sigma > 1$ if this condition above holds.

To show that it holds, let $\varphi(\beta, \gamma, \delta) = \beta \log(\frac{\beta(1+\gamma+\delta(1+\beta\gamma))}{(1+\beta\gamma)(1+\delta\beta)}) - \log(\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\gamma)(1+\delta\beta)}) + \frac{(1-\beta)(1+\delta)}{1+\gamma+\delta(1+\beta\gamma)}$. First, it is easy to show that $\lim_{\gamma\to\infty} \varphi(\beta, \gamma, \delta) = 0$. Second, we show that $\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma} < 0$ for all $\beta, \delta < 1$, which implies that $\varphi(\beta, \gamma, \delta) > 0$ for all finite $\gamma > 0$ and $\beta, \delta < 1$:

$$\begin{aligned}
\frac{d\varphi(\beta, \gamma, \delta)}{d\gamma} = \beta &\frac{\frac{(1+\delta\beta)(1+\beta\gamma) - \beta(1+\gamma+\delta(1+\beta\gamma))}{(1+\beta\gamma)^2}}{\frac{1+\gamma+\delta(1+\beta\gamma)}{1+\beta\gamma}} \\
&- \frac{\frac{(1+\delta\beta)(1+\gamma) - (1+\gamma+\delta(1+\beta\gamma))}{(1+\gamma)^2}}{\frac{1+\gamma+\delta(1+\beta\gamma)}{1+\gamma}} \\
&- \frac{(1-\beta)(1+\delta)(1+\delta\beta)}{(1+\gamma+\delta(1+\beta\gamma))^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-\beta}{1+\gamma+\delta(1+\beta\gamma)} \\
&\quad \times \left\{ \frac{\beta}{1+\beta\gamma} + \frac{\delta}{1+\gamma} - \frac{(1+\delta)(1+\delta\beta)}{1+\gamma+\delta(1+\beta\gamma)} \right\} \\
&= \frac{1-\beta}{1+\gamma+\delta(1+\beta\gamma)} \\
&\quad \times \left\{ \frac{\beta+\beta\gamma+\delta+\delta\beta\gamma}{1+\gamma+\beta\gamma+\beta\gamma^2} - \frac{1+\delta+\delta\beta+\delta^2\beta}{1+\gamma+\delta+\delta\beta\gamma} \right\}.
\end{aligned}
$$

The numerator of the term in curly brackets is

$$
\begin{aligned}
&= [\beta+\beta\gamma+\delta+\delta\beta\gamma] + [\beta\gamma+\beta\gamma^2+\delta\gamma+\delta\beta\gamma^2] \\
&\quad + [\delta\beta+\delta\beta\gamma+\delta^2+\delta^2\beta\gamma] \\
&\quad + [\delta\beta^2\gamma+\delta\beta^2\gamma^2+\delta^2\beta\gamma+\delta^2\beta^2\gamma^2] \\
&\quad - [1+\delta+\delta\beta+\delta^2\beta] - [\gamma+\delta\gamma+\delta\beta\gamma+\delta^2\beta\gamma] \\
&\quad - [\beta\gamma+\delta\beta\gamma+\delta\beta^2\gamma+\delta^2\beta^2\gamma] \\
&\quad - [\beta\gamma^2+\delta\beta\gamma^2+\delta\beta^2\gamma^2+\delta^2\beta^2\gamma^2] \\
&= \beta+\beta\gamma+\delta^2+\delta^2\beta\gamma-1-\delta^2\beta-\gamma-\delta^2\beta^2\gamma \\
&= (\beta-1)+\delta^2(1-\beta)+\gamma(\beta-1)+\delta^2\beta\gamma(1-\beta) \\
&= (1-\beta)[\delta^2+\delta^2\beta\gamma-1-\gamma] \\
&= (1-\beta)[\delta^2(1+\beta\gamma)-(1+\gamma)].
\end{aligned}
$$

Using this expression in $\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma}$, we obtain

$$
\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma} = \frac{(1-\beta)^2}{(1+\gamma+\delta(1+\beta\gamma))^2}\frac{\delta^2(1+\beta\gamma)-(1+\gamma)}{(1+\gamma)(1+\beta\gamma)}.
$$

Note that $\delta^2(1+\beta\gamma) < 1+\gamma$ for all $\delta, \beta < 1$. As a result, $\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma} < 0$ for all δ, $\beta < 1$.

Next, we show that $\frac{d}{d\sigma}(\frac{c_2(\tau_i(\sigma))}{c_1(\tau_i(\sigma))})|_{\sigma=1} > 0$. For this purpose, let $k_2(\tau_i(\sigma))$ be the competitive-equilibrium savings associated with the optimal tax policy $\tau_i(\sigma)$ and let $k_2^c(\sigma)$ be the commitment savings for a given σ. We show that $\frac{d}{d\sigma}(k_2(\tau_i(\sigma)) - k_2^c(\sigma))|_{\sigma=1} > 0$. Thus, the competitive-equilibrium savings under the optimal policy is higher than commitment savings when σ is marginally higher than 1. To see this, first consider the consumer's optimality conditions under commitment and in competitive equilibrium.

$$\delta f'(k_2^c(\sigma))\left(\frac{y_1 - k_2^c(\sigma)}{f(k_2^c(\sigma))}\right)^{\sigma} = 1 \quad \text{under commitment},$$

$$\frac{\delta(1+\beta\gamma)}{(1+\gamma)(1+\tau_i(\sigma))} f'\big(k_2(\tau_i(\sigma))\big) \times \left(\frac{y_1 - k_2(\tau_i(\sigma))}{f(k_2(\tau_i(\sigma)))}\right)^{\sigma} = 1 \quad \text{in competitive equilibrium.}$$

We can rewrite this problem as

$$F(k_2^c(\sigma), \sigma) = 1 \quad \text{under commitment},$$

$$\frac{(1+\beta\gamma)}{(1+\gamma)(1+\tau_i(\sigma))} F(k_2^c(\sigma), \sigma) = 1 \quad \text{in competitive equilibrium},$$

where $F(k_2, \sigma) = \delta f'(k_2)(\frac{y_1 - k_2}{f(k_2)})^{\sigma}$. We know that

$$k_2^c(1) = k_2(\tau_i(1)),$$

$$\frac{(1+\beta\gamma)}{(1+\gamma)(1+\tau_i(\sigma))} = 1,$$

$$\tau_i'(1) < 0.$$

Next, take the derivative of the commitment and competitive-equilibrium optimality condition with respect to σ to obtain

$$\left.\frac{dk_2^c(\sigma)}{d\sigma}\right|_{\sigma=1} = -\frac{F_2(k_2^c(1), 1)}{F_1(k_2^c(1), 1)},$$

$$\left.\frac{dk_2(\tau_i(\sigma))}{d\sigma}\right|_{\sigma=1} = -\frac{F_2(k_2(\tau_i(1)), 1)}{F_1(k_2(\tau_i(1)), 1)} + \frac{\tau_i'(1)F(k_2(\tau_i(1)), 1)}{(1+\tau_i(1))F_1(k_2(\tau_i(1)), 1)}.$$

Since $k_2^c(1) = k_2(\tau_i(1))$, taking the difference yields

$$\frac{d}{d\sigma}\big(k_2(\tau_i(\sigma)) - k_2^c(\sigma)\big)|_{\sigma=1} = \frac{\tau_i'(1)F(k_2(\tau_i(1)), 1)}{(1+\tau_i(1))F_1(k_2(\tau_i(1)), 1)}.$$

Note that $\tau_i'(1) < 0$ and it is easy to see that $F_1(k_2(\tau_i(1)), 1) < 0$. As a result, $\frac{d}{d\sigma}(k_2(\tau_i(\sigma)) - k_2^c(\sigma))|_{\sigma=1} > 0$. This directly implies that $\frac{d}{d\sigma}(\frac{c_2(\tau_i(\sigma))}{c_1(\tau_i(\sigma))})|_{\sigma=1} > 0$. *Q.E.D.*

PROOF OF PROPOSITION 7: To prove this proposition, we solve the consumer's problem backward, find her optimal consumption choices, and use those decision rules to obtain her value function.

Problem at time $T-1$: The consumer's problem reads

$$\max_{c_{T-1},c_T} (1+\gamma)\log(c_{T-1}) + \delta(1+\beta\gamma)\log(c_T) \\ -\gamma \max_{\tilde{c}_{T-1},\tilde{c}_T} \log(\tilde{c}_{T-1}) + \delta\beta\ \log(\tilde{c}_T)$$

subject to the budget constraints

$$c_{T-1} + (1+\tau_{i,T-1})k_T = r(\bar{k}_{T-1})k_T + w(k_T) + s_T \quad \text{and} \\ c_T = Y_T = r(\bar{k}_T)k_T + w(\bar{k}_T).$$

The rest-of-lifetime budget constraint is thus

$$c_{T-1} + c_T \frac{1+\tau_{i,T-1}}{r(\bar{k}_T)} = r(\bar{k}_{T-1})k_{T-1} + w(\bar{k}_{T-1}) + s_{T-1} \\ + w(\bar{k}_T)\frac{1+\tau_{i,T-1}}{r(\bar{k}_T)} = Y_{T-1}.$$

The first-order condition is $\frac{1}{c_{T-1}} = \frac{\delta(1+\beta\gamma)}{1+\gamma}\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\frac{1}{c_T}$. Inserting c_T into the rest-of-lifetime budget constraint, we obtain

$$c_{T-1} = \frac{1+\gamma}{1+\gamma+\delta(1+\beta\gamma)}Y_{T-1} \quad \text{and} \\ c_T = \frac{\delta(1+\beta\gamma)}{1+\gamma+\delta(1+\beta\gamma)}\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}Y_{T-1}.$$

This implies

$$\tilde{c}_{T-1} = \frac{1}{1+\delta\beta}Y_{T-1} \quad \text{and} \quad \tilde{c}_T = \frac{\delta\beta}{1+\delta\beta}\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}Y_{T-1}.$$

Notice that the c and the $\tilde{c}$ are constant multiples of each other. As a result, the value function becomes

$$U_{T-1}(k_{T-1}, \bar{k}_{T-1}, \tau) = \log(c_{T-1}) + \delta\log(c_T) + \text{a constant}.$$

Now rewrite the value function in period $T-1$ to be used in the problem of the consumer in period $T-2$ by inserting the consumption allocations as functions of Y_{T-1}. This delivers

$$U_{T-1}(k_{T-1}, \bar{k}_{T-1}, \tau) = (1+\delta)\log(Y_{T-1}) + \delta\log(r(\bar{k}_T)/(1+\tau_{i,T-1})) \\ + \text{a constant}.$$

Problem at time $T-2$: Using the $T-2$ budget constraint and the rest-of-lifetime budget constraint at time $T-1$ for the consumer, we obtain the rest-of-lifetime budget constraint at time $T-2$ as

$$\begin{aligned} c_{T-2} + \frac{1+\tau_{i,T-2}}{r(\bar{k}_{T-1})} Y_{T-1} = Y_{T-2} & \\ &= r(\bar{k}_{T-2})k_{T-2} + w(\bar{k}_{T-2}) + s_{T-2} \\ &\quad + \frac{w(\bar{k}_{T-1}) + s_{T-1}}{r(\bar{k}_{T-1})}(1+\tau_{i,T-2}) \\ &\quad + \frac{w(\bar{k}_T)}{r(\bar{k}_{T-1})r(\bar{k}_T)}(1+\tau_{i,T-2})(1+\tau_{i,T-1}). \end{aligned}$$

The objective of the government is to maximize

$$\begin{aligned} &\max_{c_{T-2}, Y_{T-1}} (1+\gamma)\log(c_{T-2}) \\ &\quad + \delta(1+\beta\gamma)\left[(1+\delta)\log(Y_{T-1}) + \delta\log\left(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\right) + \text{ a constant}\right] \\ &\quad - \gamma \max_{\tilde{c}_{T-2}, \tilde{Y}_{T-1}} \log(\tilde{c}_{T-2}) \\ &\quad + \delta\beta\left[(1+\delta)\log(\tilde{Y}_{T-1}) + \delta\log\left(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\right) + \text{ a constant}\right]. \end{aligned}$$

The first-order condition is

$$\frac{1}{c_{T-2}} = \frac{\delta(1+\delta)(1+\beta\gamma)}{1+\gamma} \frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}} \frac{1}{Y_{T-1}}.$$

Using the budget constraint, we obtain

$$c_{T-2} = \frac{1+\gamma}{1+\gamma+\delta(1+\delta)(1+\beta\gamma)} Y_{T-2} \quad \text{and}$$

$$Y_{T-1} = \frac{\delta(1+\delta)(1+\beta\gamma)}{1+\gamma+\delta(1+\delta)(1+\beta\gamma)} \frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}} Y_{T-2}.$$

Inserting Y_{T-1} in terms of c_{T-1} into the consumer's problem, we obtain the Euler equation

$$\frac{1}{c_{T-2}} = \frac{\delta(1+\delta)(1+\beta\gamma)}{1+\gamma+\delta(1+\beta\gamma)} \frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}} \frac{1}{c_{T-1}}.$$

The temptation allocations are given by

$$\tilde{c}_{T-2} = \frac{1}{1+\delta\beta(1+\delta)} Y_{T-2} \quad \text{and}$$

$$\tilde{Y}_{T-1} = \frac{\delta\beta(1+\delta)}{1+\delta\beta(1+\delta)} \frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}} Y_{T-2}.$$

The objective function of the government is

$$U_{T-2}(k_{T-2}, \bar{k}_{T-2}, \tau_i) = \log(c_{T-2}) + \delta(1+\delta)\log(Y_{T-1}) + \delta^2 \log\left(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\right) + \text{a constant}.$$

Since c_{T-1} is a multiple of Y_{T-1} and c_T is a multiple of $(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}})Y_{T-1}$, by inserting them we obtain

$$U_{T-2}(k_{T-2}, \bar{k}_{T-2}, \tau_i) = \log(c_{T-2}) + \delta\log(c_{T-1}) + \delta^2 \log(c_T) + \text{a constant}.$$

Problem at $T-3$: The first-order condition for the consumer is

$$\begin{aligned}\frac{1}{c_{T-3}} &= \frac{\delta(1+\delta+\delta^2)(1+\beta\gamma)}{1+\gamma} \frac{r(\bar{k}_{T-2})}{1+\tau_{i,T-3}} \frac{1}{Y_{T-2}} \\ &= \frac{\delta(1+\delta+\delta^2)(1+\beta\gamma)}{1+\gamma+\delta(1+\delta)(1+\beta\gamma)} \frac{r(\bar{k}_{T-2})}{1+\tau_{i,T-3}} \frac{1}{c_{T-2}},\end{aligned}$$

$$U_{T-3}(k_{T-2}, \bar{k}_{T-3}, \tau_i) = \log(c_{T-3}) + \delta\log(c_{T-2}) + \delta^2 \log(c_{T-1}) + \log(c_T) + \text{a constant}.$$

Continuing this procedure backward completes the proof. *Q.E.D.*

PROOF OF PROPOSITION 9: We solve the problem of the consumer and find tax rates that implement the commitment allocation. Proposition 6 implies that the problem of a consumer at age t is given by

$$\max_{c_t, Y_{t+1}} \frac{c_t^{1-\sigma}}{1-\sigma} + \delta\beta U_{t+1}(Y_{t+1})$$

subject to

$$c_t + \frac{1+\tau_{i,t}}{r_{t+1}} Y_{t+1} = Y_t,$$

where

$$U_t(Y_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \delta\beta U_{t+1}(Y_{t+1}).$$

We guess and verify that $U_{t+1}(Y_t) = b_t \frac{Y_t^{1-\sigma}}{1-\sigma}$, where $b_T = 1$. The optimality condition for the consumer is given by

$$c_t^{-\sigma} = \delta\beta b_{t+1} \frac{r_{t+1}}{1+\tau_{i,t}} Y_{t+1}^{-\sigma}.$$

Inserting this into the budget constraint, we obtain

$$c_t = \frac{Y_t}{1 + (\delta\beta b_{t+1})^{1/\sigma}\left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}},$$

$$Y_{t+1} = \frac{\left(\delta\beta b_{t+1}\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{1/\sigma} Y_t}{1 + (\delta\beta b_{t+1})^{1/\sigma}\left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}}.$$

Using these decision rules, we obtain

$$b_t = \frac{1 + \frac{1}{\beta}(\delta\beta b_{t+1})^{1/(\sigma)}\left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}}{\left(1 + (\delta\beta b_{t+1})^{1/(\sigma)}\left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}\right)^{1-\sigma}}.$$

Note that the optimality condition for the consumer can be written as

$$c_t^{-\sigma} = \delta r_{t+1}\frac{\beta b_{t+1}}{1+\tau_{i,t}}\left(1 + (\delta\beta b_{t+2})^{1/\sigma}\left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}\right)^{-\sigma} c_{t+1}^{-\sigma}.$$

Inserting b_{t+1} yields

$$c_t^{-\sigma} = \delta r_{t+1}\frac{\beta}{1+\tau_{i,t}}\frac{1 + \frac{1}{\beta}(\delta\beta b_{t+2})^{1/\sigma}\left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}}{1 + (\delta\beta b_{t+2})^{1/\sigma}\left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}} c_{t+1}^{-\sigma}.$$

To implement the commitment allocation, the government should set

$$\frac{\beta}{1+\tau_{i,t}}\frac{1+\frac{1}{\beta}(\delta\beta b_{t+2})^{1/\sigma}\left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}}{1+(\delta\beta b_{t+2})^{1/\sigma}\left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}}=1,$$

where r_t for all t is the equilibrium interest rate that arises under commitment, that is, $r_t = r(\bar{k}_t)$.

The recursive formulas for b_t and $\tau_{i,t}$ jointly determine the sequence of optimal tax rates. We solve these formulas backward noting that $b_T = 1$ and $b_{T+1} = 0$. Thus, $\tau_{i,T-1} = \beta - 1$ and $b_{T-1} = \frac{1+\delta^{1/\sigma}r_T^{(1-\sigma)/\sigma}}{(1+\beta\delta^{1/\sigma}r_T^{(1-\sigma)/\sigma})^{1-\sigma}}$. Continuing backward, we obtain $\tau_{i,T-2} = \frac{\beta-1}{1+\beta\delta^{1/\sigma}r_T^{(1-\sigma)/\sigma}}$, $b_{T-2} = \frac{1+\delta^{1/\sigma}r_{T-1}^{(1-\sigma)/\sigma}(1+\delta^{1/\sigma}r_T^{(1-\sigma)/\sigma})}{(1+\beta\delta^{1/\sigma}r_{T-1}^{(1-\sigma)/\sigma}(1+\delta^{1/\sigma}r_T^{(1-\sigma)/\sigma}))^{1-\sigma}}$,

$$\tau_{i,T-3}=\frac{\beta-1}{1+\beta(\delta^{1/\sigma}r_{T-1}^{(1-\sigma)/\sigma}+\delta^{2/\sigma}r_{T-1}^{(1-\sigma)/\sigma}r_T^{(1-\sigma)/\sigma})},$$

and

$$\begin{aligned}\tau_{i,T-4}&=(\beta-1)/\big(1+\beta\big(\delta^{1/\sigma}r_{T-2}^{(1-\sigma)/\sigma}+\delta^{2/\sigma}r_{T-2}^{(1-\sigma)/\sigma}r_{T-1}^{(1-\sigma)/\sigma}\\&\quad+\delta^{3/\sigma}r_{T-2}^{(1-\sigma)/\sigma}r_{T-1}^{(1-\sigma)/\sigma}r_T^{(1-\sigma)/\sigma}\big)\big).\end{aligned}$$

One can notice the pattern in the expressions above, which implies the optimal tax for period t is given by

$$\tau_{i,t}=\frac{\beta-1}{1+\beta\sum_{m=t+2}^{T}\left\{(\delta^{1/\sigma})^{m-(t+1)}\prod_{n=t+2}^{m}r(\bar{k}_n)^{(1-\sigma)/\sigma}\right\}}.$$

We can also show that as $T\to\infty$, the optimal tax rate converges to a negative value. To see this, let $\{c_t^c\}_{t=0}^{\infty}$ be the consumption sequence associated with the commitment solution. Inserting the commitment Euler equation $\frac{c_{t+1}^c}{c_t^c} = (\delta r_{t+1})^{1/\sigma}$ into the tax expression, we obtain

$$\tau_{i,t}=\frac{\beta-1}{1+\frac{\beta}{c_{t+1}^c}\left[\frac{c_{t+2}^c}{r_{t+2}}+\frac{c_{t+3}^c}{r_{t+2}r_{t+3}}+\cdots+\frac{c_T^c}{r_{t+2}r_{t+3}\ldots r_T}\right]}.$$

Note that

$$c_{t+1}^c+\frac{c_{t+2}^c}{r_{t+2}}+\frac{c_{t+3}^c}{r_{t+2}r_{t+3}}+\cdots+\frac{c_T^c}{r_{t+2}r_{t+3}\cdots r_T}=Y_{t+1}^c,$$

where Y_t^c is the lifetime income at time t associated with the commitment solution. Thus, the optimal tax rate can be written as

$$\tau_{i,t} = \frac{(\beta - 1)\dfrac{c_{t+1}^c}{Y_{t+1}^c}}{(1-\beta)\dfrac{c_{t+1}^c}{Y_{t+1}^c} + \beta}.$$

Note that since $c_{t+1}^c / Y_{t+1}^c > 0$ for any t and T, we obtain that $\tau_{i,t} < 0$ for all t. Moreover, since the equilibrium allocation under the optimal tax sequence is the same as the allocation associated with the commitment solution and since self-control cost is zero, the optimal tax policy delivers first-best welfare. *Q.E.D.*

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