Asset prices in a Huggett economy [☆]

Per Krusell [a,b,c,d], Toshihiko Mukoyama [e,f], Anthony A. Smith, Jr. [g,*]

[a] *IIES, Sweden*
[b] *CAERP, Spain*
[c] *CEPR, UK*
[d] *NBER, United States*
[e] *University of Virginia, United States*
[f] *CIREQ, Canada*
[g] *Department of Economics, Yale University, 28 Hillhouse Avenue, Room 306, New Haven, CT 06520, United States*

Abstract

This paper explores asset pricing in economies where there is no direct insurance against idiosyncratic risks but other assets can be used for self-insurance, subject to exogenously-imposed borrowing limits. We analyze an endowment economy, based on Huggett (1993) [11], both with and without aggregate risk. Our main innovation is that we obtain full analytical tractability by studying the case with "maximally tight" borrowing constraints. We illustrate by looking at riskless bonds, equity, and the term structure of interest rates, and we show that the model can reproduce many features of observed asset prices when idiosyncratic risks are quantitatively reasonable.
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1. Introduction

Are some of the striking features of asset prices—in particular, the high premium for risk in asset markets and the low return on risk-free assets—a result of market incompleteness and, in particular, of missing markets for consumers' idiosyncratic risks? This possibility was raised in the concluding remarks of the seminal paper by Mehra and Prescott [22], and it was subsequently investigated by many researchers, among them Mankiw [20], Heaton and Lucas [9,10], Huggett [11], Telmer [27], Lucas [18], den Haan [7], Constantinides and Duffie [6], Krusell and Smith [13], Marcet and Singleton [21], Storesletten, Telmer, and Yaron [25]. Some of these analyses suggest that the effects of market incompleteness can be quantitatively important—e.g., the work by Constantinides and Duffie and by Storesletten, Telmer, and Yaron—but the "average view" in this literature is probably closer to concluding that no major aspects of asset prices are overturned if market incompleteness is taken into account. In fact, a recent study by Krueger and Lustig [12] demonstrates that in a range of interesting cases, risk premia will not be affected at all by market incompleteness, even though the risk-free rate might be.

Though it had a different substantive question in focus, the early work of Bewley [3,4] contains the key ingredients that have since become central elements used in studies of asset pricing with incomplete markets. Bewley assumed that consumers face less than fully uninsurable shocks but that there are a large number of consumers of each type across whom these shocks are independent, so that these shocks have no direct aggregate consequences. Thus, each consumer can be studied in a stationary environment. Literatures then developed around this setting, examining not only asset pricing but also a range of other issues. A general challenge here has been that multiperiod equilibrium models are hard to analyze, even with the aid of numerical methods, especially when there is aggregate uncertainty. In the present paper, we show how some simplifying assumptions allow us to study asset pricing with analytical precision. In particular, we study the setting in Huggett [11], which lays out a canonical incomplete-markets asset-pricing model. This setting can be viewed as an extension of the seminal work of Lucas [19] to the case where insurance markets against idiosyncratic risks are missing.

A central element of Huggett's setting is that consumers can borrow as a partial way of insuring. There is a borrowing constraint, and Huggett's numerical results show how the tightness of this constraint affects the degree of de-facto insurance and, thus, the risk-free rate. Our main contribution is to study a special case of the Huggett economy both without and with aggregate risk; Huggett studied the only former and, in addition, looked only at the risk-free rate. The special case is that where the borrowing constraints are "maximally tight," i.e., so tight that they induce autarky. For this case, we obtain closed-form solutions for asset prices. The case is of particular interest, because the tighter is the borrowing constraint, the more "bite" will the market incompleteness have in terms of producing asset prices that are different from those obtaining in the standard representative-agent, Lucas-style model. One could thus view our present setting as one that allows us to examine the *potential* of incomplete-markets settings for explaining asset prices. We demonstrate how the different primitives of the model—the discount rate, the preference curvature, and the endowment process—influence prices. In particular, we show that the model allows a very rich set of asset-price predictions, including large equity premia, a low risk-free rate, and a yield curve that is qualitatively different than in the standard model.

The Huggett economy is the simplest form of endowment economy. In Huggett's [11] paper, only a riskless asset is available to agents, who are thus using this asset for precautionary saving against endowment shocks. There is no aggregate risk: the aggregate endowment is constant over time. Huggett shows, using numerical analysis, that with high curvature in utility and

a tight enough borrowing constraint, the risk-free rate can be significantly below the discount rate: agents value the riskless asset not only for its direct return but for its value as an insurance instrument. Our analytical power comes from the insight that if the borrowing constraint is maximally tight, implying that *no borrowing at all* is possible, the equilibrium has to be autarky. In autarky, the bond price will have to be equal to that of the agent in the population who values the bond the most: all other consumers would like to hold a negative amount of bonds (but cannot), and the bond-pricing consumer is just indifferent at zero bond holdings. In very simple settings with a two-state endowment process, which we spend most of the paper analyzing, it is obvious who values the bond the most—it is the consumer with the high endowment—but for general endowment processes it is not obvious.

Moreover, in a Huggett economy with aggregate risk (like that studied by den Haan), we can similarly look at the case with maximally tight borrowing constraints by assuming that for every asset contingent on the aggregate state, no negative holding is allowed, again implying that equilibrium is autarky and that each state-contingent asset is priced by the agent in the population who values it the most. Thus, overall, asset prices in our Huggett economy are determined in the same manner as are those in Lucas's [19] exchange economy—so that all markets for contingent claims clear at zero—although here only one type of consumer has an interior asset demand for each asset, whereas in Lucas's setting all agents (i.e., the representative agent) have interior solutions for all assets. We thus derive explicit, easy-to-interpret formulas for all claims contingent on aggregate shocks, and thus any assets with payoffs contingent on aggregate shocks can be priced. As an illustration, we show how to derive predictions for the term structure of interest rates; as in Lucas's work, these are easily priced using the contingent-claims prices. Few existing studies in this literature manage to look at a very rich set of assets, since the numerical analysis of portfolio choice of models with incomplete markets with aggregate risk is quite challenging. A case of special interest is that where the long-term bonds are not fully liquid; for illustration we analyze the case where the secondary markets are absent. For this economy we show that, even in the absence of aggregate risk, there is a nontrivial yield curve, and this yield curve is upward-sloping under a reasonable calibration.

Other studies before ours manage to characterize equilibria analytically in specific incomplete-markets settings. Krueger and Lustig are able to characterize risk premia by assuming a form of independence between idiosyncratic and aggregate shocks; Constantinides and Duffie, on the other hand, use a setting with literally permanent shocks and are able to characterize prices that way (since autarky is an equilibrium in that case as well). Krueger and Lustig's results apply in a special case of our setting, if we make the appropriate independence assumptions. Constantinides and Duffie's results are different than ours in a couple of ways. First, in their setting, all agents have interior solutions whereas in our case a subset of the agents price the assets. Second, we model endowment shocks using stationary Markov chains. We emphasize stationary processes not only because some argue that this is more realistic, but because it allows us to show explicitly how the degree of persistence in individual endowments influences asset prices. Using a judicious choice of the process for idiosyncratic risk we can then reproduce *any* pricing kernel, provided it satisfies the same restriction as in Constantinides and Duffie's work. Finally, our work is also related to Alvarez and Jermann [1,2], who study asset pricing with endogenous solvency constraints. For some settings of parameter values, they find that the equilibrium allocation features less-than-full, or even no, risk sharing.

Our model allows very general preference and endowment settings; we discuss extensions in the final section of our paper. We begin the analysis in the paper in Section 2 with the simplest case: the revisiting of Huggett's analysis without aggregate uncertainty, obtaining an analytical

solution for the price of the riskless bond. We then look at aggregate uncertainty in Section 3 and explore the quantitative implications of our model, including how they relate to the literature. For simplicity, throughout both of these sections we restrict attention to the case with only two possible states for individual risk. In Section 4 we look at more than two states, which is a case of interest because it is then less clear who prices each asset. Section 5 concludes.

2. The economy without aggregate uncertainty

In this section, we analyze the model of Huggett [11]. Consider an exchange economy where each consumer receives a random nondurable endowment every period. There is a continuum of consumers with total population of one. In this section, we focus on a steady state where the aggregate variables are constant and the distribution of the individual states is stationary. In this and the following section, we assume that the endowment follows a two-point process. In Section 4, we extend our analysis to a setting where there are more than two possible values of the endowment.

2.1. Model

The consumers cannot write contracts that depends on individual idiosyncratic states. Instead, they are allowed to borrow and lend through selling and buying bonds. The bond holding is denoted by a and the price of a bond that delivers one unit of consumption good next period is denoted by q.

A consumer maximizes the expected present-value of utility:

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right],$$

where $u(\cdot)$ is the momentary utility function, c_t is the consumption at period t, and $\beta \in (0, 1)$. Following Huggett [11], we consider the specification

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

where $\sigma > 1$.

Using recursive notation (a "$'$" denotes next period's value),

$$c + qa' = a + \epsilon,$$

where ϵ is the random endowment. We let ϵ take on two values, ϵ_ℓ and ϵ_h, where $\epsilon_\ell < \epsilon_h$. It follows a Markov process with transition probability $\Pr[\epsilon' = \epsilon_{s'} | \epsilon = \epsilon_s] = \pi_{ss'}$. We impose a borrowing constraint

$$a' \geqslant \underline{a},$$

where $\underline{a} \leqslant 0$ is a given constant.

The consumer's Bellman equation is

$$V_s(a) = \max_{c,a'} \frac{c^{1-\sigma}}{1-\sigma} + \beta\left[\pi_{sh} V_h(a') + (1 - \pi_{sh}) V_\ell(a')\right]$$

subject to

$$c = a + \epsilon_s - qa'$$

and

$$a' \geqslant \underline{a}.$$

Here, $V_s(a)$ is the value function of a consumer with the endowment ϵ_s and the bond holding a. Let the decision rule of the consumer be $a' = \psi(a; s)$.

The stationary equilibrium is defined by the consumer's optimization and the value of q, where

$$\sum_{s=\ell,h} \int \psi(a; s)\Gamma_s(da) = 0, \tag{1}$$

and where $\Gamma_s(a)$ is the stationary distribution of asset holdings for the consumers with endowment ϵ_s.

In the following, we consider the special case of $\underline{a} = 0$. The implication of this assumption is that, since nobody can borrow and (1) has to hold, nobody can save in equilibrium: $\psi(0; s) = 0$ for all s. One does not need to characterize $\psi(a; s)$ for other values of a, since the stationary distribution over a has all its mass on 0 in this special case. Thus, in equilibrium, for each s, consumption equals ϵ_s for all agents in state s. Thus, $\underline{a} = 0$ is maximally tight in that it is the highest value such that an equilibrium exists or, alternatively, so tight that autarky is induced.

2.2. Determination of the equilibrium bond price

In this section, we characterize the equilibrium bond price. Let $\lambda_s \geqslant 0$ be the Lagrange multiplier for the borrowing constraint when $\epsilon = \epsilon_s$.

The first-order conditions for the consumers are

$$-qc_s^{-\sigma} + \beta\big[\pi_{sh}V_h'(a') + (1 - \pi_{sh})V_\ell'(a')\big] + \lambda_s = 0, \tag{2}$$

for $s = h, \ell$. Here, c_s is the optimal c when $\epsilon = \epsilon_s$. $V_s'(a)$ is the derivative of $V_s(a)$ with respect to a. The envelope condition is

$$V_s'(a) = c_s^{-\sigma}.$$

Noting that $c_s = \epsilon_s$ in equilibrium, (2) can be rewritten as

$$\frac{q}{\beta} - \frac{\lambda_h}{\beta\epsilon_h^{-\sigma}} = \pi_{hh} + (1 - \pi_{hh})\left(\frac{\epsilon_\ell}{\epsilon_h}\right)^{-\sigma}, \tag{3}$$

and

$$\frac{q}{\beta} - \frac{\lambda_\ell}{\beta\epsilon_\ell^{-\sigma}} = \pi_{\ell h}\left(\frac{\epsilon_h}{\epsilon_\ell}\right)^{-\sigma} + 1 - \pi_{\ell h}. \tag{4}$$

Since the right-hand side of (3) is larger than one and the right-hand side of (4) is less than one,

$$\frac{\lambda_h}{\beta\epsilon_h^{-\sigma}} < \frac{\lambda_\ell}{\beta\epsilon_\ell^{-\sigma}}$$

follows. Therefore, $\lambda_\ell > 0$ and the borrowing constraint is always binding for the consumers with $s = \ell$. Thus, it is sufficient for an equilibrium that $\lambda_h \geqslant 0$ is satisfied. From (3), we can characterize the equilibrium level of the bond price q as follows.
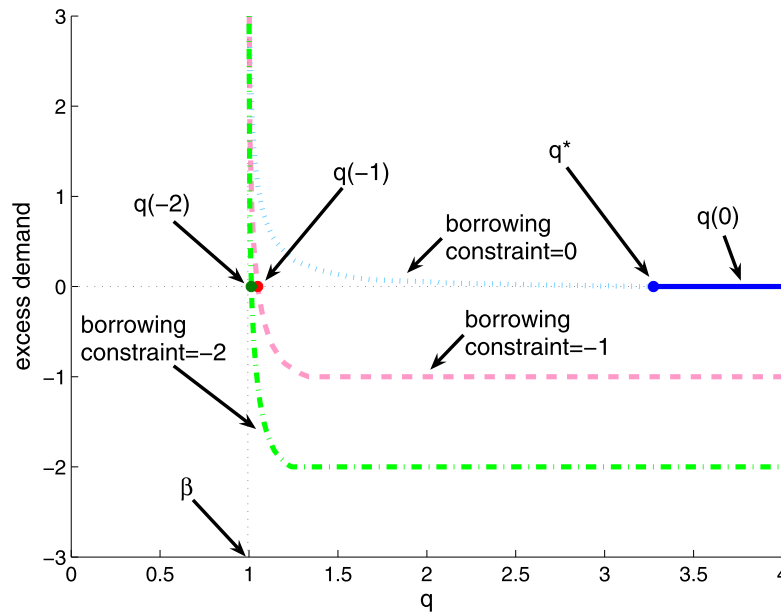
Fig. 1. Excess demand functions for $\underline{a} = 0$, $\underline{a} = -1$, and $\underline{a} = -2$; and the equilibrium prices $q(0)$, $q(-1)$, and $q(-2)$. $q(0)$ is any $q \geqslant q^*$.

Proposition 1. *The equilibrium bond price q satisfies*

$$q \geqslant q^* \equiv \beta \left[\pi_{hh} + (1 - \pi_{hh}) \left(\frac{\epsilon_h}{\epsilon_\ell} \right)^\sigma \right]. \tag{5}$$

Any bond price q that satisfies (5) is consistent with the consumers' optimization and the bond-market equilibrium (1), and thus constitutes an equilibrium. Note that the right-hand side of (5) is always strictly larger than β, so the risk-free rate is always strictly less than $1/\beta - 1$.

The lower bound of the equilibrium bond price, denoted as q^* in (5), is of particular interest since it is the limit of the equilibrium bond prices when $\underline{a}$ approaches zero from below. We formalize this claim in Proposition 11 in Appendix A. The proof of the proposition also makes clear that q^* can be thought of as a global upper bound on the equilibrium bond price in this class of economies. Fig. 1 illustrates: it draws the excess demand for the bond under $\underline{a} = 0$, $\underline{a} = -1$, and $\underline{a} = -2$ as functions of the bond price q. The parameter values are identical to the ones used by Huggett [11].[1] The figure also depicts the equilibrium price (that is, the point where excess demand equals zero) for each value of the borrowing constraint. The excess demand function shifts upwards as $\underline{a}$ increases. The equilibrium price increases monotonically as $\underline{a}$ increases, and converges to q^* as $\underline{a} \to 0$ from below. When $\underline{a} = 0$, any price that is larger than q^* is consistent with the bond market equilibrium. In what follows, we assume that the equilibrium value of q is equal to q^*, and characterize q^*.

2.3. Comparative statics

Eq. (5) shows that q^* can be characterized by the marginal utility of the high-endowment consumers. It also shows the role of various parameters in determining the equilibrium bond price. q^* is increasing in β, σ, and ϵ_h/ϵ_ℓ. It is decreasing in π_{hh}: when the high-endowment state is permanent, there is less of a precautionary-saving motive and q^* becomes small.

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[1] That is, $\beta = 0.99322$, $\sigma = 1.5$, $\epsilon_h = 1.0$, $\epsilon_\ell = 0.1$, $\pi_{hh} = 0.925$, and $\pi_{\ell h} = 0.5$.
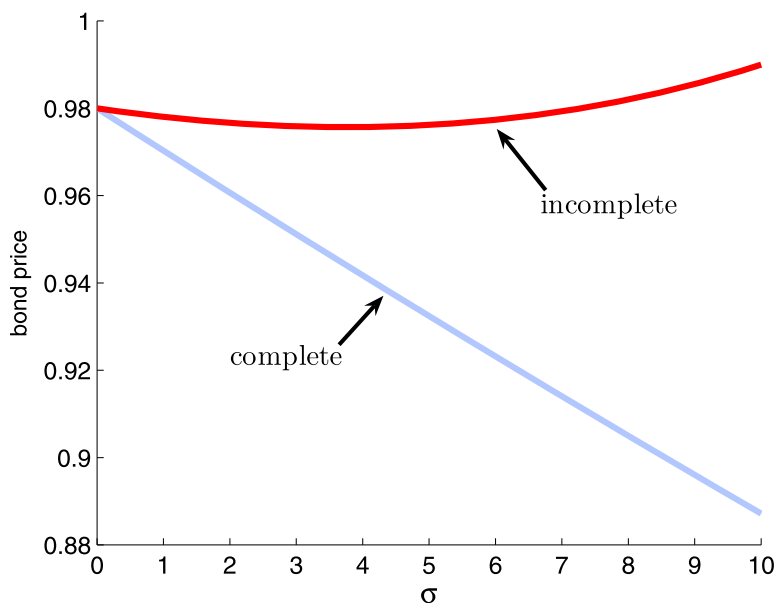
Fig. 2. Bond prices for various values of σ.

In the context of Huggett [11], this expression clarifies the role of σ in determining the precautionary-saving motive. This mechanism helps solve the risk-free rate puzzle by Weil [28]; here, a high σ is consistent with a low risk-free rate. To see this, suppose that the endowment grows over time: let the endowment be $(1 + g)^t \epsilon$, where ϵ has the same properties as before. Then, the equilibrium price of the bond becomes $(1 + g)^{-\sigma} q^*$, where q^* corresponds to the price in the absence of growth. In the complete-markets model, q^* would equal β, and therefore a positive g and a large value of σ imply a very low bond price. Thus, since the risk-free rate is the inverse of the bond price, when we consider a growing economy, the complete-markets risk-free rate would be very large, contradicting observation (this is the risk-free rate puzzle). In the current model, however, the precautionary-saving motive increases the bond price, and this can offset the effect of growth. In the incomplete-markets case, q^* is increasing in σ. Thus, with growth, the bond price can either be increasing or decreasing in σ in the incomplete-markets model. Fig. 2 plots the equilibrium bond prices for various values of σ, when $\beta = 0.98$, $g = 0.01$, $\pi_{hh} = 0.9$, and $\epsilon_h/\epsilon_\ell = 1.08$, illustrating that the riskfree rate is increasing in σ for low σ and decreasing in σ for higher σ.

2.4. A note on transactions costs in "secondary" markets

With this model, one can also price other kinds of assets. When there is no aggregate risk, another kind of asset that might be priced is a long-term riskless bond, i.e., a bond that pays one unit of consumption for sure in a future period n. Consider $n = 2$ for simplicity: what is the issue price of a two-period riskless bond? To the extent it is traded in the intermediate period, it must be $(q^*)^2$, from the usual arbitrage arguments. However, suppose that there are transactions costs, so that the two-period bond cannot be re-traded in the intermediate period: the "secondary" market is not operative. Suppose, moreover, that the two-period bond has the same kind of maximally tight borrowing constraint as does the one-period bond: it cannot be issued by individuals (they cannot use it to borrow), but it can be held in positive amounts. Given a zero net supply, no one will hold the two-period bond in equilibrium, however, and equilibrium is still autarky, allowing us to price the assets as easily as before. The one-period riskless bond will, as before, be priced

by the rich agent, so it will command the price q^*. The two-period bond will also be priced by the rich agent, who is the only agent with the chance of a consumption drop between now and two periods from now. Thus, it will have a price $q^{(2)}$ satisfying

$$q^{(2)} = \beta^2 \left[\pi_{hh}^{(2)} + \left(1 - \pi_{hh}^{(2)}\right) \left(\frac{\epsilon_h}{\epsilon_\ell}\right)^\sigma \right],$$

where $\pi_{hh}^{(2)} \equiv \pi_{hh}^2 + (1 - \pi_{hh})\pi_{\ell h}$ is the probability of transiting from h to h in two periods. Longer-period bonds subject to no re-trading can be priced similarly.

What will the term structure of interest rates look like in our incomplete-markets economy without secondary markets for bonds? We see that $q^{(2)} < (q^*)^2$, so that the longer-period bond gives a higher return (the yield curve is upward-sloping), if and only if

$$\pi_{hh}^{(2)} + \left(1 - \pi_{hh}^{(2)}\right) \left(\frac{\epsilon_h}{\epsilon_\ell}\right)^\sigma < \left[\pi_{hh} + (1 - \pi_{hh}) \left(\frac{\epsilon_h}{\epsilon_\ell}\right)^\sigma \right]^2.$$

Thus, we obtain a nontrivial yield curve. Inspecting the expression, if the endowment process has positive serial correlation (which is reasonable to assume), implying $\pi_{hh} \geqslant \pi_{hh}^{(2)}$, then for any given value of $(\frac{\epsilon_h}{\epsilon_\ell})^\sigma > 1$, the yield curve is upward-sloping if the process is not mean-reverting too quickly. Also note that for an iid process, we always obtain a positive slope of the yield curve.

2.5. Assets in positive net supply

Note that above, as in Huggett [11], we assume that assets are in zero net supply. Many interesting cases, however, involve assets in positive net supply, e.g., government bonds, a "Lucas tree," physical capital, or money. In this section we show how cases with positive asset supply can be analyzed using the framework above. In particular, one can map a case with positive asset supply into one with a zero asset supply but with a looser borrowing constraint. Thus, if an outside asset is introduced in a context where previously the borrowing constraint was maximally tight, the equilibrium real interest rate would need to rise.

For concreteness, consider a case where η, a part of the aggregate endowment, is capitalized, i.e., traded in the market as an asset. Thus, we can think of this asset as a Lucas tree with dividend η each period. We let the individual's remaining non-tradable endowment be such that the aggregate endowment is what it was before: $\tilde{\epsilon}_s \equiv \epsilon_s - \eta$ for each idiosyncratic state s. Using p to denote the price of the tree, we can write the consumer's budget constraint as

$$c = a + (\eta + p)x + \tilde{\epsilon}_s - qa' - px',$$

where x denotes the consumer's share of the tree. In equilibrium, thus, the sum of asset holdings across all consumers has to equal 1 (and, as before, the sum of borrowing and lending has to equal 0).

In equilibrium since both assets are riskless, they have to deliver equal returns if they are both held by unconstrained consumers: $p = q(p + \eta)$. Now define $\hat{a} \equiv a + (\eta + p)(x - 1)$. Using the rate-of-return equalization and some simple algebra, we can rewrite the budget as

$$c = \hat{a} + \epsilon - q\hat{a}'.$$

In equilibrium, the sum of $\hat{a}$ across all consumers has to equal 0. Thus, the budget constraint and the equilibrium condition look identical to those in the previous sections. However, the borrowing

constraint is not the same. Considering $\underline{a} \leqslant 0$, as before, to be the lower bound for lending, and using a similar lower bound for tree holdings, $\underline{x} \leqslant 0$, the borrowing constraint for $\hat{a}'$ would read

$$\hat{a}' \geqslant \underline{\hat{a}} \equiv \underline{a} + (\eta + p)(\underline{x} - 1) < 0.$$

Thus, because $\underline{\hat{a}} < 0$, this transformed consumer problem allows strictly positive borrowing even if $\underline{a} = 0$. Given the results from the previous sections, we conclude that the lowest possible interest rate will not be obtained whenever there is a positive net supply of assets.

Several remarks are in order. First, the transformed model here is not identical to that in the previous sections, where the lower bound on borrowing was exogenous; here, the lower bound is a function of p. The main purpose of the transformation, however, was simply to show that the lower bound on saving is strictly negative (so long as $\eta > 0$ or $p > 0$).[2] Second, in the following section, an economy with aggregate uncertainty will be studied, also using a setting with assets in zero net supply, and there as well one can map economies with positive asset supplies into the zero-supply setting by appropriate transformation of the borrowing constraints, which become looser.[3] Third, some cases of positive asset supplies are not identical to the one just studied. One is the case where the outside asset is physical capital explicitly used in production—the present model is a model with exogenous output. However, in any steady state of such a model, the consumer's problem can be transformed as it was in the case of the tree above, and there would be a corresponding equilibrium condition of zero net assets (with a transformed borrowing constraint); thus, the equilibrium real-return implications of such an economy would not differ conceptually from those here.

A final case of interest is the Bewley [3,4] model of money: consumers have precautionary-savings needs due to uninsurable income shocks, and money obtains value as a storage vehicle used for buffer-stock saving. The Bewley version of the present economy can be thought of as another case with an outside asset: one where money is a tree with zero dividend ($\eta = 0$) and where the stationary return on the asset is given exogenously by central-bank policy. To the extent that there is active policy either in the form of paying interest on money or in the form of changes in the total stock of money, there is also a tax/transfer appearing in the consumer's budget; for example, if the central bank pays interest on money, it needs to tax consumers to obtain the resources to do so. Thus, the consumer's budget reads $c = a + pm + p\mu + \epsilon_s - qa' - p(1 + \mu)m'$, where m is the consumer's holdings as a fraction of the tree/total stock of money and μ (possibly negative) is the net stock of money growth/inflation rate. Thus, assuming that money has value here, it would have to be that q adjusts to equal $1 + \mu$. With the transformation $\hat{a} \equiv a + p(m - 1)$ and $\underline{\hat{a}} = \underline{a} - p$, it is easy to verify that this reproduces the transformed problem above. Bewley's main interest was to characterize optimal monetary policy, possibly involving paying interest on currency (or deflation). Here, however, the focus is on asset returns, which are exogenous in the steady state of Bewley's model. Bewley's model, instead, makes the value of the asset, p, endogenous: it may or may not be strictly positive, and it will typically increase in $\underline{a}$ (allowing private borrowing drives down the value of money).

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[2] The presence of a price (here, p) in the borrowing constraint does not introduce additional complications in numerical computation of an equilibrium in the transformed economy relative to the case of literally zero asset supply. One simply guesses on p, obtains the implied return $1/q$, solves the consumer's problem—which depends on q and p—given these values, simulates and then varies p in order to obtain zero asset holdings on average in the simulation.

[3] In a separate Technical Appendix to the present paper, Krusell, Mukoyama, and Smith [15], we go through all the necessary transformations. In that document we also explicitly make the point that if the borrowing constraint is turned into a sufficiently high minimum-savings constraint—for the case above, $\underline{x} = 1$—it will literally reproduce the maximally tight borrowing constraint in the zero-net-asset economy and deliver an identical interest rate.

3. The economy with aggregate uncertainty

In this section, we extend the basic model by incorporating aggregate uncertainty. Suppose that there are two aggregate states, $Z \in \{g, b\}$. We assume that there are two Arrow securities: a state-z' security, purchased at the price $Q_{zz'}$ when the current aggregate state is z, delivers one unit of consumption good next period when the next period's aggregate state is z'. Note that the aggregate states are spanned by these securities, but that the market is still incomplete: idiosyncratic risks cannot be insured away.[4]

The introduction of Arrow securities has two virtues. First, any asset whose returns depend only on the aggregate state can be priced uniquely by the prices of these securities. Second, we can introduce the borrowing constraint in a natural manner, since the holdings of these securities is directly linked to the total asset balance in the following period.[5] Finally, as in the case without aggregate uncertainty, we focus on the case with zero net supply of each of the Arrow securities.[6]

3.1. Model

The consumer's problem is now to maximize

$$E\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}\right]$$

subject to the (recursively stated) constraints

$$c + Q_{Zg}a_g' + Q_{Zb}a_b' = a_Z + \epsilon$$

and

$$a_g' \geqslant 0, \qquad a_b' \geqslant 0.$$

Here, $a_{z'}$ is the amount of state-z' security held by the consumer. Asset-market equilibrium requires the sum of net demands for $a_{z'}'$ to be zero for $z' = g, b$. As in the previous section, in equilibrium nobody can borrow, so nobody can save, and therefore the equilibrium is autarky. In what follows we presume, though do not prove (as we did in the economy without aggregate

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[4] Any two independent securities can replicate the Arrow securities here. For example, suppose that there is infinitely-lived "equity" which delivers e_z in state z and a one-period bond which delivers 1 unit in both future states. Denoting x as the equity demand, p_z as the equity price, and y as the bond demand,

$$a_g' = (p_g + e_g)x' + y'$$

and

$$a_b' = (p_b + e_b)x' + y'$$

hold. From these equations, it can be seen that a g-state Arrow security can be replicated by combining $1/(p_g + e_g - p_b - e_b)$ units of equity and $(-p_b - e_b)/(p_g + e_g - p_b - e_b)$ units of bond, and a b-state Arrow security can be replicated by combining $1/(p_b + e_b - p_g - e_g)$ units of equity and $(-p_g - e_g)/(p_b + e_b - p_g - e_g)$ units of bond.

[5] Following Alvarez and Jermann [1,2], it would perhaps be more appropriate to refer to these constraints as "solvency constraints" because the assets are contingent claims. For consistency with the rest of the paper, we use instead the term borrowing constraints.

[6] As discussed in Section 2.5 above, assets in net positive supply can be thought of as loosening the borrowing constraints in the zero-net-supply economy.

uncertainty), that the prices we derive are the limit prices for economies with increasingly tight constraints on the Arrow securities.[7]

We assume, as before, that the endowment can only have two values: $\epsilon \in \{\epsilon_h, \epsilon_\ell\}$ where $\epsilon_h > \epsilon_\ell$. Let $\Pr[Z' = z'|Z = z] = \phi_{zz'}$ and $\Pr[\epsilon = \epsilon_{s'}|\epsilon = \epsilon_s, \ Z = z, \ Z' = z'] = \pi_{ss'|zz'}$. Then the consumer's Bellman equation is

$$V(a; s, z) = \max_{c, a'_g, a'_b} \frac{c^{1-\sigma}}{1 - \sigma}$$
$$+ \beta \left[\sum_{z'=g,b} \phi_{zz'} \left[\pi_{sh|zz'} V(a'_{z'}; h, z') + (1 - \pi_{sh|zz'}) V(a'_{z'}; \ell, z') \right] \right]$$

subject to

$$c = a + \epsilon_s - Q_{zg} a'_g - Q_{zb} a'_b$$

and

$$a'_g \geqslant 0, \qquad a'_b \geqslant 0.$$

Let $\lambda^{z'}_{sz}$ be the Lagrange multiplier for the borrowing constraint for the state-z' security when the current state is s and z. The first-order condition is

$$-Q_{zz'} c_{sz}^{-\sigma} + \beta \phi_{zz'} \left[\pi_{sh|zz'} V'(a'_{z'}; h, z') + (1 - \pi_{sh|zg}) V'(a'_{z'}; \ell, z'^{z'}_{sz}) \right] + \lambda^{z'}_{sz} = 0, \qquad (6)$$

where c_{sz} is consumption of the consumer whose individual state is s when the aggregate state is z.

The envelope conditions are

$$V'(a; s, z) = c_{sz}^{-\sigma}.$$

3.2. The prices of contingent claims

Recall that $c_{sz} = \epsilon_s$ in equilibrium. To determine $Q_{zz'}$, let us look at (6). For each (z, z'), there are two first-order conditions (for $s = h$ and $s = \ell$). They can be rewritten as

$$\frac{Q_{zz'}}{\beta \phi_{zz'}} - \frac{\lambda^{z'}_{hz}}{\beta \phi_{zz'} \epsilon_h^{-\sigma}} = \pi_{hh|zz'} + (1 - \pi_{hh|zz'}) \left(\frac{\epsilon_\ell}{\epsilon_h} \right)^{-\sigma} \qquad (7)$$

and

$$\frac{Q_{zz'}}{\beta \phi_{zz'}} - \frac{\lambda^{z'}_{\ell z}}{\beta \phi_{zz'} \epsilon_\ell^{-\sigma}} = \pi_{\ell h|zz'} \left(\frac{\epsilon_h}{\epsilon_\ell} \right)^{-\sigma} + (1 - \pi_{\ell h|zz'}). \qquad (8)$$

Using the logic employed in the previous section, we conclude that

$$\frac{\lambda^{z'}_{hz}}{\beta \phi_{zz'} \epsilon_h^{-\sigma}} < \frac{\lambda^{z'}_{\ell z}}{\beta \phi_{zz'} \epsilon_\ell^{-\sigma}}$$

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[7] It is very difficult to characterize how the law of motion for the wealth distribution, a mapping from the set of distributions (and the aggregate state) into itself, varies with parameters. This makes it extremely challenging to even demonstrate existence, let alone examine comparative statics.

holds. Therefore, $\lambda_{\ell z}^{z'} > 0$ and the borrowing constraint is binding for the consumers with $s = \ell$.[8] To satisfy $\lambda_{hz}^{z'} \geqslant 0$, $Q_{zz'}$ has to satisfy

$$Q_{zz'} \geqslant Q_{zz'}^* \equiv \beta \phi_{zz'} \left[\pi_{hh|zz'} + (1 - \pi_{hh|zz'}) \left(\frac{\epsilon_h}{\epsilon_\ell} \right)^\sigma \right].$$

Again, we focus on the case where the asset prices are determined by the lower bound: $Q_{zz'} = Q_{zz'}^*$. Then, $Q_{zz'}$ is increasing in β, $\phi_{zz'}$, σ, and $\epsilon_h / \epsilon_\ell$. It is decreasing in $\pi_{hh|zz'}$. In the following, we denote $\omega \equiv (\epsilon_h / \epsilon_\ell)^\sigma$. Note that $\omega > 1$ and it is increasing in σ and $\epsilon_h / \epsilon_\ell$. Also define

$$m_{zz'} \equiv \beta \left[\pi_{hh|zz'} + (1 - \pi_{hh|zz'}) \omega \right]. \tag{9}$$

3.2.1. The bond price

Now we investigate the properties of bonds and stock in this economy. The bond price at state z is

$$q_z = \sum_{z'=g,b} Q_{zz'} = \sum_{z'=g,b} \phi_{zz'} m_{zz'} = E[m_{zz'}]. \tag{10}$$

In this section, all the expectations $E[\cdot]$, variances $Var[\cdot]$, and covariances $Cov(\cdot, \cdot)$ are with respect to z', conditional on z. The (gross) return from the bond (that is, risk-free rate) is $R_z^f = q_z^{-1}$. Thus,

$$1 = E\left[R_z^f m_{zz'} \right]. \tag{11}$$

Note that from the definition of $m_{zz'}$,

$$q_z = \beta \left(\omega - (\omega - 1) E[\pi_{hh|zz'}] \right). \tag{12}$$

This expression clarifies that, similarly to the previous section, the (average) level of $\pi_{hh|zz'}$ is an important determinant of the bond price.

As a general proposition, in our environment, prices are not a function of the process for aggregate consumption, in contrast to representative-agent models. For example, in our model, the cyclicality of the bond price q_z depends on how $E[\pi_{hh|zz'}]$ behaves. In contrast, in a complete-markets environment, where we can identify a "representative agent," q_z is always pro-cyclical: denoting the total endowment as C_z in state z, we would obtain $q_z = \beta \sum_{z'=b,g} \phi_{zz'} (C_{z'}/C_z)^{-\sigma}$. In our model, on the other hand, q_z is pro-cyclical if and only if $E[\pi_{hh|gz'}] < E[\pi_{hh|bz'}]$, i.e., if the future endowment prospects of a rich consumer are better in bad aggregate times than in good aggregate times. More broadly, q's properties depend on the individual endowment process in a particular way, picking out a marginal rate of substitution of a specific individual at each point in time, and this individual is also not necessarily the same person over time. In the simple environment discussed in the present section, it is always the richest agent; in more complex environments (see Section 4), it may be an agent with an intermediate endowment level.

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[8] This is not the case if we extend the model to allow the values of ϵ_ℓ and ϵ_h to vary across aggregate states. In such a case, it is possible that high-endowment consumers are constrained when the aggregate state switches.

3.2.2. The term structure of interest rates

Any asset that depends only on the aggregate state can be priced by $Q_{zz'}$. Here we consider a long-term riskless bond, in order to examine the implications for the term structure of interest rates. We assume, in contrast to the approach discussed in Section 2.4, that the secondary markets for long-term bonds are perfect. Let $q_z^{(n)}$ be the price of n-period bond when the aggregate state is z. Recall that the price of a one-period bond is

$$q_z^{(1)} = Q_{zg} + Q_{zb}. \tag{13}$$

We can construct the price of an n-period bond by combining the Arrow-securities and lower-horizon bonds from

$$q_z^{(n)} = Q_{zg} q_g^{(n-1)} + Q_{zb} q_b^{(n-1)} \tag{14}$$

recursively, with the known expression for $q^{(1)}$ above as starting condition.

To analyze the term structure, and focusing on the relation between a one- and a two-period bond, note that the net, per-period returns of these bonds are

$$r_z^{(n)} \equiv \left(\frac{1}{q_z^{(n)}} \right)^{\frac{1}{n}} - 1,$$

for $n = 1, 2$. Therefore, we will have $r_z^{(1)} < r_z^{(2)}$ and an *upward-sloping yield curve* if and only if $q_z^{(2)} < (q_z^{(1)})^2$, that is, if and only if

$$\frac{q_z^{(2)}}{q_z^{(1)}} < q_z^{(1)},$$

or, using (14) for $n = 2$ on the left-hand side as well as (13) on the right-hand side, if and only if

$$\frac{Q_{zg} q_g^{(1)} + Q_{zb} q_b^{(1)}}{q_z^{(1)}} < Q_{zg} + Q_{zb}.$$

Applying this expression for $z = g$ and $z = b$ separately, the yield curve is upward-sloping in state $z = g$ if and only if

$$q_g^{(1)} > q_b^{(1)},$$

whereas it is upward-sloping in state $z = b$ if and only if

$$q_g^{(1)} < q_b^{(1)}.$$

Thus, if $q_g^{(1)} > q_b^{(1)}$, so that the short-term bond price is pro-cyclical (the short-term interest rate is counter-cyclical), the yield curve is upward-sloping in booms and downward-sloping in recessions. Alternatively, if $q_g^{(1)} < q_b^{(1)}$ (the short-term interest rate is pro-cyclical), the yield curve must be downward-sloping in booms and upward-sloping in recessions. Note that this result follows from simple manipulation of the prices of contingent claims, and thus it follows whether or not there are incomplete markets for idiosyncratic risks (as long as there are complete markets for aggregate risk).

However, when there is no idiosyncratic risk (or when this risk is fully insured), so that there is a representative agent, we also know that $q_g^{(1)} > q_b^{(1)}$ must hold given any mean-reverting process, so in a complete-markets model the yield curve must be upward-sloping in booms and downward-sloping in recessions. In this model, in contrast, we can obtain the reverse result, since
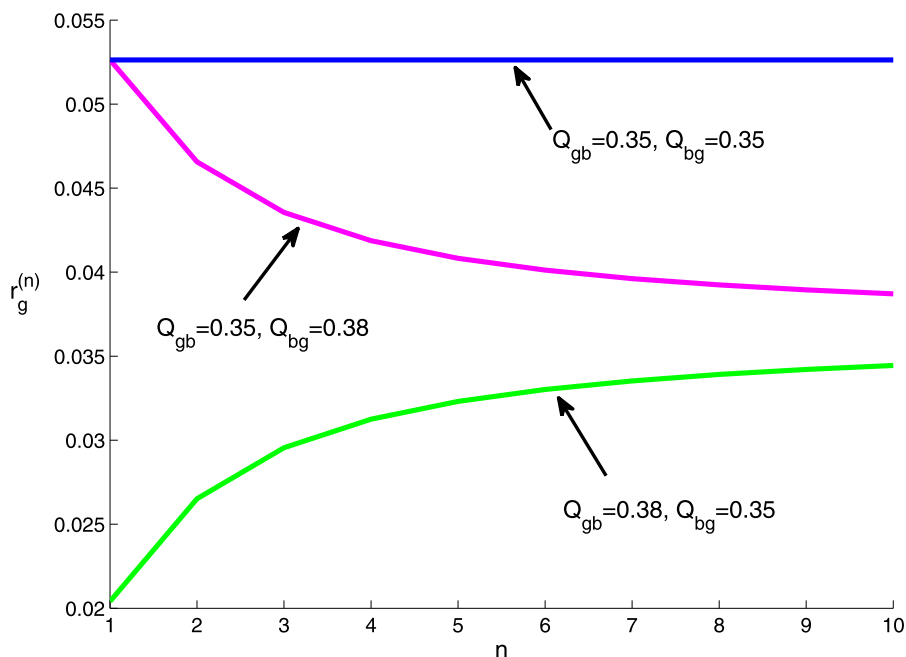
Fig. 3. Yield curves in booms: $r_g^{(n)}$.

$q_g^{(1)} < q_b^{(1)}$ is possible: as the last section showed, it is the expected growth in consumption of the rich agent that matters for bond pricing, and not expected aggregate consumption growth. Thus, if rich agents have higher expected consumption growth in booms than in recessions, the short-term interest rate will be pro-cyclical in this model, and the yield curve will slope upward in recessions and downward in booms.

It is also possible to examine the magnitude of the slope: it depends on the relative magnitudes of Q_{zg} and Q_{zb}. These, in turn, depend on any possible (a)symmetry in the cycle, in the case without idiosyncratic risks, and on details of the consumption process of the rich, in the case of incomplete markets studied here. Figs. 3 and 4 depict yield curves, assuming that $Q_{gg} = Q_{bb} = 0.6$, with horizons up to 10 periods. Three curves are drawn for different combinations of Q_{gb} and Q_{bg}, thus allowing both the cases of pro-cyclical and counter-cyclical short-term rates.

Fig. 5 depicts the yield curves for symmetric ($Q_{gg} = Q_{bb} = 0.6$, $Q_{gb} = 0.38$, and $Q_{bg} = 0.35$) and asymmetric ($Q_{gg} = 0.6$, $Q_{bb} = 0.3$, $Q_{gb} = 0.38$, and $Q_{bg} = 0.65$) business cycles. Here, we keep Q_{gg} and Q_{gb} the same and change Q_{bb} and Q_{bg} holding $Q_{bg} + Q_{bb}$ constant.

We also note that there are individual endowment processes for which the yield curve is non-monotonic. Fig. 6 illustrates with an example.

3.2.3. The equity risk premium

If there is an asset that provides Y_g when the next-period aggregate state is good and Y_b when the next-period aggregate state is bad, then its price is

$$P_z = \sum_{z'=g,b} Y_{z'} Q_{zz'} = \beta \sum_{z'=g,b} Y_{z'} \phi_{zz'} m_{zz'} = E[Y_{z'} m_{zz'}].$$

The ex-post (gross) return is $R_{zz'} \equiv Y_{z'}/P_z$. Therefore,

$$1 = E[R_{zz'} m_{zz'}] \tag{15}$$

holds. This implies that $m_{zz'}$ is the pricing kernel in this economy.

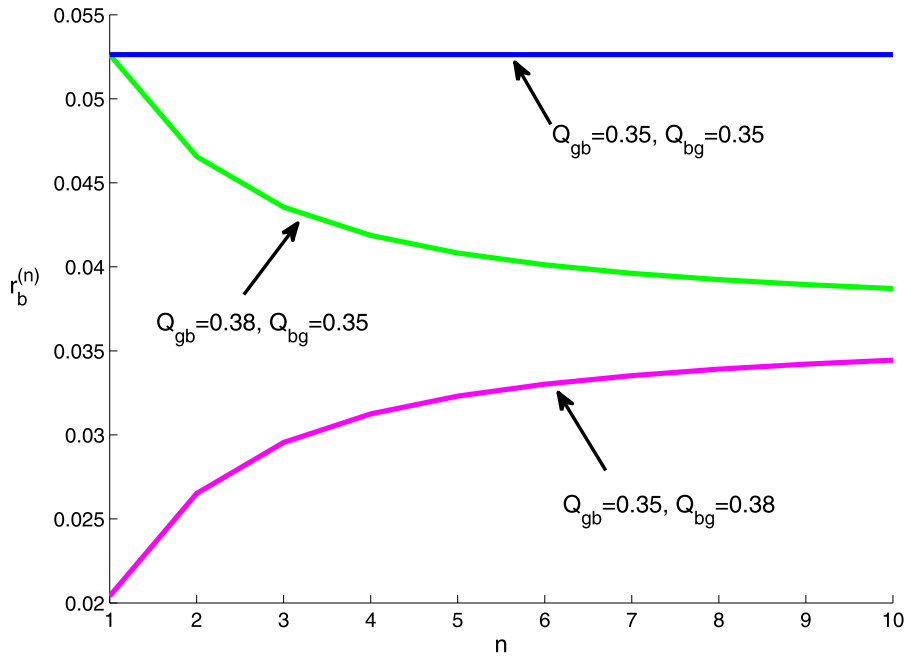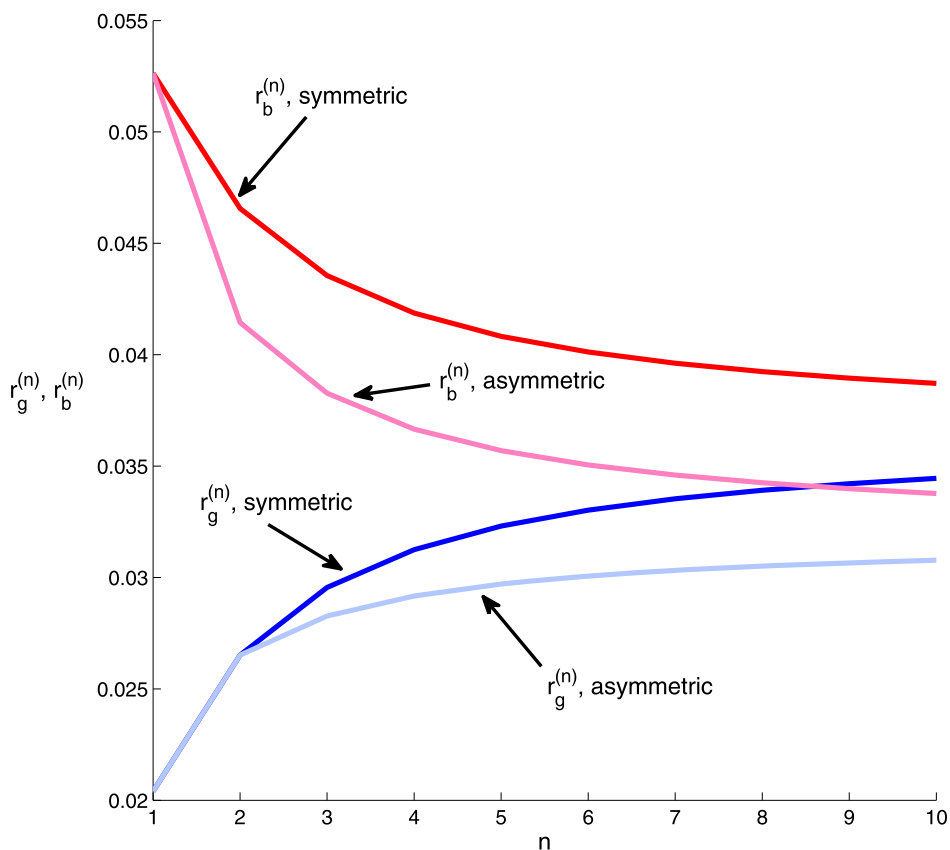Fig. 4. Yield curves in recessions: $r_b^{(n)}$.

Fig. 5. $r_g^{(n)}$ and $r_b^{(n)}$ for symmetric and asymmetric business cycles.

Define the risk premium as $R_{zz'}^e \equiv R_{zz'} - R_z^f$. In the following, we analyze the risk premium in this economy using a method similar to that used in Krusell and Smith [13], thus exploiting the two-state nature of the endowment process for simple analytics.
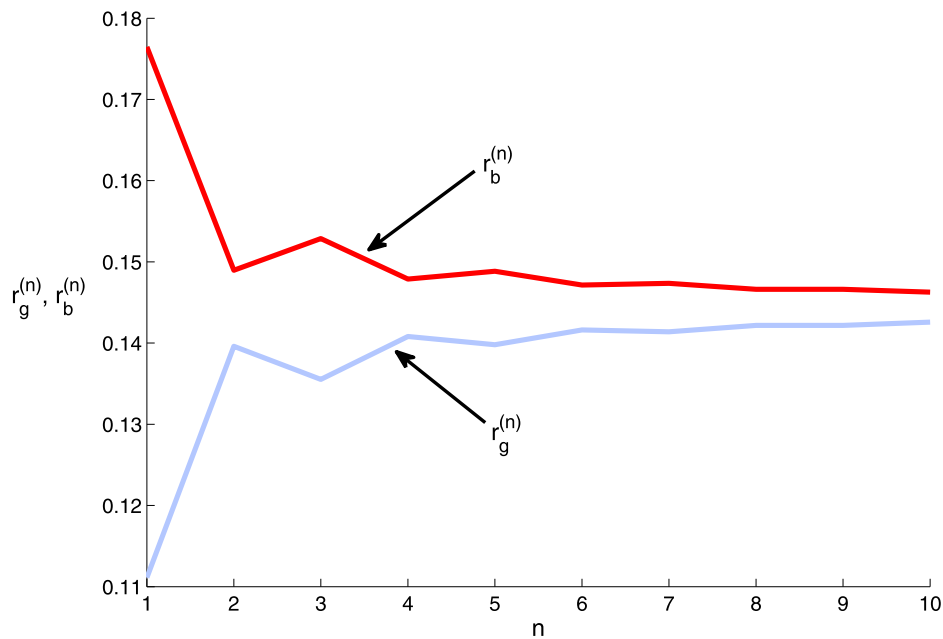
Fig. 6. $r_g^{(n)}$ and $r_b^{(n)}$ for $Q_{gg} = 0.1$, $Q_{bb} = 0.15$, $Q_{gb} = 0.8$, and $Q_{bg} = 0.7$.

From (11) and (15),

$$E[R_{zz'}^e m_{zz'}] = 0.$$

Since

$$E[R_{zz'}^e m_{zz'}] = E[R_{zz'}^e]E[m_{zz'}] + Cov(R_{zz'}^e, m_{zz'}),$$

the following holds:

$$E[R_{zz'}^e]E[m_{zz'}] = -Cov(R_{zz'}^e, m_{zz'}). \tag{16}$$

Now we are able to state and prove the following proposition.

Proposition 2. *Suppose that $Y_g > Y_b$. The expected value of risk premium, $E[R_{zz'}^e]$, is positive if and only if $\pi_{hh|zg} - \pi_{hh|zb} > 0$.*

Proof. See the Technical Appendix. □

Again, the persistence of the endowment process for the rich consumer, $\pi_{hh|zz'}$, plays a key role. Now, let us investigate how our model can be helpful in addressing the equity premium puzzle. Suppose that $\pi_{hh|zg} - \pi_{hh|zb} > 0$. Following Krusell and Smith [13], now we show that the Sharpe ratio for an asset with $Y_g > Y_b$ is exactly equal to the market price of risk. From (16),

$$E[R_{zz'}^e]E[m_{zz'}] = -\rho(R_{zz'}^e, m_{zz'})\sigma[R_{zz'}^e]\sigma[m_{zz'}].$$

Here, $\rho(A, B)$ denotes the correlation coefficient between random variables A and B and $\sigma[A]$ denotes the standard deviation of A (both conditional on z). Since

$$\rho(R_{zz'}^e, m_{zz'}) = -\rho(Y_{z'}, \pi_{hh|zz'}) = -1,$$

we find

$$\frac{E[R^e_{zz'}]}{\sigma[R^e_{zz'}]} = \frac{\sigma[m_{zz'}]}{E[m_{zz'}]}.$$

The left-hand side is the Sharpe ratio, and the right-hand side is the market price of risk.

From the definition of $m_{zz'}$, the market price of risk can be calculated as

$$\frac{\sigma[m_{zz'}]}{E[m_{zz'}]} = \frac{(\omega-1)(\pi_{hh|zg} - \pi_{hh|zb})\sqrt{\phi_{zg}(1-\phi_{zg})}}{\omega - (\omega-1)E[\pi_{hh|zz'}]}. \qquad (17)$$

Note that From (9),

$$m_{zb} - m_{zg} = \beta(\omega-1)(\pi_{hh|zg} - \pi_{hh|zb})$$

holds. Using this and (12), (17) can also be expressed as

$$\frac{\sigma[m_{zz'}]}{E[m_{zz'}]} = \frac{(m_{zb} - m_{zg})\sqrt{\phi_{zg}(1-\phi_{zg})}}{q_z}.$$

3.3. Can incomplete markets explain asset prices?

The main purpose of this section is to obtain a quantitative assessment of the prices that can be achieved with the incomplete-markets model when the borrowing constraints are maximally tight. In Section 3.3.2, we show that by judicious choice of the process for idiosyncratic risk, our model with tight (binding) borrowing constraints can, in fact, reproduce *any* pricing kernel and, hence, any set of asset prices. This result requires that the pricing kernel satisfy a restriction identical to the one in Constantinides and Duffie [6], who obtain an analogous result in a different environment (one in which idiosyncratic shocks to income are permanent). The required process for idiosyncratic risk, however, need not be quantitatively reasonable. We begin, therefore, in Section 3.3.1 with a quantitative model in the spirit of Mehra and Prescott [22] in which we calibrate the process for idiosyncratic risk to match observed data. For this quantitatively reasonable model, we show that our model with tight borrowing constraints can in fact come close to matching the first and second moments of returns on riskfree bonds and on equity in US data.

3.3.1. A quantitative investigation

In this section, we examine the extent to which our model with tight borrowing constraints can reproduce first- and second-moment features of US asset prices when the process for idiosyncratic shocks is calibrated in a quantitatively reasonable way. As in the Lucas asset pricing model, total consumption equals total endowments (there is neither production nor physical investment). Following Mehra and Prescott [22], we extend the model with aggregate uncertainty developed in Section 3.1 to allow the aggregate endowment to grow stochastically. In particular, letting y_t be the aggregate endowment in period t, we assume that $y_t = \gamma_t y_{t-1}$, where $\gamma_t \in [\gamma^g, \gamma^b]$ is the growth rate of the aggregate endowment. The growth rate γ_t depends on the aggregate state z_t: $\gamma_t = \gamma^{z_t}$.

In this section, we model individual idiosyncratic shocks as stochastically fluctuating shares of the aggregate endowment. In particular, in any period fraction χ of consumers receives a "high" multiple, ϵ_h, of the aggregate endowment and fraction $1 - \chi$ of consumers receives a "low" multiple, $\epsilon_\ell < \epsilon_h$, of the aggregate endowment, where we impose the adding-up restriction that $\chi\epsilon_h + (1-\chi)\epsilon_\ell = 1$. Note that the interpretation of ϵ in this section differs slightly from its

interpretation in Section 3.1; here, it is a multiple of the aggregate endowment, whereas there it is the individual endowment itself. We impose a set of constraints on the transition probabilities $\pi_{ss'|zz'}$ to ensure that the fraction of consumers receiving the shock ϵ_h in any period is indeed equal to χ: in particular, for all pairs (z, z'), we require $\chi \pi_{hh|zz'} + (1 - \chi)\pi_{\ell h|zz'} = \chi$.

Adapting the arguments from Section 3.2, it is straightforward to show that with tight borrowing constraints the high-endowment consumer prices the contingent claims, in which case the pricing kernel is:

$$m_{zz'} = \beta\left(\gamma^{z'}\right)^{-\sigma}\left[\pi_{hh|zz'} + (1 - \pi_{hh|zz'})\left(\frac{\epsilon_h}{\epsilon_\ell}\right)^\sigma\right].$$

In the absence of growth ($\gamma^g = \gamma^b = 1$), this expression is identical to the corresponding expression in Section 3.2 for the pricing kernel.[9] The pricing kernel can be used exactly as in Mehra and Prescott [22] to price any assets whose payoffs depend on realizations of the aggregate state.

We assume that a period corresponds to one year. For purposes of comparison, we calibrate the aggregate growth rate process exactly as in Mehra and Prescott [22]. Specifically, this process is chosen to match the mean, standard deviation, and first-order autocorrelation of the growth rate of US per capita consumption over the time period 1889–1978, yielding $\gamma^g = 1.054$, $\gamma^b = 0.982$, and $\phi_{gg} = \phi_{bb} = 0.43$.

Asset pricing in our economy involves only a subset of the population at any point in time. Thus, if one allows significant heterogeneity the data might only place very weak constraints on asset prices. To discipline our analysis, we interpret individual data as coming from a single process like the one above, to which all consumers are subjected. Thus, we set χ, the fraction of consumers receiving the high labor endowment, equal to one-half, implying that $\pi_{hh|zz'} = \pi_{\ell\ell|zz'}$ for all (z, z'): that is, the probability of remaining in the current idiosyncratic state is the same for both high and low idiosyncratic states. We choose the remaining parameters of the idiosyncratic shock process (the ratio of ϵ_h/ϵ_ℓ and the four transition probabilities $\pi_{hh|zz'}$) to match features of the dynamics of household-level labor income in the Panel Study of Income Dynamics (PSID), as documented by Storesletten, Telmer, and Yaron [24], subject to the restrictions imposed by requiring χ to be time-invariant and equal to one-half.[10] Specifically, Storesletten et al. [24] find that (log) labor income is highly persistent, with autocorrelation coefficient equal to 0.963, and that the conditional variance of (log) labor income varies countercyclically with the state of the aggregate economy: in "good" times it is 8.8% and in "bad" times it is 16.3%, almost twice as large. As we describe below, we choose the labor-income process in our model to replicate these numbers.[11]

We assume that the probability of remaining in the high idiosyncratic state depends on tomorrow's aggregate state, but not on today's, leaving three parameters—ϵ_h/ϵ_ℓ, $\pi_{hh|zg}$, and $\pi_{hh|zb}$—to match the three observed facts about household labor income. Over the long run, the economy is in the good aggregate state half of the time and in the bad aggregate state half the time, so we set $0.5\pi_{hh|zg} + 0.5\pi_{hh|zb}$ equal to 0.963, that is, averaging across aggregate states the first-order

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[9] See, in particular, Eq. (9).

[10] We ignore asset income when calibrating the idiosyncratic shock process, but this will not matter if labor's share of income is relatively stable over the business cycle, as it is in US aggregate data.

[11] Heaton and Lucas [10] find that aggregate shocks are not very important in explaining the conditional mean of household-level labor income in the PSID data, so we ignore this dependence in our calibration, though we do allow the conditional *variance* of idiosyncratic shocks to depend on the aggregate state, for which Storesletten et al. [24] do find compelling evidence.

Table 1

	Data	Baseline model	Complete markets	Homoskedastic
Mean risk-free rate	0.80%	0.80%	0.80%	0.80%
Mean equity premium	6.18%	2.93%	0.72%	0.72%
Std. dev. risk-free rate	5.67%	5.25%	2.08%	2.08%
Std. dev. equity premium	16.67%	7.94%	4.95%	4.95%
Sharpe ratio	0.37	0.37	0.15	0.15
β		0.59	1.06	0.61
σ		4.20	4.20	4.20

autocorrelation of the idiosyncratic shock is 0.963. In addition, we require that the coefficient of variation of tomorrow's idiosyncratic shock, conditional on today's idiosyncratic state and on tomorrow's aggregate state, is equal to 8.8% if tomorrow's aggregate state is good equal to 16.3% if tomorrow's aggregate state is bad.[12] The calibrated parameters, then, are $\epsilon_h/\epsilon_\ell = 2.06$, $\pi_{hh|zg} = 0.984$, and $\pi_{hh|zb} = 0.942$.

We now use the model to price assets, in particular, a riskless bond in zero net supply that pays one unit of consumption in all aggregate states in the next period and "equity," which we define to be a claim to the entire future stream of aggregate endowments (itself in zero net supply). The first column (labeled "Data") of Table 1 reports unconditional moments of asset prices in US data, as calculated in Mehra and Prescott [22] using annual data for 1889–1978.[13] The second column (labeled "Baseline model") reports the asset prices for our calibrated economy with tight borrowing constraints when the discount factor $\beta = 0.59$ and the coefficient of relative risk aversion $\sigma = 4.2$. We choose these two preference parameters to match the average risk-free rate and the Sharpe ratio (i.e., the expected equity risk premium divided by its standard deviation) in the data. A success of our quantitative model is that it can reproduce the observed Sharpe ratio with a relatively low coefficient of relative risk aversion, though the discount factor is smaller than conventional values (see the next paragraph for additional discussion of the low discount factor).[14]

The average equity premium (the average difference between the return on equity and the risk-free rate) in our quantitative model is roughly 3%, about half of the observed value but over two full percentage points larger than in the complete-markets model of Mehra and Prescott [22], as reported in the third column (labeled "Complete markets") of Table 1. In this column, for purposes of comparison, we again choose the discount factor to match the risk-free rate, holding fixed the coefficient of relative risk aversion at its value in the second column. The required discount factor is larger than one, reflecting the well-known "risk-free rate puzzle" that the risk-free rate in observed data appears to be "too low."[15] Another success of our quantitative

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[12] We target the coefficient of variation because the idiosyncratic shock is in levels, not logs, in our setup; the coefficient of variation is approximately equal to the standard deviation of the log of the idiosyncratic shock. It turns out that this (conditional) coefficient of variation depends on today's idiosyncratic state, so we compute an average across the two idiosyncratic states, which occur equally frequently over the long run.

[13] In Mehra and Prescott [22], nominal returns are adjusted for inflation using ex post realizations of inflation rather than expected inflation; this adjustment accounts for part of the reported volatility of returns.

[14] Despite these successes, it is important to note that although the process for idiosyncratic labor income risk is calibrated to observed data, the resulting process for consumption displays too much variability: income equals consumption in our model, but in observed data, by contrast, consumption is typically less variable than income, both across time for a single household and in a cross-section of households at a point in time.

[15] The main effect of changing the discount factor is to change the risk-free rate; its effects on risk prices are small and would be zero in a continuous-time version of our model.

model, then, is that it can "solve" the risk-free rate puzzle with a discount factor smaller than one. The required discount factor is, in fact, smaller than values typically used in calibrated dynamic general-equilibrium macroeconomic models, but is in the same range as the "short-run" discount rates often used in quantitative consumption-savings models with hyperbolic (or quasi-geometric) discounting. Finally, relative to the Mehra–Prescott model, our quantitative model with reasonably calibrated idiosyncratic risk generates substantial increases in the volatility of both the risk-free rate and the equity premium—though the volatility of the latter remains much smaller (by a factor of two) than in the data—and in the Sharpe ratio.

The fourth and last column (labeled "Homoskedastic") of Table 1 reports asset prices when $\pi_{hh|zg} = \pi_{hh|zb} = 0.963$ (in which case the conditional variance of income shocks does not vary with the aggregate state of the economy), where we again adjust the discount factor (by a small amount as it turns out) to keep the risk-free rate equal to its value in the observed data. Here we see that risk prices are exactly the same as they are when markets for idiosyncratic risk are complete. The intuition for this result goes back to Proposition 2, where we showed in a slightly different setting—one in which the idiosyncratic shock does not depend on the aggregate state—that the risk premium is zero when the probability of remaining in the high idiosyncratic state does not depend on tomorrow's aggregate state. Here, instead, the idiosyncratic risk is defined to be a share of a fluctuating aggregate endowment, so that there is a risk premium even when markets are complete. But, here as in Proposition 2, the introduction of idiosyncratic risk does not increase the risk premium unless the transition probabilities for idiosyncratic risk depend on the aggregate state in such a way that bad idiosyncratic shocks are more likely to occur when the economy transits to a bad rather than to a good aggregate state.

In terms of relations to the quantitative literature, Lettau [17] computes "bounds" on Sharpe ratios with a similar approach to the one adopted here. He uses individual income data, interpreting it as consumption, and shows how one can obtain higher Sharpe ratios if one introduces a correlation between idiosyncratic risk and the aggregate shock. He does not obtain substantial differences compared to the representative-agent results. An important difference between our approach and his is that our theory leads us to look at the agent who, at any date and state, "likes the asset the most" in pricing risk, as opposed to just taking the individual consumption process and using it directly to obtain a kernel. Thus, in our equilibrium, different agents price risk at different moments in time.

3.3.2. Connections to some known results in the literature

Recently, Krueger and Lustig [12] demonstrated that, in certain contexts, the equity premium in an incomplete-markets model is identical to that in the complete-markets model. This finding seems to be at odds with our earlier results—that market incompleteness can have a large effect on the risk premium. However, the main result in Krueger and Lustig, whose analysis is more general than ours in the sense that they do not just study autarky equilibria, applies in our economy as well, provided we make assumptions in line with the assumptions they make, and below, we demonstrate their irrelevance result in the context of our model. We have to somewhat modify our model to allow the Krueger–Lustig special case. We derive closed-form solutions for asset prices in our modified model, and we also slightly relax one of their assumptions. The assumptions needed for the irrelevance result, even in their slightly relaxed form, are, however, rather restrictive.

A different kind of result is obtained in Constantinides and Duffie [6]. They show that given an aggregate consumption process and given specific preference parameters, it is (under a mild condition) possible to find an individual endowment process that is consistent with the aggre-

gate endowment process and that matches any asset-pricing facts. Thus, in contrast to Krueger and Lustig's argument that incomplete-markets settings are incapable (under some conditions) of explaining asset prices, Constantinides and Duffie instead point to the model's capability of matching *any* asset prices. We comment on Constantinides and Duffie's results by providing a result to the same effect: the incomplete-markets model is quite capable *in principle* of reproducing many asset-price characteristics.

Krueger and Lustig [12]:   In order to accommodate Krueger and Lustig's assumptions, we will modify our model first by allowing for the values of ϵ_h and ϵ_ℓ to vary across the aggregate states. With a slight abuse of notation, let us denote the value of ϵ_s when the aggregate state is z by ϵ_{sz}. The first assumption by Krueger and Lustig is that the individual endowment varies proportionally with the aggregate states:

$$\frac{\epsilon_{hg}}{\epsilon_{hb}} = \frac{\epsilon_{\ell g}}{\epsilon_{\ell b}}. \tag{18}$$

We let this ratio be denoted $\xi > 1$. This implies that $\epsilon_{hg}/\epsilon_{\ell g} = \epsilon_{hb}/\epsilon_{\ell b}$. To be consistent with the previous notation, we let $(\epsilon_{hg}/\epsilon_{\ell g})^\sigma = (\epsilon_{hb}/\epsilon_{\ell b})^\sigma = \omega$.

The second important assumption employed by Krueger and Lustig is that $\pi_{ss'|zz'}$ is independent of z and z' (notice, already at this point, that the key requirement in our quantitative section, i.e., a large positive value of $\pi_{hh|zg} - \pi_{hh|zb}$, is ruled out). We denote this aggregate-state-independent probability $\pi_{ss'}$. This assumption implies that there exists a stationary distribution over individual states. The first assumption, in turn, then implies that z only affects the aggregate endowment: all endowments are scaled up and down as the economy experiences a boom and a recession. We normalize the aggregate endowment in $z = b$ to 1, so that the aggregate endowment when $z = g$ is ξ. Below we consider such a stationary situation.

The consumer's first-order conditions for this economy implies

$$\frac{Q_{zz'}}{\beta\phi_{zz'}} - \frac{\lambda_{sz}^{z'}}{\beta\phi_{zz'}\epsilon_{sz}^{-\sigma}} = \pi_{sh}\left(\frac{\epsilon_{sz}}{\epsilon_{hz'}}\right)^\sigma + (1 - \pi_{s\ell})\left(\frac{\epsilon_{sz}}{\epsilon_{\ell z'}}\right)^\sigma.$$

As for our baseline model, it is straightforward to show that the borrowing constraint is binding for $s = \ell$ consumers and that $Q_{zz'}$ is determined by the first-order condition of the $s = h$ consumers.

With our assumptions on the endowment, the Arrow-security prices therefore become

$$Q_{gg} = \beta\phi_{gg}\big[\pi_{hh} + (1 - \pi_{hh})\omega\big], \qquad Q_{bb} = \beta\phi_{bb}\big[\pi_{hh} + (1 - \pi_{hh})\omega\big],$$
$$Q_{gb} = \beta\phi_{gb}\xi^\sigma\big[\pi_{hh} + (1 - \pi_{hh})\omega\big],$$

and

$$Q_{bg} = \beta\phi_{bg}\xi^{-\sigma}\big[\pi_{hh} + (1 - \pi_{hh})\omega\big].$$

From here on, define $\theta \equiv [\pi_{hh} + (1 - \pi_{hh})\omega]$.

Suppose that the current state is $z = g$. The (gross) bond return is

$$R_g^f = \frac{1}{\sum_{z'=g,b} Q_{gz'}} = \frac{1}{\beta\theta(\phi_{gg} + \phi_{gb}\xi^\sigma)}$$

and the (gross) expected return of a risky asset which provides Y_g when $z' = g$ and Y_b when $z' = b$ is

$$E\big[R^e_{gz'}\big] = \sum_{z'=g,b} \phi_{gz'} \frac{Y_{z'}}{P_g} = \frac{\sum_{z'=g,b} \phi_{gz'} Y_{z'}}{\sum_{z'=g,b} Q_{gz'} Y_{z'}} = \frac{\sum_{z'=g,b} \phi_{gz'} Y_{z'}}{\beta\theta(\phi_{gg}Y_g + \phi_{gb}\xi^\sigma Y_b)},$$

where P_z is the price of this asset when the aggregate state is z. From these, the multiplicative risk premium (equity premium) can be calculated as

$$\frac{E[R^e_{gz'}]}{R^f_g} = \frac{(\phi_{gg} + \phi_{gb}\xi^\sigma)\sum_{z'=g,b} \phi_{gz'} Y_{z'}}{(\phi_{gg}Y_g + \phi_{gb}\xi^\sigma Y_b)}.$$

An important fact here is that θ cancels out. All the parameters that appear on the right-hand side are parameters determining aggregates.

When markets are complete in this economy, the representative agent consumes 1 when $z = b$ and ξ when $z = g$. The Lucas asset pricing formula implies that

$$Q_{gg} = \beta\phi_{gg}, \qquad Q_{bb} = \beta\phi_{bb}, \qquad Q_{gb} = \beta\phi_{gb}\xi^\sigma,$$

and

$$Q_{bg} = \beta\phi_{bb}\xi^{-\sigma}.$$

Note that these differ from the incomplete-markets security prices only by the constant factor θ. Considering $z = g$, the gross bond return is:

$$R^f_g = \frac{1}{\sum_{z'=g,b} Q_{gz'}} = \frac{1}{\beta(\phi_{gg} + \phi_{gb}\xi^\sigma)}.$$

The gross return on the risky asset is

$$E\big[R^e_{gz'}\big] = \sum_{z'=g,b} \phi_{gz'} \frac{Y_{z'}}{P_g} = \frac{\sum_{z'=g,b} \phi_{gz'} Y_{z'}}{\sum_{z'=g,b} Q_{gz'} Y_{z'}} = \frac{\sum_{z'=g,b} \phi_{gz'} Y_{z'}}{\beta(\phi_{gg}Y_g + \phi_{gb}\xi^\sigma Y_b)}.$$

Therefore, the multiplicative risk premium is

$$\frac{E[R^e_{gz'}]}{R^f_g} = \frac{(\phi_{gg} + \phi_{gb}\xi^\sigma)\sum_{z'=g,b} \phi_{gz'} Y_{z'}}{(\phi_{gg}Y_g + \phi_{gb}\xi^\sigma Y_b)},$$

which is the same as in the incomplete-markets case. Therefore, we established that the risk premium is not affected by the market incompleteness, although the level of the risk-free rate is affected. The case of $z = b$ can be established similarly.

What is important in this derivation is that θ cancels out when we calculate the risk premium. For this to occur, the value $\pi_{hh} + (1 - \pi_{hh})\omega$ has to be common across different values of z'. Incomplete markets matter for the risk premium in the Krueger–Lustig version of the model when, for example, $\pi_{hh|zg}$ is different from $\pi_{hh|zb}$. This observation is closely related to our Proposition 2: $\pi_{hh|zz'}$ is a key parameter for asset pricing in the incomplete-markets model.[16]

In the following, we relax the above assumptions. First, we relax the condition (18). When the $\pi_{ss'}$'s are independent of z and z', we can derive a necessary and sufficient condition for the multiplicative equity premium to be the same across the complete-markets model and our incomplete-markets model. To be precise, first note that when the $\pi_{ss'}$'s are independent of z,

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[16] In Proposition 2, the risk premium is zero when $\pi_{hh|zg} = \pi_{hh|zb}$. Here, the risk premium can still be positive when $\pi_{hh|zg} = \pi_{hh|zb}$, since the ϵ_s's are different across different z''s.

there is a stationary distribution over the individual states. In fact, assuming that the total population is one, the fraction of the consumers with state s (denoted χ_s) is completely pinned down by the $\pi_{ss'}$'s. For example, χ_h is

$$\chi_h = \frac{\pi_{\ell h}}{1 - \pi_{hh} + \pi_{\ell h}}. \tag{19}$$

For a stationary state, the following result is attained.

Proposition 3. *Suppose that $\pi_{ss'|zz'}$ is independent of z and z'. Then, the multiplicative risk premium for any asset whose return depends only on the aggregate state in the stationary equilibrium is the same across the complete-markets model and our incomplete-markets model if and only if the condition*

$$\frac{\pi_{hh}\epsilon_{hg}^{-\sigma} + (1 - \pi_{hh})\epsilon_{\ell g}^{-\sigma}}{\pi_{hh}\epsilon_{hb}^{-\sigma} + (1 - \pi_{hh})\epsilon_{\ell b}^{-\sigma}} = \frac{(\pi_{\ell h}\epsilon_{hg} + (1 - \pi_{hh})\epsilon_{\ell g})^{-\sigma}}{(\pi_{\ell h}\epsilon_{hb} + (1 - \pi_{hh})\epsilon_{\ell b})^{-\sigma}} \tag{20}$$

holds.

Proof. See the Technical Appendix. $\square$

Clearly, when the condition (18) is satisfied, (20) is automatically satisfied. The condition (20) highlights the possibility that the irrelevance result would hold outside Krueger and Lustig's assumption: even when (18) is not satisfied, if the $\pi_{ss'}$'s satisfy the condition (20), the irrelevance result still holds.

Second, we relax the independence assumptions on the transition probabilities. When the $\pi_{ss'}$'s are not independent of z and z', in general the distribution of the idiosyncratic states moves around over time. In this case, an irrelevance result is difficult to come by, since aggregate consumption is affected not only by the $\pi_{ss'}$'s and ϵ_s's but also by the fractions χ_s, which move over time (note that the prices in the incomplete-markets model are not affected by the χ_s's). For χ_s to be constant (or just a function of z), we need some restrictions on the transition probabilities $\pi_{ss'|zz'}$'s. In particular, letting χ_{sz} denote the fraction of type-s consumers in state z,

$$\chi_{hz}\pi_{hh|zz'} + (1 - \chi_{hz})\pi_{\ell h|zz'} = \chi_{hz'}$$

has to be satisfied for any z and z'.[17] When this condition is satisfied, a necessary and sufficient condition for the irrelevance becomes (by the same logic as above)

$$\frac{\pi_{hh|zg}\epsilon_{hg}^{-\sigma} + (1 - \pi_{hh|zg})\epsilon_{\ell g}^{-\sigma}}{\pi_{hh|zb}\epsilon_{hb}^{-\sigma} + (1 - \pi_{hh|zb})\epsilon_{\ell b}^{-\sigma}} = \frac{(\chi_{hg}\epsilon_{hg} + (1 - \chi_{hg})\epsilon_{\ell g})^{-\sigma}}{(\chi_{hb}\epsilon_{hb} + (1 - \chi_{hb})\epsilon_{\ell b})^{-\sigma}}$$

for $z = g, b$. A notable feature of this condition is that it is possible to break the irrelevance result even when $\epsilon_{sg} = \epsilon_{sb}$ (when $\epsilon_{sg} = \epsilon_{sb}$, (20) is automatically satisfied).

Constantinides and Duffie [6]: Constantinides and Duffie [6] proposed a model where, similarly to ours, the equilibrium outcome is autarky. They show that (with given preference parameters and aggregate endowment process), under some conditions, one can find an individual

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[17] This type of condition is used by Krusell and Smith [14] to make the unemployment rate a function only of the current aggregate state.

endowment process for a given process of strictly positive pricing kernels. Intuitively, autarky in the Constantinides and Duffie model obtains since the individual shocks are permanent: when individual shocks do not mean-revert, they cannot be insured against with assets that are contingent only on aggregate states. Our framework yields the following result, which is similar in its conclusion, though very different in its construction, since it relies on binding borrowing constraints and is valid for shocks that mean-revert (in fact recurring permanent shocks cannot be accommodated with the simple finite-state Markov chain we consider).

Proposition 4. *In the model with aggregate uncertainty developed in Section 3.1, for any preference parameters and any given set of values for $m_{zz'} \geqslant \beta$, all z and z', we can find a set of parameters for the individual income process given by the parameters $\pi_{ss'|zz'}$, all s, s', z and z', ϵ_h, and ϵ_ℓ, that generates these $m_{zz'}$ values as an equilibrium outcome.*

Proof. See the Technical Appendix. □

This result is different from Constantinides and Duffie's in some respects. Since we do not allow the individual endowment level to depend on the aggregate endowment level, our model cannot condition $m_{zz'}$ on the aggregate endowment level. That is, the aggregate endowment process in Proposition 4 may not behave in a realistic manner. This feature can be alleviated if we allow ϵ_s to depend on the aggregate endowment level. Such an assumption would (in principle) make the analysis more complex, since it would no longer be the case that the high-endowment consumer always prices the asset. Another difference is that our lower bound for $m_{zz'}$ is β, while their lower bound for the pricing kernel is $\beta(C_{t+1}/C_t)^{-\sigma}$ (in our notation), where C_t is the aggregate endowment at time t.

In the following proposition, we allow ϵ_h and ϵ_ℓ to vary, and we find a result that is closer to Constantinides and Duffie's. Here, we slightly change the notations to express the dependence on the time and the history clearer.

Proposition 5. *Denote the history of the aggregate state z_t as $z^t \equiv \{z_0, z_1, \ldots, z_t\}$. Let the preference parameters and the aggregate consumption process, $C_t(z^t)$, be given. Suppose that we are given the values of the pricing kernel at each point in history, $m_{t+1}(z^{t+1})$. Assume that $m_{t+1}(z^{t+1})$ satisfies $m_{t+1}(z^{t+1}) \geqslant \beta(C_{t+1}(z^{t+1})/C_t(z^t))^{-\sigma}$ for all z^{t+1} and $z^t \subset z^{t+1}$, and $\frac{m_{t+1}(z^{t+1})}{\beta(C_{t+1}(z^{t+1})/C_t(z^t))^{-\sigma}}$ is bounded above. Then, for these given values of $m_{t+1}(z^{t+1})$, we can find an individual income process given by the idiosyncratic transition probabilities $\pi_{t+1}(s_{t+1}|s_t, z^{t+1})$ (the probability that the endowment $s_{t+1} \in \{h, \ell\}$ realizes at time $t+1$ given s_t and z^{t+1}), high endowment value $\epsilon_{ht}(z^t)$, low endowment value $\epsilon_{\ell t}(z^t)$, and an initial population of the consumers across the endowment values that generates these $m_{t+1}(z^{t+1})$ values as an equilibrium outcome.*

Proof. See the Technical Appendix. □

This means that, in terms of generality, our model with variable ϵ_ss can perform as well as Constantinides and Duffie's [6] model in terms of matching asset-pricing facts, when we are given the preference parameters, the aggregate consumption data, and the asset-price data.

Alvarez and Jermann [1,2]:   Alvarez and Jermann analyze a setting with complete markets against idiosyncratic risk but lack of commitment, endogenously generating solvency constraints

reminiscent of our borrowing constraints. They show that, under certain conditions, an autarky outcome is possible where asset prices are determined by a subset of consumers—those who value the asset the most. In their setting with two types, the resulting asset prices parallel those we derive, but with more types their prices can be shown to be quite different and, in fact, for identical endowment processes to those we study here, deliver even higher bond prices than we obtain. Intuitively, the difference derives from their borrowing being specific with regard to individual states—which is possible since there are markets against idiosyncratic risks—whereas our borrowing is unconditional.

4. More than two individual endowment states

In this section, we extend the model to incorporate the possibility of more than two possible values of the individual endowment. Most of the results in the two-state case extend to this new environment. The main difference is that now asset prices are not necessarily determined by the richest (highest-endowment) consumers.

4.1. The economy without aggregate uncertainty

The next proposition generalizes Proposition 1 to the case where ϵ takes on N different values $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$, with $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_N$, and where the utility function is more general.

Proposition 6. *When ϵ takes on N different values $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$ and the consumer has an increasing and strictly concave utility function $u(c)$, the equilibrium bond price can be characterized by*:

$$q \geqslant q^* \equiv \max_{i=2,\ldots,N} M_i, \tag{21}$$

where

$$M_i \equiv \beta \sum_{j=1}^{N} \pi_{ij} \frac{u'(\epsilon_j)}{u'(\epsilon_i)}.$$

The proof is omitted, since its steps are very similar to Proposition 1. Note that the smallest-endowment consumers ($\epsilon = \epsilon_1$) are excluded from the max in (21), since, as in the previous case, they are always strictly borrowing-constrained. In this case, it may not be the case that the highest-endowment consumers' marginal rates of substitution determine q^*. If M_j is larger with a $j \neq N$, a consumer with the endowment level j is a "pricer" of the bond.

4.1.1. Comparative statics

As is shown in the previous analysis, when the endowment takes on only two values, q^* is decreasing in π_{hh}. This result can be generalized to the following proposition.

Proposition 7. *Consider π_{ij} as well as $\hat{\pi}_{ij}$, where, for each $i = 2, \ldots, N$, $\hat{\pi}_{ij}$ first-order stochastically dominates π_{ij}. Then q^* is smaller under $\hat{\pi}_{ij}$ than under π_{ij}.*

Proof. See the Technical Appendix. □

Clearly, the comparative statics with respect to β are similar to those of the two-value endowment case. The comparative statics with respect to σ, when $u(c) = c^{1-\sigma}/(1-\sigma)$, are not as straightforward (it is indeed straightforward if $\arg\max_{i=1,\dots,N} M_i = N$). However, it turns out nevertheless that q^* is always increasing in σ.

Proposition 8. q^* *is increasing in* σ.

Proof. See the Technical Appendix. $\square$

The intuition is that, if a consumer with endowment i prices the bond, there are high weights (either because π is large or (ϵ_i/ϵ_j) is large) on the states where this consumer experiences large consumption drops, and an increase in σ further increases the weights on these events.

4.1.2. Who prices the bond?

As is stated above, it is not necessarily the case that the highest-endowment consumer determines the bond price. A sufficient condition for this to be guaranteed can, however, be stated.

Proposition 9. *Suppose that for all* $i = 2, \dots, N - 1$ *and* $n = 1, \dots, i - 1$,

$$\sum_{j=1}^{n} \pi_{Nj} \geqslant \sum_{j=1}^{n} \pi_{ij} \tag{22}$$

holds. Then, $N \in \arg\max_{i=2,\dots,N} M_i$.

Proof. See the Technical Appendix. $\square$

The intuition behind (22) is that the highest-endowment consumers face the "largest risk" of an endowment loss in the near future, thus giving them the strongest incentive to save for precautionary reasons. A special case is where (22) holds for all $n = 1, \dots, N$. That is, this would mean that in terms of the distribution over the next-period state s', $\{\pi_{is'}\}$ first-order stochastically dominates $\{\pi_{Ns'}\}$.

Another special case is a type of "downward-rigid" wages. Let the state $s = 1$ be the unemployment state and the other states as the employment states with different level of wages ϵ_s (ϵ_1 can be interpreted as unemployment income, home production, or the utility from leisure). Suppose that from any employed state $s = 2, \dots, N$, there is an equal probability of becoming unemployed: $\pi_{21} = \pi_{31} = \dots = \pi_{N1}$. Also suppose that the wage is "downward-rigid" except at the very top: $\pi_{ij} = 0$ for $i = 2, \dots, N - 1$ and $j = 2, \dots, i - 1$. It can easily be checked that this Markov transition matrix satisfies (22). Thus, in such an economy too the highest-endowment consumer's marginal rates of substitution determine the bond price.

When (22) is not satisfied, the bond price may be determined by the consumers other than the highest-endowment consumers. Consider the following example. Let $N = 3$ and the utility functions be given by $u(c) = c^{1-\sigma}/(1-\sigma)$. Now M_2 and M_3 are, by definition, given by

$$M_2 = \beta \left[\pi_{21} \left(\frac{\epsilon_2}{\epsilon_1} \right)^{\sigma} + \pi_{22} + \pi_{23} \left(\frac{\epsilon_2}{\epsilon_3} \right)^{\sigma} \right]$$

and

$$M_3 = \beta \left[\pi_{31} \left(\frac{\epsilon_3}{\epsilon_1} \right)^{\sigma} + \pi_{32} \left(\frac{\epsilon_3}{\epsilon_2} \right)^{\sigma} + \pi_{33} \right].$$

Here, there would be cases where $M_2 > M_3$. For example, when π_{33} is close to 1 and π_{21} is close to 1, M_3 is close to 1 and M_2 is close to $(\epsilon_2/\epsilon_1)^\sigma > 1$; thus, the consumer with the middle endowment level would determine the bond price, since this consumer faces a "larger risk" than does the rich consumer. Moreover, M_2 is increasing in ϵ_2 and decreasing in ϵ_3, while M_3 is decreasing in ϵ_2 and increasing in ϵ_3, and thus the comparative statics with respect to ϵ_2 and ϵ_3 would depend crucially on whether M_2 is larger or smaller than M_3.

4.1.3. Examples

This section illustrates two calibrated examples.

Example 1. Consider a wage process given by Castañeda, Díaz-Giménez, and Ríos-Rull [5]. To match the earnings and wealth distribution in the United States, they introduce four employment states with different productivity levels, expressed as the effective unit of labor. They also introduce retirement and life cycle effect, but we ignore them here. Their productivity levels are not exactly the same as the income because of the endogenous labor supply and the existence of taxes. We abstract from these factors as well, and we use their calibrated productivity level as the endowment level in our framework. There are four levels of the productivity: $\epsilon_1 = 1.0$, $\epsilon_2 = 3.15$, $\epsilon_3 = 9.78$, and $\epsilon_4 = 1061.00$. The transition probabilities are (after adjusting for the retirement probability):

$$
\begin{pmatrix}
\pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\
\pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\
\pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \\
\pi_{41} & \pi_{42} & \pi_{43} & \pi_{44}
\end{pmatrix}
=
\begin{pmatrix}
0.98429 & 0.00117 & 0.00399 & 0.00006 \\
0.03140 & 0.96482 & 0.00378 & 0.00000 \\
0.01534 & 0.00440 & 0.98006 & 0.00020 \\
0.10903 & 0.00501 & 0.06249 & 0.82346
\end{pmatrix}.
$$

We use $\sigma = 1.5$ and $\beta = 0.924$, following Castañeda, Díaz-Giménez, and Ríos-Rull [5]. Then, when there is no borrowing/saving, M_i can be calculated by

$$
M_i = \beta \left(\pi_{i1} \left(\frac{\epsilon_1}{\epsilon_j} \right)^{-\sigma} + \pi_{i2} \left(\frac{\epsilon_2}{\epsilon_j} \right)^{-\sigma} + \pi_{i3} \left(\frac{\epsilon_3}{\epsilon_j} \right)^{-\sigma} + \pi_{i4} \left(\frac{\epsilon_4}{\epsilon_j} \right)^{-\sigma} \right). \tag{23}
$$

From the above calibration, $(M_1, M_2, M_3, M_4) = (0.91, 1.05, 1.36, 3576.38)$. Therefore, the highest-endowment consumer prices the bond in this example.

Example 2. Floden and Lindé [8] estimate the following AR(1) wage process (after controlling for the observable characteristics and measurement errors) from

$$
\log(\epsilon_{t+1}) = a + \rho \log(\epsilon_t) + \eta_{t+1}, \quad \eta \sim (0, \sigma^2).
$$

Floden and Lindé estimate that $\rho = 0.9136$ and $\sigma = 0.2064$. We use this process as the wage process and approximate this by 4-state Markov process using Tauchen's [26] method. We set the maximum and minimum of $\log(\epsilon)$ as plus and minus four (unconditional) standard deviation of $\log(\epsilon)$. This leads to $\epsilon_1 = 1.0$, $\epsilon_2 = 3.87$, $\epsilon_3 = 14.99$, and $\epsilon_4 = 58.02$ and

$$
\begin{pmatrix}
\pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\
\pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\
\pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \\
\pi_{41} & \pi_{42} & \pi_{43} & \pi_{44}
\end{pmatrix}
=
\begin{pmatrix}
0.99243 & 0.00757 & 0.00000 & 0.00000 \\
0.00018 & 0.99844 & 0.00137 & 0.00000 \\
0.00000 & 0.00137 & 0.99844 & 0.00018 \\
0.00000 & 0.00000 & 0.00757 & 0.99243
\end{pmatrix}.
$$

We maintain the assumption of $\sigma = 1.5$ and $\beta = 0.924$. From (23), $(M_1, M_2, M_3, M_4) = (0.92, 0.92, 0.93, 0.97)$. Again, the highest-endowment consumer prices the bond. Now the bond price is at a reasonable value. This is because the highest-endowment consumer's probability of "falling" is very small. The equilibrium price changes if, for example, we use finer grid points. If we approximate by six-point process with $\epsilon_1 = 1.0$, $\epsilon_2 = 2.25$, $\epsilon_3 = 5.08$, $\epsilon_4 = 11.43$, $\epsilon_5 = 25.76$, and $\epsilon_6 = 58.02$, the resulting M's are $(M_1, M_2, M_3, M_4, M_5, M_6) = (0.83, 0.89, 0.94, 0.99, 1.08, 1.21)$.

4.2. The economy with aggregate uncertainty

Similarly to Proposition 6, it is possible to extend the model with aggregate uncertainty to allow for N different values of $\epsilon \in \{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$, where $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_N$. Again, the utility function can also be generalized to any increasing and strictly concave function $u(c)$. In this case, the security price $Q_{zz'}$ is, assuming again that it is determined by the lower bound, given by

$$Q_{zz'} = \phi_{zz'} \max_{i=2,\ldots,N} M_{i|zz'},$$

where

$$M_{i|zz'} \equiv \beta \sum_{j=1}^{N} \pi_{ij|zz'} \frac{u'(\epsilon_j)}{u'(\epsilon_i)}.$$

Note that it is possible that $Q_{zz'}$ can be based on the marginal rates of substitution of different consumers for different aggregate states (z, z'). Therefore, it is possible, for example, that $q_z = \sum_{z'} Q_{zz'}$ is stable across different zs even though the individual values for $\sum_{z'} \phi_{zz'} M_{i|zz'}$ are volatile.

To illustrate this point in more detail, let us consider the following example:

$$[\pi_{ij|zz'}] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } z = g \text{ and } z' = g, b,$$

$$[\pi_{ij|zz'}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{for } z = b \text{ and } z' = g, b.$$

Here, when $z = g$, ϵ_1 and ϵ_2 switch with each other, and when $z = b$, ϵ_2 and ϵ_3 switch with each other. It can readily be verified that

$$(M_{2|gz'}, M_{3|gz'}) = \left(\beta \frac{u'(\epsilon_1)}{u'(\epsilon_2)}, \beta \right),$$

and that

$$(M_{2|bz'}, M_{3|bz'}) = \left(\beta \frac{u'(\epsilon_3)}{u'(\epsilon_2)}, \beta \frac{u'(\epsilon_2)}{u'(\epsilon_3)} \right).$$

Assume that $u'(\epsilon_2)/u'(\epsilon_3) = u'(\epsilon_1)/u'(\epsilon_2) = \kappa(> 1)$. Then, the bond price would satisfy $q_z = \beta\kappa$ and it is independent of z, while at the individual level

$$\left(\sum_{z'} \phi_{zz'} M_{2|gz'}, \sum_{z'} \phi_{zz'} M_{3|gz'} \right) = \left(\beta \frac{u'(\epsilon_1)}{u'(\epsilon_2)}, \beta \right)$$

and

$$\left(\sum_{z'} \phi_{zz'} M_{2|bz'}, \sum_{z'} \phi_{zz'} M_{3|bz'} \right) = \left(\beta \frac{u'(\epsilon_3)}{u'(\epsilon_2)}, \beta \frac{u'(\epsilon_2)}{u'(\epsilon_3)} \right)$$

are different across $z = g, b$. In this example, the individuals with ϵ_1 are always borrowing-constrained, ϵ_2 individuals are constrained only when $z = b$, and ϵ_3 individuals are constrained only when $z = g$. Here, *all* individuals are strictly borrowing-constrained at some point in time, while the risk-free rate is stable over time.

5. Conclusion

In this paper, we analyzed a variant of the Huggett [11] model in order to illustrate how asset pricing may depend on asset-market incompleteness: it focuses on the case when direct insurance against idiosyncratic risks is completely unavailable but insurance against aggregate risk is fully available (so that there is, at each point in time, one asset for each aggregate state next period). In so doing, we also extend the basic Huggett model to one with many idiosyncratic states, aggregate uncertainty, and a general utility function. The limitation of our setting is clear— we require the borrowing constraint to be maximally strict, implying an autarkic equilibrium— but this limitation is also a strength in that it allows closed-form solutions. Because we adopt an extreme borrowing constraint, we can compute analytical expressions for asset prices in the Huggett model that apply as limits when the borrowing constraint approaches its maximally strict value. Our closed-form solutions allow us to conduct comparative statics straightforwardly. For example, we gain an understanding of how asset prices are influenced by the dependence of individual transition probabilities (and hence idiosyncratic risk) on the aggregate state. Here, we draw a connection to the work by Krueger and Lustig [12] by deriving their main result (under slightly more general assumptions) as a special case. We also derive a result similar to the one in Constantinides and Duffie [6], of the sort "give me any asset prices and I will find an endowment process in this economy generating those prices," though, unlike them, we do not need to assume that idiosyncratic shocks are permanent. Such a result, of course, has to be subjected to quantitative scrutiny, and we argue that our model with tight borrowing constraints can indeed generate realistic-looking asset prices in many (though not all) respects when the process for idiosyncratic risk is calibrated in a quantitatively reasonable way.

Most of our results are derived in a very simple setting: there are only two states for idiosyncratic risk, subject to a stationary Markov chain, and these states are the same independently of the aggregate state. We also consider simple extensions of this case, the case of having N shock values being particularly interesting, but do not attempt a broad generalization. Generalizations are possible, however, and should be straightforward. We could also consider the case where individual shocks are permanent (as in Constantinides and Duffie's work) and where the support of the shocks is more complex and aggregate-state dependent. Since our equilibrium construction— indeed the main "trick" of the paper—is to set the borrowing constraints on all assets so tightly as to (just) induce autarky as an equilibrium, it is a much more ambitious extension to consider production and, in particular, a means of physically saving/investing. In the latter case, however, similar equilibrium constructions are sometimes possible, as in Krusell and Smith [13] and Leduc [16], where a model with only capital is first solved—in a nontrivial manner, involving numerically computed solutions—and where other assets, such as bonds, with maximally strict borrowing constraints are then priced along the same lines as in this paper. Finally, it should be

mentioned that all of our main results are derived in the case where assets are in zero net supply. As Section 2.5 shows, if assets are in net positive supply, one is in some sense further from the asset prices we derive in this paper based on maximally tight borrowing constraints, since a positive supply can equivalently be thought of as a looser borrowing constraint.

Appendix A

This appendix gives the proof of Proposition 11, which builds on Lemmas 1 and 2 and Proposition 10.

Lemma 1. *When the borrowing constraint $\underline{a} = 0$ and the bond price is $\tilde{q} < q^*$, the bond demand a' is strictly positive for a consumer with $s = h$ and $a = 0$.*

Proof. Consider the first-order condition for a consumer with $s = h$ and $a = 0$:

$$q(\epsilon_h - qa')^{-\sigma} = \beta\left(\pi_{hh} V_h'(a'; q) + (1 - \pi_{hh}) V_\ell'(a'; q)\right) + \lambda_h.$$

This is satisfied with equality with $a' = 0$ and $\lambda_h = 0$ when $q = q^*$. Here, the value functions are explicitly indexed with q to clarify the dependence on q. Thus, for $\tilde{q} < q^*$,

$$
\begin{aligned}
\tilde{q}(\epsilon_h)^{-\sigma} &< q^*(\epsilon_h)^{-\sigma} \\
&= \beta\left(\pi_{hh} V_h'(0; q^*) + (1 - \pi_{hh}) V_\ell'(0; q^*)\right) \\
&= \beta\left(\pi_{hh}(\epsilon_h)^{-\sigma} + (1 - \pi_{hh})(\epsilon_\ell)^{-\sigma}\right) \\
&\leqslant \beta\left(\pi_{hh}\left(\epsilon_h - \tilde{q}a_h'(0; \tilde{q})\right)^{-\sigma} + (1 - \pi_{hh})\left(\epsilon_\ell - \tilde{q}a_\ell'(0; \tilde{q})\right)^{-\sigma}\right) \\
&= \beta\left(\pi_{hh} V_h'(0; \tilde{q}) + (1 - \pi_{hh}) V_\ell'(0; \tilde{q})\right) \\
&\leqslant \beta\left(\pi_{hh} V_h'(0; \tilde{q}) + (1 - \pi_{hh}) V_\ell'(0; \tilde{q})\right) + \lambda_h
\end{aligned}
$$

for any $\lambda_h \geqslant 0$. Here, $a_s'(0; \tilde{q})$ is the decision rule for a' when the current income state is s, the current asset holding is 0, and the bond price is $\tilde{q}$. The first line follows from $\tilde{q} < q^*$. The second and the third lines are from the definition of q^*. The fourth line comes from the fact that $a_s'(0; \tilde{q}) \geqslant 0$ given the borrowing constraint. The fifth line is from the envelope condition at $a = 0$ under the bond price $\tilde{q}$. The final line follows from $\lambda_h \geqslant 0$. Therefore, the first-order condition for this consumer is not satisfied with $a' = 0$ when $\tilde{q} < q^*$. Moreover, since the utility function and the value function are both strictly concave in a (strict concavity of the value function can be shown in a standard manner), the bond demand a' has to be strictly positive to satisfy the first-order condition. $\square$

Lemma 2. *When the borrowing constraint $\underline{a} = 0$ and bond price is $\tilde{q} > q^*$, the borrowing constraint is strictly binding* (*the Lagrange multiplier on the borrowing constraint is strictly positive*) *for a consumer with $a = 0$.*

Proof. Consider a sequence of functions $V_s(a; q : i)$ where $i = 0, 1, 2, \ldots$:

$$V_s(a; \tilde{q} : i + 1) = \max_{a'} \frac{(a + \epsilon_s - \tilde{q}a')^{1-\sigma}}{1 - \sigma} + \beta\left[\pi_{sh} V_h(a'; \tilde{q} : i) + (1 - \pi_{sh}) V_\ell(a'; \tilde{q} : i)\right]$$

subject to

$$a' \geqslant 0.$$

Let $V_s(a; \tilde{q} : 0) = 0$. The mapping from $V_s(a; \tilde{q} : i)$ to $V_s(a; \tilde{q} : i+1)$ is a contraction mapping. For $i = 0$, clearly the borrowing constraint is strictly binding for $a = 0$. It is also straightforward to check that it is the case for $i = 1$ when $\tilde{q} > q^*$. Now suppose that the borrowing constraint is (weakly) binding for the optimization problem for some $i = k \geqslant 1$. Below we will show that it is also strictly binding for the optimization problem for $i = k + 1$.

Consider the first-order condition for a consumer with $a = 0$:

$$q(\epsilon_s - \tilde{q}a')^{-\sigma} = \beta(\pi_{sh} V_h'(a'; \tilde{q} : k) + (1 - \pi_{sh})V_\ell'(a'; \tilde{q} : k)) + \lambda_s.$$

Suppose, by contradiction, that this holds with $\lambda_s = 0$ and some $a' = \bar{a}' \geqslant 0$ when $q = \tilde{q}$. Then,

$$
\begin{aligned}
\tilde{q}(\epsilon_s - \tilde{q}\bar{a}')^{-\sigma} &= \beta(\pi_{sh} V_h'(\bar{a}'; \tilde{q} : k) + (1 - \pi_{sh})V_\ell'(\bar{a}'; \tilde{q} : k)) \\
&\leqslant \beta(\pi_{sh} V_h'(0; \tilde{q} : k) + (1 - \pi_{sh})V_\ell'(0; \tilde{q} : k)) \\
&= \beta(\pi_{sh}(\epsilon_h)^{-\sigma} + (1 - \pi_{sh})(\epsilon_\ell)^{-\sigma}) \\
&\leqslant q^*(\epsilon_s)^{-\sigma}
\end{aligned}
$$

and this is a contradiction since $q^*(\epsilon_s)^{-\sigma} < \tilde{q}(\epsilon_s - \tilde{q}\bar{a}')^{-\sigma}$. The first line is from the supposition. The second line comes from the concavity of the value function. The third line follows from the envelope condition and the optimal decision rule of $i = k$. The fourth line is from the definition of q^*. From the contraction mapping theorem, the policy function converges pointwise, and therefore at the unique fixed point of V_s the borrowing constraint is weakly binding for $a = 0$ consumers. Running through the above argument again with this fixed point establishes that the borrowing constraint is in fact strictly binding. $\square$

Proposition 10. *Suppose that $\pi_{hh} > \pi_{\ell h}$. Consider a sequence of economies with different values of $\underline{a} < 0$. Suppose that an equilibrium bond price exists for each $\underline{a}$, and denote it $q(\underline{a})$. If $q(\underline{a})$ converges to a limit as $\underline{a} \to 0$ from below, the limit of $q(\underline{a})$ is q^*.*

Proof. Label the limit $\tilde{q}$. We show that $\tilde{q} = q^*$ by ruling out $\tilde{q} < q^*$ and $\tilde{q} > q^*$. Suppose first that $\tilde{q} < q^*$. Let w be the optimal amount of steady-state saving a' for a consumer with $s = h$ and $a = 0$ in an economy where $q = \tilde{q}$ and $\underline{a} = 0$. Lemma 1 shows that $w > 0$. Since the decision rule is continuous in q and $\underline{a}$, in the neighborhood of $\underline{a} = 0$, the optimal saving of a consumer with $s = h$ and $a = \underline{a}$ is strictly positive. Denote $a_h'(a; q(\underline{a}), \underline{a})$ the steady-state bond demand for a consumer with $s = h$ and asset a when the bond price is $q(\underline{a})$ and the borrowing constraint is $\underline{a}$. With this notation, $a_h'(0; q(0), 0) = w$. Since $a_h'(a; q(\underline{a}), \underline{a})$ is continuous in all three terms, we can choose a value $\hat{a} < 0$ so that

$$a_h'(\underline{a}; q(\underline{a}), \underline{a}) > \frac{w}{2} \quad \text{for all } \underline{a} \in [\hat{a}, 0].$$

Then, for an economy with the borrowing constraint $\underline{a} \in [\hat{a}, 0]$, the total bond demand by the consumers with $s = h$ is larger than $\underline{S} \equiv w\chi_h/2$ (where χ_h is the total mass of the consumers with $s = h$), since the bond demand is increasing in a. For $s = \ell$, the total bond demand can be negative but it is bounded below by $\underline{a}\chi_\ell$ (where χ_ℓ is the total mass of consumers with $s = \ell$). Since $\underline{S} + \underline{a}\chi_\ell > 0$ for $\underline{a}$ close enough to zero, the bond market cannot be in equilibrium.

Second, suppose that $\tilde{q} > q^*$. Lemma 2 shows that a consumer with $a = 0$ is strictly borrowing constrained (the Lagrange multiplier on the borrowing constraint is strictly positive) when $q = \tilde{q}$ and $\underline{a} = 0$. From the continuity of the optimal decision in q and $\underline{a}$ (and since $q(\underline{a})$ is convergent), a consumer with $a = 0$ is still borrowing-constrained under $\underline{a} < 0$ and $q(\underline{a})$ if $\underline{a}$ is

sufficiently close to zero. Pick such an $\underline{a}$ and call it $\bar{\underline{a}}$. From the monotonicity of the decision rules of a' in a, all the consumers with $a \leqslant 0$ are borrowing constrained when $\underline{a} = \bar{\underline{a}}$. Huggett [11] (Theorem 2) shows that when $q > \beta$ and $\pi_{hh} > \pi_{\ell h}$, the stationary distribution of the individual states is unique. Here, with $\underline{a} = \bar{\underline{a}}$, everyone having $a = \bar{\underline{a}}$ is a stationary distribution, and from the uniqueness ($\tilde{q} > \beta$ holds since $q^* > \beta$), this is the only candidate for the stationary equilibrium. But this cannot be compatible with the bond market equilibrium condition when $\underline{a} = \bar{\underline{a}} < 0$, since the bond demand has to sum up to zero in equilibrium. $\quad\square$

The following proposition establishes the existence and the convergence of $q(\underline{a})$ with an additional assumption.

Proposition 11. *Suppose that $\pi_{hh} > \pi_{\ell h}$ and $\sigma > 1$. Then the equilibrium bond price $q(\underline{a})$ exists for $\underline{a} < 0$ sufficiently close to zero, and a sequence of equilibrium prices $q(\underline{a})$ converges to q^* as $\underline{a} \to 0$ from below.*

Proof. To show existence, we can appeal to Miao [23] (Theorem 4.1). It is straightforward to check that the assumptions required are satisfied given the above assumptions when $\underline{a}$ is sufficiently close to zero. Note that the equilibrium price is not necessarily unique. However, from Miao [23] (Theorem 3.6), the ergodic measure of the bond holding is unique for a given $(q, \underline{a})$. Moreover, Miao [23] (Theorem 3.9) shows that the aggregate excess demand of the bond generated from this ergodic measure is continuous in $(q, \underline{a})$ and increasing in $\underline{a}$. Denote by $e(q, \underline{a})$ the excess demand for a given $(q, \underline{a})$. We also know that $e(q, \underline{a}) \to \underline{a}$ as $q \to \infty$ (note that one can find a large enough q for any given a to make $a' = \underline{a}$ the optimal choice and then use a similar logic as in the second part of Proposition 10) and $e(q, \underline{a}) \to \infty$ as $q \to \beta$ (Miao [23] (Theorem 3.7)). Therefore, the set of equilibrium prices (that is, $\{q | e(q, \underline{a}) = 0\}$) is a compact set for a given $\underline{a}$. Denote it $Q(\underline{a})$.

Let $\bar{q}(\underline{a}) \equiv \max\{q \mid q \in Q(\underline{a})\}$ and $\underline{q}(\underline{a}) \equiv \min\{q \mid q \in Q(\underline{a})\}$. From the above argument, at $q = \bar{q}(\underline{a})$ and $q = \underline{q}(\underline{a})$, $e(q, \underline{a})$ is decreasing in q. Since $e(q, \underline{a})$ is increasing in $\underline{a}$ for a given q, it follows that both $\bar{q}(\underline{a})$ and $\underline{q}(\underline{a})$ are increasing in $\underline{a}$.

Since $\bar{q}(\underline{a})$ is monotonically increasing in $\underline{a}$, in order to show that $\bar{q}(\underline{a})$ is convergent, it is sufficient to show that $\bar{q}(\underline{a})$ does not diverge to infinity as $\underline{a} \to 0$. Suppose, by contradiction, that $\bar{q}(\underline{a}) \to \infty$ as $\underline{a} \to 0$. This implies that there exists an $A < 0$ such that $\bar{q}(\underline{a}) > q^*$ for all $\underline{a} \geqslant A$. However, with the same logic as in the second part of Proposition 10, for $\underline{a}$ sufficiently close to zero, such a $\bar{q}(\underline{a})$ cannot satisfy the bond-market equilibrium condition. The same logic can be applied to show the convergence of $\underline{q}(\underline{a})$.

Thus far we have shown that $\bar{q}(\underline{a})$ and $\underline{q}(\underline{a})$ are both convergent. Using $\bar{q}(\underline{a})$ and $\underline{q}(\underline{a})$ as the sequences of prices in Proposition 10, it follows that $\bar{q}(\underline{a}) \to q^*$ and $\underline{q}(\underline{a}) \to q^*$ as $\underline{a} \to 0$ from below. Thus, any equilibrium price sequence $q(\underline{a})$ converges to q^* as $\underline{a} \to 0$ from below. $\quad\square$

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