Question 1. Consider two firms, 1 and 2, who produce in a duopoly market. Firm 1 chooses its quantity of output $x_1$ and firm 2 chooses $x_2$. Given these outputs, the market price is given by $p = A - B(x_1 + x_2)$. There are no costs of production.

1.1 (10 points) Write the profit functions for firms 1 and 2. Find the optimal quantity of output $x_2$ for firm 2, as a function of $x_1$ (i.e., find firm 2’s best response function).

1.2 (10 points) Find the equilibrium outputs for firm 1 and 2 and the attendant profits for each firm.

1.3 (10 points) Find the level of output $x^*$ that would maximize the sum of firm 1 and firm 2’s profits if they each produce $x^*$. How does this compare to your answer from question [1.2], and why? Explain clearly why producing $x^*$ is not a best response for either firm.

1.4 (10 points) Now return to the setting of question [1.2]. In equilibrium, firm 2 chooses an output that is a best response to firm 1’s output. Suppose that firm 2 is concerned that it might be wrong about how much output 1 is going to produce, and so hires an industrial spy who can (reliably) report firm 1’s output to firm 2 before 2 makes a decision. Moreover, firm 1 knows that firm 2 has hired such a spy. As a result, firm 1 knows that whatever value of $x_1$ it produces, firm 2 will produce a best response. Hence, from firm 1’s point of view, $x_2$ is no longer to be viewed as fixed when 1 maximizes 1’s profits, but rather $x_2$ is effectively a function of $x_1$, with this function given by firm 2’s best response function from question [1.1]. To analyze this situation, start with the profit function for firm 1 from question [1.1], replace $x_2$ in this profit function by firm 2’s best response function, so that 1’s profits are now entirely a function of $x_1$, and then take a derivative and solve to find the (new) equilibrium quantity $x_1$. Insert this optimal quantity into firm 2’s best response function to find firm 2’s equilibrium quantity. How do these compare to the quantities you found in question [1.2]? Explain the differences in your answers. Which firm benefits from firm 2’s spy and which one loses?
**Question 2.** Consider an economy with two consumers with utility functions \( u^A(x_1^A, x_2^A) \) and \( u^B(x_1^B, x_2^B) \). Each consumer is endowed with 1 unit of time that she can sell to firms. There are two firms in the economy, firm 1, using \( L_1 \) units of the consumers’ time to produce good \( x_1 \) according to production function \( x_1 = f_1(L_1) \) and firm 2, using \( L_2 \) units of the consumers’ time to produce good \( x_2 \) according to production function \( x_2 = f_2(L_2) \).

2.1 (10 points) Explain what it means for an allocation in this economy to be (Pareto) efficient. Formulate a maximization problem whose solution would give an efficient allocation. Find the first-order conditions for this maximization and interpret (but do not solve) them.

2.2 (5 points) Suppose the firms’ production functions are given by \( x_1 = k_1 L_1 \) and \( x_2 = k_2 L_2 \) (where \( k_1 \) and \( k_2 \) are constants). Explain why these firms will earn zero profits in equilibrium.

2.4 (15 points) Define a competitive equilibrium. In addition to the production functions given in [2.2], suppose the consumers’ utility functions are given by \( u^A(x_1^A, x_2^A) = \ln x_1^A + \ln x_2^A \) and \( u^B(x_1^B, x_2^B) = \ln x_1^B + \ln x_2^B \). Find the competitive equilibrium of the economy. (First, set the price paid for labor equal to 1, then look at the firms’ profit maximization problems and think about prices \( p_1 \) and \( p_2 \), then....) Explain why this competitive equilibrium is efficient.

2.4 (10 points) Now suppose firm 2’s production function is given by \( x_2 = f_2(L_1, L_2) = \frac{L_2}{L_1} \). Explain why this is an externality. Explain why the competitive equilibrium of this economy will be inefficient and identify a change in the competitive allocation that would make both consumers better off. Briefly describe one way of addressing this externality.

**Question 3.** Suppose there are two agents, \( A \) and \( B \), who have one unit each of goods 1 and 2 to consume.

3.1 (10 points) Suppose the agents’ utility functions are \( u^A(x_1^A, x_2^A) = x_1^A + x_2^A \) and \( u^B(x_1^B, x_2^B) = x_1^B x_2^B \). Draw an Edgeworth box for these two agents and identify the contract curve of efficient outcomes.

3.2 (10 points) Now suppose that A’s utility function is given by \( u^A(x_1^A, x_2^A) = \frac{1}{2} (x_1^A + x_2^A) \) if \( x_1^A \leq x_2^A \) and by \( u^A(x_1^A, x_2^A) = x_1^A \) if \( x_1^A \geq x_2^A \). Hence, person A doesn’t care about good 2 unless A consumes more of it than good 1. Draw A’s indifference curves. (Notice that \( \frac{1}{2} (x_1^A + x_2^A) = x_1^A \) when \( x_1^A = x_2^A \).) Then draw an Edgeworth box showing the contract curve of efficient outcomes. For which of these efficient outcomes are the agents’ marginal rates of substitution equal, and for which are they not? Which outcomes will correspond to competitive equilibria, and which will not?