1. (a) (5 points) Describe the budget line analytically and then present a carefully labeled diagram (using all of the above information) for a faculty member who chooses to receive day care support from the university but cannot get tax deduction?

Solution

\[ 12d + c = 1200 \]

(b) (5 points) Describe the budget line analytically and then present a carefully labeled diagram (using all of the above information) for a faculty member who chooses not to receive day care support from the university but gets tax deduction instead?

Solution
10d + c = 1000

(c) (5 points) Find analytically Kim’s optimal consumption of day-care and other goods if s/he chooses the university support.

Solution
Kim solves the following maximization problem
\[ \max_{c \geq 0, d \geq 0} c^3 d \]
s.t. \[ 12d + c = 1200 \]

The first order conditions together with the budget constraint yield the following system of two equations and two unknowns

\[ \frac{c}{3d} = 12 \]
\[ 12d + c = 1200 \]

\[ \Rightarrow \]
\[ d^* = 25 \]
\[ c^* = 900 \]

(d) (5 points) Find analytically Kim’s optimal consumption of day-care and other goods if s/he chooses the tax deduction instead.

Solution
Kim solves the following maximization problem

\[ \min_{c \geq 0, d \geq 0} c^3 d \]
s.t. \[ 12d + c = 1200 \]
\[
\begin{align*}
\max_{c \geq 0, d \geq 0} & \quad c^3d \\
\text{s.t.} & \quad 10d + c = 1000
\end{align*}
\]

The first order conditions together with the budget constraint yield the following system of two equations and two unknowns

\[
\begin{align*}
\frac{c}{3d} &= 10 \\
10d + c &= 1000
\end{align*}
\]

\[
\Rightarrow
\]

\[
\begin{align*}
d^* &= 25 \\
c^* &= 750
\end{align*}
\]

(e) (5 points) Evaluate the indirect utility function of Kim under each option analytically. Would Kim prefer to get the university support for day-care or the tax deduction?

Solution

\[
\begin{align*}
&u_{\text{support}} = (900^3)(25) \\
&u_{\text{subsidy}} = (750^3)(25)
\end{align*}
\]

Thus, Kim prefers to receive the university support.

(f) (5 points) With respect to the goals of the university given by (i) and (ii), does the university achieve its policy goal. Briefly explain why or why not?

Solution

The University achieves goal (i) of improving faculty’s utility since \(u_{\text{support}} > u_{\text{subsidy}}\) as found in part (e), yet the hours of day care consumption are the same across two policies: \(d^* = 25\) under both programs as found in parts (c) and (d).

2. (a) (5 points) Give a definition of a feasible allocation \((b_A, c_A, b_B, c_B)\) for this endowment economy.

Solution

A feasible allocation in this endowment economy is any \((b_A, c_A, b_B, c_B)\) such that all quantities are weakly positive and \(b_A + b_B \leq 2\) and \(c_A + c_B \leq 1\).
Comments: It is difficult to give partial credit on this question since your answer is either correct or incorrect but I awarded up to 2 points if there was some mention of the total endowment. I did not penalize for answers that were just in words. However, I took off 1 point if the definition specified that the total allocation has to equal the total endowment.

(b) (7 points) Draw the Edgeworth box of this endowment economy and label it carefully (axis', direction, etc.) for the feasible consumption allocation:

\[
\begin{align*}
b_A &= 1, c_A = \frac{3}{4}, b_B = 1, c_B = \frac{1}{4},
\end{align*}
\]

and with indifference curves through the feasible allocation defined above for Ann and Bob.

Solution

Note that the width of the box is 2, the total endowment of bread, and the height is 1, the total endowment of cheese.

Comments: The above diagram shows that the indifference curves going through the suggested allocation intersect, but I also accepted tangent indifference curves since they are when $\gamma = 3$. Up to 2 points were awarded for correct/complete axis labels, 1 point for incorporating the total endowment in the answer, 1 point for using the suggested allocation, and 3 points for drawing sensible indifference curves that both pass through the allocation.

(c) (5 points) Define the notion of a Pareto efficient allocation for an endowment economy (without referring to marginal rate of substitution.)

Solution

A feasible allocation $x$ is Pareto efficient if there is no other feasible allocation $y$ such that $u_n(y) \geq u_n(x)$ for all $n$ and $u_i(y) > u_i(x)$ for at least one $i$.
Comments: I accepted answers that were just in words but I took off up to 3 points if the phrasing of the definition was too loose or sloppy.

(d) (8 points) Describe graphically the conditions for a Pareto efficient allocation in the Edgeworth box and give the corresponding analytic conditions for a Pareto efficient allocation in this economy.

Solution
The following point in the Edgeworth Box, labeled \( PE \), is a Pareto efficient allocation.

In general, for some allocation \( x \geq 0 \) for Ann and \( \omega - x \geq 0 \) for Bob to be Pareto efficient, the indifference curves going through this allocation for Ann and Bob must be tangent. In other words, the slopes of the indifference curves at this allocation must be equal so

\[
\frac{MU_{Ab}(x)}{MU_{Ac}(x)} = \frac{MU_{Bb}(\omega-x)}{MU_{Bc}(\omega-x)}.
\]

If the indifference curves are not tangent, that means that they intersect at two points and so the upper contour sets for Ann and Bob overlap at some allocations. If we moved to one these allocations, both Ann and Bob would be strictly better off and so our initial allocation could not have been Pareto efficient.

Comments: Up to 4 points were awarded each for the correct graphical and analytical condition for a Pareto efficient allocation. However, up to 2 points were deducted if there was no justification of the tangency condition and 1 point was deducted if the analytical condition did not somehow contain the notion of feasibility, namely if Ann gets \( x \) then Bob must get what is left over, which is given by \( \omega - x \).

(e) (12 points) Ann and Bob have agreed that Ann receives a weight of \( \alpha \) and Bob a weight of \( 1 - \alpha \) in the social welfare function with \( 0 \leq \alpha \leq 1 \) given by:

\[
\alpha u_A(b_A, c_A) + (1 - \alpha) u_B(b_B, c_B)
\]
Derive the Pareto efficient allocation as a function of $\alpha$ and $\gamma$ (taking account of the resource constraints). Describe briefly the objective function, the choice variables and the method you use to derive the Pareto efficient allocation. Briefly describe how the Pareto efficient allocation varies in $\gamma$ for a given $\alpha$.

Solution

In order to find the Pareto efficient allocation for a given value of $\alpha$, we must solve

$$
\max_{b_A, c_A, b_B, c_B} \alpha u_A(b_A, c_A) + (1 - \alpha) u_B(b_B, c_B)
$$

subject to the constraints that $b_A + b_B \leq \omega_b = 2$ and $c_A + c_B \leq \omega_c = 1$. Our utility functions are $\ln b_A + \ln c_A$ for Ann and \(\gamma \ln b_B + \ln c_B\) for Bob. Since both utility functions demonstrate strictly monotonic preferences, we know that our endowment constraints will hold with equality at the optimal solution. This tells us that $b_B = 2 - b_A$ and $c_B = 1 - c_A$. Substituting these constraints into our constrained maximization problem we get the following equivalent unconstrained maximization problem:

$$
\max_{b_A, c_A} \alpha u_A(b_A, c_A) + (1 - \alpha) u_B(2 - b_A, 1 - c_A).
$$

Substituting in our utility functions, our specific problem becomes:

$$
\max_{b_A, c_A} \alpha(\ln b_A + \ln c_A) + (1 - \alpha)(\gamma \ln(2 - b_A) + \ln(1 - c_A))
$$

and our associated FOCs for $b_A$ and $c_A$, respectively, are:

$$
\frac{\alpha}{b_A} - \frac{\gamma(1 - \alpha)}{2 - b_A} = 0
$$

$$
\frac{\alpha}{c_A} - \frac{(1 - \alpha)}{1 - c_A} = 0
$$

The first FOC can be rearranged into $\alpha(2 - b_A) = b_A\gamma(1 - \alpha)$ and solving for $b_A$ yields $b_A = \frac{2\alpha}{\gamma(1 - \alpha)}$. Our second FOC can be rearranged into $\alpha(1 - c_A) = c_A(1 - \alpha)$ and solving for $c_A$ yields $c_A = \alpha$. Using our endowment constraints and our solution for Ann’s Pareto efficient allocation, our complete Pareto efficient allocation is given by:

$$
x_A = \left( \frac{2\alpha}{\alpha + \gamma(1 - \alpha)}; \alpha \right)
$$

$$
x_B = \left( \frac{2\gamma(1 - \alpha)}{\alpha + \gamma(1 - \alpha)}; 1 - \alpha \right)
$$

Note that $c_A$ and $c_B$ do not depend on $\gamma$ but $\frac{\partial}{\partial \gamma} b_A = -\frac{2\alpha(1 - \alpha)}{(\alpha + \gamma(1 - \alpha))^2} < 0$ and $\frac{\partial}{\partial \gamma} b_B = -\frac{\partial}{\partial \gamma} b_A > 0$. So for a given $\alpha$, as Bob’s preference
for bread increases, his allocation of bread in the Pareto efficient allocation increases while Ann's decreases but the allocation of cheese is unaffected.

Comments: Up to 5 points were awarded for the correct set up of the problem (using either the substitution or Lagrange method), 5 points were awarded for the correct work/math and answer (2 for finding the FOCs, 2 for correct work shown, 1 for the solution), and 2 points were awarded for the correct description of how $\gamma$ affects the Pareto efficient allocation (which could have been based on either an intuitive or analytical argument).

(f) (8 points) Now describe graphically the Pareto efficient allocations in the Edgeworth box as a function of $\alpha$ when $\alpha$ varies from 0 to 1 (for a given $\gamma$, say $\gamma = 2$) and relate the graph to the preferences of the agents.

Solution

As $\alpha$ varies from 0 to 1, the associated Pareto efficient allocation also varies. When $\alpha = 0$, $x_A = (0,0)$ and when $\alpha = 1$, $x_A = (2,1)$, with Bob getting the residual of Ann's allocation. When $\alpha = \frac{1}{2}$, $x_A = \left(\frac{2}{1+\gamma}, \frac{1}{1+\gamma}\right)$. Since $\gamma > 1$, when Ann is getting half the total endowment of cheese, she is getting less than half the total endowment of bread since Bob's marginal utility for it is higher. This tells us that the set of Pareto efficient allocations lies to the left of (or above) the line that connects the two origins of the Edgeworth box. Graphically, the set of all Pareto efficient allocations is given by the curve labeled $PE$.

The set of Pareto efficient allocations for $\gamma' > \gamma$ would lie strictly to the left (or above) the one shown in the graph except at the two origins, where they would intersect.

Comments: Up to 6 points were awarded for the correct graph and 2 points for somehow relating the shape of the graph to $\gamma$. For the graph, I awarded up to 3 points if you provided a sensible diagram of what the contract curve might look like but the remaining 3 points
were reserved for being able to show how you found your diagram (even if you happened to draw the correct shape for the curve).