General Comments:

The problem set was generally well attempted. However, there were some common mistakes which are listed below:

- A few students were not able to calculate the expected surplus from trade in question 1.2

- In question 1.4, when finding the total expected surplus from weakly (or strongly) efficient trades, some students multiplied the expected surplus from one trade by 20 (or 16) instead of 36.

- Question 4 was generally well attempted. However, it is recommended that all students read through the explanations provided in the solution.

- When computing the mean and variance of prices in question 5.4, a common mistake was to include the prices recorded as zero. Price was recorded as zero when no trade took place and should not have been included in the calculation.
1. (a) A given pair has a weakly efficient trading opportunity if \( v \geq c \), and a strongly efficient trading opportunity if \( v > c \). So

\[
\Pr(\text{Weakly Efficient Trading}) = \Pr(v \geq c) = \sum_{i=2}^{10} \Pr(v = i) \Pr(v \geq c | v = i)
\]

But by the uniform assumption we know that \( \Pr(v = i) = \frac{1}{9} \) for any \( i \) between 2 and 10. We also know that the probability that \( c \) is less than or equal to \( v \) is \( \Pr(v \geq c | v = i) = \frac{i-1}{9} \). So

\[
\Pr(\text{Weakly Efficient Trading}) = \frac{1}{9} \sum_{i=2}^{10} \frac{i-1}{9} = \frac{1}{81} \sum_{i=2}^{10} (i-1)
\]

Notice that we used the formula \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \) to calculate the sum.

Similarly, since the probability \( \Pr(v > c | v = i) = \frac{i-2}{9} \),

\[
\Pr(\text{Strictly Efficient Trading}) = \frac{1}{9} \sum_{i=2}^{10} \frac{i-2}{9} = \frac{1}{81} \sum_{i=2}^{10} (i-2) = \frac{1}{81} \sum_{j=1}^{9} (j-1)
\]

Because of the symmetry of this particular problem a faster (although less general way) to do this question would be to notice that \( \Pr(v > c) = \Pr(v < c) \) and that \( \Pr(v = c) = \frac{1}{9} \). So \( \Pr(v > c) = \frac{1}{2} (1 - \frac{1}{9}) = \frac{4}{9} \) and \( \Pr(v \geq c) = \frac{4}{9} + \frac{1}{9} = \frac{5}{9} \).

(b) First notice that the expected surplus from all weakly efficient trades taking place is the same as the surplus from all strongly efficient trades taking place. Similar to part 1, we can calculate the expected surplus from each trade as

\[
E(\text{Surplus}) = \sum_{i=2}^{10} \Pr(v = i) E(\text{Surplus} | v = i)
\]

and we can calculate the expected surplus given that the buyer’s valuation is \( i \) as

\[
E(\text{Surplus} | v = i) = \sum_{j=2}^{i} \Pr(c = j) (i - j) = \frac{1}{9} \sum_{j=1}^{i-1} (i - j - 1)
\]

\[
= \frac{1}{9} [(i-1)^2 - \frac{(i-1)i}{2}] = \frac{1}{9} (i-1)(i-2).
\]
Plugging this into the original summation, we have

\[
E(\text{Surplus}) = \sum_{i=2}^{10} \frac{10}{9}\left(\frac{i}{2} - 1\right)
\]

\[
= \frac{1}{81}\left\{\frac{1}{2} \sum_{i=2}^{10} i^2 - \frac{3}{2} \sum_{i=2}^{10} i + \sum_{i=2}^{10} 1\right\}
\]

\[
= \frac{1}{81}\{192 - 81 + 9\} = \frac{120}{81} = \frac{40}{27}.
\]

(c) The probability a specific pair has the possibility of weakly efficient trade is \(\frac{5}{9}\), and strongly efficient trade with probability \(\frac{4}{9}\). There are 36 pairs, so the expected number of weakly efficient trades is \(36 \times \frac{5}{9} = 20\), and the expected number of strongly efficient trades is \(36 \times \frac{4}{9} = 16\).

(d) Since the surplus from weakly and strongly efficient trades is the same and there are a total of 36 pairs, the expected surplus in both cases is \(36 \times \frac{40}{27} = 53\frac{1}{3}\).

2. (a) Each buyer faces the entire market supply and each seller the entire market demand. The easiest way calculate the exact number of efficient trades, plot the valuation/demand curve and the cost/supply curve as follows:

There are 4 buyers and sellers with each of the nine types. From the graph the number of weakly efficient trades is 20, and the number of strongly efficient trades 16.

An alternate way to arrive at the same result is to first notice that in the pit market the price will be 6, and all buyers with valuation higher than 6 will buy. So if all weakly efficient trades take place, then all buyers with valuation at least 6 will buy. This is \(\frac{5}{9}\) of the sample, so the number of trades will be \((36)\frac{5}{9} = 20\). If only the strictly efficient trades take place, then buyers with valuation greater than 6 will buy, which is \(\frac{4}{9}\) of the sample. So the number of trades will be \((36)\frac{4}{9} = 16\).

(b) Notice that in this case the total market surplus will be given by the number of buyer and seller pairs of each type (as lined up as in the graph above) times the surplus generated by each pair.

\[
\text{Market Surplus from weakly efficient trades} = 4[(10 - 2) + (9 - 3) + (8 - 4) + (7 - 5) + (6 - 6)] = 80
\]
Clearly the market surplus generated by strongly efficient trades is also 80.

An alternate way to calculate the gains from trade is by considering the aggregate utility before and after trading. Since all transfers are zero-sum this is just the sum of the valuations of those who have the object. Before trade takes place, the sellers have the objects and there are six sellers of each type. Hence the aggregate utility is $4(\sum_{i=2}^{10} i) = 4(54) = 216$. After trading, all those with valuations over 6 hold an object, and half those with valuations of 6 hold the object. Hence the aggregate utility is $8(7 + 8 + 9 + 10) + 4(6) = 296$. So the gains from trading are $296 - 216 = 80$. This is the same whether all weakly efficient trades take place or just the strongly efficient, since the trades which are weakly, but not strongly, efficient have no effect on the surplus.

3. Notice that the static market structure of the double auction market is exactly the same as that of the pit market (even though the dynamics differ). Hence the answers to 3(a) and 3(b) are exactly the same as 2(a) and 2(b).

4. (a) The expected number of trades in bilateral is the same as the actual number of trades in the pit auction (and double auction). In the bilateral trading framework a given buyer will buy an object if their valuation is higher than the valuation of the seller they are matched with. This probability is roughly $\frac{1}{2}$, depending on whether trade occurs when the valuations are the same (in this situation, there is a $\frac{4}{9}$ chance of being strictly higher, and $\frac{5}{9}$ chance of being weakly higher). In the pit market (and double auction), those with valuations higher than the median of all buyers and sellers will end up buying the object. The probability that one buyer’s valuation is higher than
the average is roughly $\frac{1}{2}$ (again, there is a $\frac{4}{9}$ chance of being strictly higher than the mean, and a $\frac{5}{9}$ chance of being weakly higher).

(b) The expected surplus in bilateral trade is less than that in pit auction (and double auction). In the pit auction (and double auction) those with the highest valuations are the ones who end up with the object. In the bilateral trading model, this may not happen (e.g. a buyer with valuation 9 won't be able to buy an object if the seller they are matched with has valuation 10, but a buyer with valuation 3 will end up with the object if a seller has valuation 2) and some individuals without an object will have a higher valuation than others who do have one (i.e. the allocation from bilateral trading may not be Pareto efficient). So the expected surplus from the double auction is higher than under bilateral trading.

5. (a, b, and c) Total number of bilateral trades, pit market trades and double auction trades that take place are 22, 21.5 and 19. This is roughly the same as the theoretical prediction which is 20 trades for weakly efficient trades and 16 for strongly efficient trades. The surplus generated by bilateral trade, pit market and double auction are 45.5, 91.75 and 84. Again this is very close to the theoretical prediction of 53.13 for the bilateral trade, and 80 for the pit and double auction.

The magnitude of the difference is not surprising given the small sample size. There were also a few efficient trades that didn’t take place, and inefficient trades that did, perhaps because of confusion during the first experiment.

(d) The observed mean and variance are (6.14, 2.06) for the bilateral trading experiment, (5.50, 1.13) for the pit auction and (5.13, 0.47) for the double auction.

In the double auction, there is one price in the entire market so there is little variance in the prices. In the bilateral trading market the price could be very high (for example if both the seller and the buyer have high valuations), or could be very low (if both the buyer and the seller have low valuations). In aggregate, these average out to an average price similar to a price in the middle of the potential valuations, roughly the same as the price in the double auction. Here, there is little variation in price since there is one price for the entire market, and the initial price was close to the market clearing price. Notice that while by the same token the mean of the pit market is no different from that of the double auction and bilateral trade, it’s variance lies somewhere in between. When the pit market experiment was conducted in class multiple informal hubs of trading were formed. As a result we observe a convergence in the price in comparison to bilateral trade, but search frictions experienced by buyers and sellers
keep the variance of the price from collapsing as in the case of double auction.

6. Suppose initially that on the first island there is a seller with valuation 8 and a buyer with valuation 10. On the second island there is a seller with valuation 1 and a buyer with valuation 3. If the islands are autarkic (no trade between islands is possible), then there will be trade between the seller and the buyer on both islands. However, when both sellers and buyers are put on the same island, there will be only one trade between the seller with valuation 1 and the buyer with valuation 10. The resulting price will be between 3 and 8.

More generally, whenever the valuation of the buyer on the first island is higher than the seller \( (v_1 > c_1) \) and the valuation of the buyer on the second island is higher than the seller on the island \( (v_2 > c_2) \) but the valuation of the seller on the first island is higher than the valuation of the buyer on the second island, the result of integrating the islands will be to reduce the number of trades from two to one.

NOTE: Even though the fewer trades take place when the islands are integrated (in this case) the total surplus from trading is still higher.