Economics 501b: Problem Set 11
Suggested Solution

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1 Social Welfare Functions

(a) The egalitarian Social Welfare function is

$$\max_{\{x_i\}_{i=0}^I} \sum_{i=0}^{I} u_i(x_i)$$

(b) The Utilitarian Social Welfare function is

$$\max_{\{x_i\}_{i=0}^I} \sum_{i=0}^{I} \lambda_i u_i(x_i)$$

for some set of positive constants $\{\lambda_i\}_{i=0}^I$.

2 Groves Mechanism

(a) If truth is not a dominant strategy for some agent i then

$$u_i(f^*(\theta_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} u_j(f^*(\theta_i, \theta_{-i}), \theta_j) < u_i(f^*(\theta_i', \theta_{-i}), \theta_i) + \sum_{j \neq i} u_j(f^*(\theta_i', \theta_{-i}), \theta_j)$$

which in turn implies

$$\sum_{i} u_j(f^*(\theta_i, \theta_{-i}), \theta_j) < \sum_{i} u_j(f^*(\theta_i', \theta_{-i}), \theta_j),$$

which contradicts the fact that $f^*(\theta)$ maximizes the egalitarian social welfare function. It therefore follows that truth telling is a dominant strategy.

(b) The ex post participation constraint for agent i is

$$u_i(f^*(\theta), \theta_i) + \sum_{j \neq i} u_j(f^*(\theta), \theta_j) - h_i(\theta_{-i})$$

whether this is greater than or less than zero depends on h_i, in particular if h_i is very large ex post utility will never be greater than 0 and we cannot satisfy the constraint.

3 Pivotal Mechanism

(a) Yes, the more general proof above covers this case.

(b) Given truthful announcements by all agents, individual i's ex post utility is

$$u_i(f^*(\theta_i, \theta_{-i}), \theta_i) - \sum_{j \neq i} u_j(f^*_{-i}(\theta_{-i}), \theta_j) + \sum_{j \neq i} u_j(f^*(\theta_i, \theta_{-i}), \theta_j),$$

which we can rewrite to be

$$\sum_{i=0}^{I} u_i(f^*(\theta_i, \theta_{-i}), \theta_i) - \sum_{j \neq i} u_j(f^*_{-i}(\theta_{-i}), \theta_j). \tag{1}$$

This comparison invites us to consider whether the addition of agent i changes the set of feasible allocations. If the allocation that is optimal if i is not present is still feasible with i then, given our assumption that $u \geq 0$ this last expression must be positive. As nothing in the question indicates that the addition agent i renders any allocations feasible, we conclude that the mechanism is ex post IR.

(c) Equation (1) shows this.

(d) For the first case, we have

$$\sum_{i=1}^{I} t_i(\theta') = \sum_{i=1}^{I} \sum_{j \neq i} [u_j(f^*_{-i}(\theta), \theta_j) - u_j(f^*(\theta), \theta_j)].$$

For this expression to be greater than zero it is sufficient that, for all i

$$\sum_{j \neq i} [u_j(f^*_{-i}(\theta), \theta_j) - u_j(f^*(\theta), \theta_j)],$$

but this follows directly from the definition of f^*_{-i}.

For the second case, I see not reason to think that it would be greater than zero.

4 n unit allocation problem

For $n = 1$ the allocation rule specifies that the good goes to individual i, where i is such that $\theta_i > \theta_j$ for all $j \neq i$. The net utility when θ_i is the highest type is

$$U_i = \theta_i - \max_{j \neq i}\{\theta_j\}$$

and everyone else (except the designer) has zero net utility, therefore the transfer is

$$t_i = \max_{j \neq i}\{\theta_j\}$$

For the case $n > 1$, the allocation rule specifies that the goods go to those with the n highest θs. The transfer rule is as follows: denote C to the the set of agents receiving an item, then for all $i \in C$

$$U_i = \theta_i - \max_{j \notin C}\{\theta_j\}$$

and therefore the transfer rule is or all $i \in C$:

$$t_i = \max_{j \notin C}\{\theta_j\}$$

5 Two Part Tariff

(a) (note that i do this assume a constant marginal cost) Given (F, p) the agent chooses

$$q(\theta, p, F) = arg\max_q v(\theta, q) - pq$$

or

$$q = 1 - \frac{p}{\theta}$$

if she decides to consume and $q = 0$ otherwise. Therefore, assuming that both types are served, the monopolist problem is

$$\max_{F,p}\left[(p-c)\left(\lambda\left(1-\frac{p}{\theta_h}\right)+(1-\lambda)\left(1-\frac{p}{\theta_l}\right)\right)+F\right]$$

subject to

$$v\left(\theta_i,\left(1-\frac{p}{\theta_i}\right)\right)-p\left(1-\frac{p}{\theta_i}\right)\geq F.$$

We can rewrite the objective as

$$(p-c)\left(1-p\left(\frac{1-\lambda}{\theta_l}+\frac{\lambda}{\theta_h}\right)\right)+F$$

and the first order condition gives us

$$p = \frac{1}{2}\left(c + \frac{\theta_l \theta_h}{(1-\lambda)\theta_h + \lambda\theta_l}\right)$$

and F is set such that the constraints binds for the low guy:

$$F = \frac{1}{2}\frac{(p-\theta_l)^2}{\theta_l}.$$

(b) i. We know from what we did in class that there is no distortion at the top, therefore

$$q_h = 1 - \frac{c}{\theta_h}$$

also with price discrimination we can compute:

$$q_l = 1 - \frac{c}{\theta_l} - \frac{\lambda}{\theta_l(1-\lambda)}(\theta_h - \theta_l).$$

The welfare implications depend on how far the quality provisions are distrorted away form the first best. In the linear tariff case we have

$$q_h = 1 - \frac{c}{2\theta_h} - \frac{\theta_l}{(1-\lambda)\theta_h + \lambda\theta_l}$$

$$q_l = 1 - \frac{c}{2\theta_l} - \frac{\theta_h}{(1-\lambda)\theta_h + \lambda\theta_l}$$

Not sure which of these is bigger!

ii. The monopolists revenue must be weakly less in this case as there is an additional constraint.

6 Local and Global Constraints

(a) Let us do parts (a) and (b) together as this will avoid some replication. There are two conceptually distinct parts to the problem. First, we must show that *if* the optimum occurs where IR_1 binds, the local downward incentive constraints bind and quality provision is increasing *then* all the other IR and IC constraints hold. Second, we must show that at the optimum IR_1 binds, the local downward incentive constraints bind and quality provision is increasing.

So, assuming the three conditions, first we will show that all other IR constraints are satisfied. We have

$$\begin{aligned} u(\theta_i, q_i)) &= u(\theta_i, q_{i-1}) - t_{i-1} \quad (IC_i) \\ &\geq u(\theta_{i-1}, q_{i-1}) - t_{i-1} \quad \text{(first part of SM)} \\ &= u(q_{i-1}, q_{i-2}) - t_{i-2} \quad (IC_{i-1}) \\ &\geq \dots \\ &\geq u(\theta_1, q_1) - t_1 = 0 \end{aligned} \tag{2}$$

Now, lets show that all the other IC constraints are satisfied. Let's look first at the downward IC constraints. We have

$$U(\theta_k) = u(\theta_k, q(\theta_k) - t(\theta_k) = u(\theta_k, q(\theta_{k-1})) - t(\theta_{k-1}) = U(\theta_k, \theta_{k-1})$$

and we want to show that

$$U(\theta_k) - U(\theta_k, \theta_l) \geq 0 \quad \forall l < k$$

For the case $l = k - 2$

$$\begin{aligned} U(\theta_k) - U(\theta_k, \theta_{k-2}) &= u(\theta_k, q(\theta_{k-1})) - t(\theta_{k-1}) - u(\theta_k, q(\theta_{k-2})) + t(\theta_{k-2}) \\ &= [u(\theta_k, q(\theta_{k-1})) - u(\theta_k, q(\theta_{k-2}))] - [u(\theta_{k-1}, q(\theta_{k-1})) - u(\theta_{k-1}, q(\theta_{k-2}))] \\ &\geq 0 \end{aligned}$$

where the last line comes from increasing differences. The same argument extends easily to the cases with $l < k - 2$.

Now let us consider the upward IC constraints.

$$\begin{aligned} U(\theta_k) - U(\theta_k, \theta_{k+1}) &= u(\theta_k, q(\theta_k)) - t(\theta_k) - u(\theta_k, q(\theta_{k+1})) + t(\theta_{k+1}) \\ &= u(\theta_k, q(\theta_k)) - u(\theta_k, q(\theta_{k+1})) + u(\theta_{k+1}, q(\theta_{k+1})) - u(\theta_{k+1}, q(\theta_k)) \\ &\geq 0 \end{aligned}$$

where the last line, again, comes from increasing differences. Again it is easy to extend this to the other surrounding constraints.

Now we can turn to the second part of the proof. First, lets show that IR1 is binding at the optimum. Suppose IR1 does not bind, then because all the IC must hold we have (for all $j \neq 1$)

$$\theta_j q_j - t_j \geq \theta_j q_1 - t_1 \geq \theta_1 q_1 - t_1 > 0$$

and consequently, if IR1 is not binding neither are any of the other IR constraints and we can simply increase all t_js by the same amount, preserving IC and increasing the monopolists profits.

Next, let us assume that one or more of the downward adjacent constraints is not binding. Because their are a finite number of agents there must be a highest j such that the constraint is not binding, taking that j we have

$$\theta_j q_j - t_j > \theta_j q_{j-1} - t_{j-1} \geq \theta_{j-1} q_{j-1} - t_{j-1} \geq \ldots \geq \theta_1 q_1 - t_1 = 0$$

and, because j is the highest individual such that this is the case (and we know that q is increasing from above), we also have

$$\theta_j [q_k - q_j] < t_k - t_j$$

for all $k > j$ and it follows that we can increase t_j without breaking any IC constraints and also preserving IRj. This cannot be optimal and therefore we conclude that all the downward adjacent ICs bind.

Finally, the fact that q is increasing can be shown the same as it is in Salanie for the two type case.

(b) We already did it above.

(c) i. After imposing the downward ICs and IR_1 one obtains

$$\begin{aligned} U(\theta_k) &= U(\theta_{k-1}) + q(\theta_{k-1})(\theta_k - \theta_{k-1}) \\ &= U(\theta_{k-2}) + q(\theta_{k-2})(\theta_{k-1} - \theta_{k-2}) q(\theta_{k-1})(\theta_k - \theta_{k-1}) \\ &= \ldots \\ &= \sum_{j=1}^{k-1} q(\theta_j)(\theta_{j+1} - \theta_j) \end{aligned}$$

ii. Consider the profits of the monopolist from type θ_k

$$\begin{aligned} \Pi(\theta_k) &= \theta_k q_k - \frac{1}{2} q_k^2 - U_{k-1} - q_{k-1}(\theta_k - \theta_{k-1}) \\ &= \theta_k q_k - \frac{1}{2} q_k^2 - \sum_{j=1}^{k-1} q(\theta_j)(\theta_{j+1} - \theta_j) \end{aligned}$$

and expected profits are

$$E\Pi = \sum_k \pi_k \left(\theta_k q_k - \frac{1}{2} q_k^2 - \sum_{j=1}^{k-1} q(\theta_j)(\theta_{j+1} - \theta_j) \right).$$

We can then maximize with respect to $\{q_k\}$, the first order condition is then

$$\pi_k(\theta_k - q_k) - \sum_{j=1}^{K-1} \pi_{j+1}(\theta_{j+1} - \theta_j) = 0$$

which gives us

$$q_k^* = \theta_k - \frac{\sum_{j=1}^{K-1} \pi_{j+1}(\theta_{j+1} - \theta_j)}{\pi_k}.$$