

Microeconomic Theory (501b)

Suggested Solutions

Problem Set 12

05/01/08

1. **(Moral Hazard and Renegotiation)** Consider the regular moral hazard model with a risk-neutral principal and a risk averse agent. The agent can choose between two effort levels, $a_i \in \{\underline{a}, \bar{a}\}$ with associated cost $c_i \in \{\underline{c}, \bar{c}\} = \{0, c\}$, with $c > 0$. Each action generates stochastically one of two possible profit levels, $x_i \in \{\underline{x}, \bar{x}\}$ with $p(\underline{x} | \underline{a}) > p(\underline{x} | \bar{a})$. The utility function of the agent is

$$u(w, c_i) = \ln w - c_i.$$

The value of the outside option is normalized to 0. (Risk-neutrality of the principal implies that his payoff function is $x - w$.)

- (a) Carefully describe the principal-agent problem when the principal wishes to implement the high effort level \bar{a} .

Solution: The principal solves the following problem

$$\begin{aligned} \max_{w(\underline{x}), w(\bar{x})} \quad & p(\bar{a})(\underline{x} - w(\underline{x})) + (1 - p(\bar{a}))(\bar{x} - w(\bar{x})) \\ \text{s.t.} \quad & p(\bar{a}) \ln(w(\underline{x})) + (1 - p(\bar{a})) \ln(w(\bar{x})) - \bar{c} \geq 0 \quad (\text{IR}) \\ \text{s.t.} \quad & p(\bar{a}) \ln(w(\underline{x})) + (1 - p(\bar{a})) \ln(w(\bar{x})) - \bar{c} \geq p(\underline{a}) \ln(w(\underline{x})) + (1 - p(\underline{a})) \ln(w(\bar{x})) \quad (\text{IC}) \end{aligned}$$

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- (b) Solve explicitly for the optimal wage schedule to be offered to the agent which implements the high effort level \bar{a} .

Solution: Consider the first-order conditions for the principal's problem in (a) :

$$\begin{aligned} -p(\bar{a}) + \lambda p(\bar{a}) \frac{1}{w(\underline{x})} + \mu [p(\bar{a}) - p(\underline{a})] \frac{1}{w(\underline{x})} &= 0 \\ -(1 - p(\bar{a})) + \lambda (1 - p(\bar{a})) \frac{1}{w(\bar{x})} - \mu [p(\bar{a}) - p(\underline{a})] \frac{1}{w(\bar{x})} &= 0. \end{aligned}$$

from which you can obtain

$$\begin{aligned} w(\underline{x}) &= \frac{\lambda p(\bar{a}) + \mu (p(\bar{a}) - p(\underline{a}))}{p(\bar{a})} = \lambda + \mu \frac{p(\bar{a}) - p(\underline{a})}{p(\bar{a})} \\ w(\bar{x}) &= \frac{\lambda (1 - p(\bar{a})) - \mu [p(\bar{a}) - p(\underline{a})]}{1 - p(\bar{a})} = \lambda - \mu \frac{p(\bar{a}) - p(\underline{a})}{1 - p(\bar{a})} \end{aligned}$$

which implies $w(\underline{x}) < w(\bar{x})$ since $\mu > 0$.

Notice however, that since both constraints are binding, you can directly solve for $w(\underline{x}), w(\bar{x})$ from the constraints (since the objective function is decreasing in wages). Equating the RHS of

the IR,IC:

$$\begin{aligned} p(\underline{a}) \ln(w(\underline{x})) + (1 - p(\underline{a})) \ln(w(\bar{x})) &= 0 \\ \Rightarrow \ln(w(\underline{x})) &= -\frac{1 - p(\underline{a})}{p(\underline{a})} \ln(w(\bar{x})) \end{aligned}$$

one then obtains (from IR)

$$\begin{aligned} -\frac{1 - p(\underline{a})}{p(\underline{a})} p(\bar{a}) \ln(w(\bar{x})) + (1 - p(\bar{a})) \ln(w(\bar{x})) &= \bar{c} \\ \Rightarrow \ln(w(\bar{x})) &= \bar{c} \frac{p(\underline{a})}{p(\underline{a}) - p(\bar{a})} \\ \Rightarrow \ln(w(\underline{x})) &= -\bar{c} \frac{1 - p(\underline{a})}{p(\underline{a}) - p(\bar{a})}. \end{aligned}$$

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- (c) **(Renegotiation 1)** Consider now the following extension to the moral hazard problem. **After** the principal has offered an (arbitrary) wage schedule and the agent has chosen and performed an (arbitrary) effort level, but **before** x is revealed, the principal has the possibility to offer a new contract to the agent. The agent can either accept or reject the new offer. If he accepts the new contract, then it replaces the old contract, if he rejects the new contract, then the old one remains in place. Show that there is no subgame perfect equilibrium of the game where $\Pr(a = \bar{a}) = 1$. (Hint: Consider the optimal contract after a has been chosen but before x has been realized.)

Solution: Consider an equilibrium candidate in which the agent chooses \bar{a} . In such a scenario, the principal can make a renegotiation offer by which he perfectly insures the agent at the level u^* the principal had previously offered him. For example, if the IR was binding in the first stage, $u^* = \bar{c}$. A full-insurance contract maximizes the principal's profits and delivers the same expected utility level (under \bar{a}) to the agent. Hence, it Pareto-dominates any (previous) contractual arrangement. Anticipating this offer, the agent will shirk (cost=0) in the action choice stage and earn a guaranteed profit of u^* in the following stage. Since the principal cannot credibly commit *not* to renegotiate (and offer full insurance) there is no equilibrium in which the agent will choose to work hard. ■

- (d) **(Renegotiation 2)** Consider now the following modification to the renegotiation problem above. Suppose now that the agent can make the new proposal and the principal can either accept or reject the new offer. Suppose further that the principal can observe the action at the time of the new proposal but that the contract can only depend on x and not on a . The timing is otherwise unchanged. Derive the subgame perfect equilibrium of the principal-agent problem. What can you say about the efficiency of the arrangement.

Solution: Backward induction. Knowing the action level (hence the actual probability distribution over outcomes), the principal will accept any offer that gives him a higher profit level than under the previous contract. For a given contract w_0 the agent will therefore demand perfect insurance at a wage $w(a, w_0) = p(a)w_0(\underline{x}) + (1 - p(a))w_0(\bar{x})$. Depending on w_0 , the agent will choose whether or not to work hard. In particular, he will choose \bar{a} iff

$$\ln(p(\bar{a})w_0(\underline{x}) + (1 - p(\bar{a}))w_0(\bar{x})) - \bar{c} \geq \ln(p(\underline{a})w_0(\underline{x}) + (1 - p(\underline{a}))w_0(\bar{x}))$$

Assume the principal still wants to induce high effort. The principal's problem in the first stage is now given by:

$$\begin{aligned} \max_{w_0(\underline{x}), w_0(\bar{x})} & p(\bar{a})(\underline{x} - w_0(\underline{x})) + (1 - p(\bar{a}))(\bar{x} - w_0(\bar{x})) \\ \text{s.t.} & \ln(p(\bar{a})w_0(\underline{x}) + (1 - p(\bar{a}))w_0(\bar{x})) - \bar{c} \geq 0 \quad (\text{IR}) \\ \text{s.t.} & \ln(p(\bar{a})w_0(\underline{x}) + (1 - p(\bar{a}))w_0(\bar{x})) - \bar{c} \geq \ln(p(\underline{a})w_0(\underline{x}) + (1 - p(\underline{a}))w_0(\bar{x})). \quad (\text{IC}) \end{aligned}$$

Again, solving directly from the constraints,

$$\begin{aligned} p(\underline{a}) w_0(\underline{x}) + (1 - p(\underline{a})) w_0(\bar{x}) &= 0 \\ \Rightarrow w_0(\underline{x}) &= -\frac{1 - p(\underline{a})}{p(\underline{a})} w_0(\bar{x}), \end{aligned}$$

one then obtains (from IR)

$$\begin{aligned} \Rightarrow w(\bar{x}) &= \bar{c} \frac{p(\underline{a})}{p(\underline{a}) - p(\bar{a})} \\ \Rightarrow w(\underline{x}) &= -\bar{c} \frac{1 - p(\underline{a})}{p(\underline{a}) - p(\bar{a})}. \end{aligned}$$

The agent thus receives the same ex-ante utility level as in the no-renegotiation case. Ex-post, he is perfectly insured, hence efficiency is improved. Note also that the principal is much better off, since he appropriates the entire ex-ante surplus gain deriving from post-renegotiation insurance (to see this, just compare the expected payments under IR in the two cases - use Jensen's inequality). ■

2. **(Moral Hazard in Teams)** Consider the following moral hazard problem with many agents. Suppose output is one-dimensional, deterministic and concave in a_i and depends on the effort of the n agents in the team:

$$x = x(a_1, a_2, \dots, a_n).$$

Each agent i has convex effort cost $c_i(a_i)$. Each agent observe his effort and the joint output x . A contract among the agents is a set of wages $\{w_i(x)\}_{i=1}^n$, which depend only on the publicly observable output x . The set of wages have to be budget balanced, or

$$\sum_{i=1}^n w_i(x) = x$$

for all x . (Think of the team as a cooperative or partnership). The utility function of each agent is $w_i(x) - c_i(a_i)$.

- (a) Describe the first-best allocation policy a^* .

Solution: The social planner's problem is

$$\begin{aligned} \max_{(a_i)_{i=1}^n} \sum_i (w_i(x(a_1, \dots, a_n)) - c_i(a_i)) \\ \text{s.t. } \sum_i w_i(x) = x \\ \text{s.t. } w_i(x) - c_i(a_i) \geq \bar{u}_i \quad \forall i \end{aligned}$$

rewriting this program neglecting for now the participation constraints gives

$$\max_{a_i} \left[x(a_1, \dots, a_n) - \sum_i c_i(a_i) \right]$$

The 1st order condition is

$$\frac{\partial x(a^*)}{\partial a_i} = c'_i(a_i^*) \quad \forall i$$

which simply prescribes marginal productivity=marginal cost. Note that - absent incentive problems - the wage allocation among the agents does not influence the planner's choice of a^* . Any vector of $(w_i^*)_{i=1}^n$ that verifies the participation and resource constraints will do. ■

- (b) Suppose (without loss of generality) that the team is restricted to using differentiable wages, or $w'_i(x)$ exists for all x and i . Show that there is no wage schedule which allows the team to realize the first best policy.

Solution: Suppose wage schedules are differentiable. Since each individual maximizes $w_i(x(a)) - c_i(a_i)$, she will choose

$$a_i : w'_i(x(a^*)) \frac{\partial x(a^*)}{\partial a_i} - c'_i(a_i^*) = 0$$

therefore $w'_i(x(a^*)) = 1 \forall i$. However, since the resource constraint must hold for any x , you can differentiate both sides and obtain $\sum_i w'_i(x) = 1$. This means that in order to induce the first best actions, each agents would have to be able to appropriate any production gains associated to anyone's effort. This is obviously not feasible. ■

- (c) Next, introduce an $n + 1 - th$, who does not deliver any effort to the team, but can be entitled to transfers (the "principal" or "budget-breaker"). Show that you can now design a wage schedule, not necessarily differentiable, such that

$$\sum_{i=1}^{n+1} w_i(x) = x$$

for all x . In fact, you can design the contract such that even

$$\sum_{i=1}^n w_i(x) = x$$

holds on the equilibrium path, but not off the equilibrium path.

Solution: The idea is to punish the agents if they do not exert the optimal effort level. Hence, set the effort levels at a_i^* . Then let the wages be given by

$$w_i(x) = \left\{ \begin{array}{l} w_i^* \text{ if } x = x(a^*) \\ 0 \text{ if } x \neq x(a^*) \end{array} \right\} \quad \forall i = 1, \dots, n$$

$$w_{n+1} = \left\{ \begin{array}{l} 0 \text{ if } x = x(a^*) \\ x(a) \text{ if } x \neq x(a^*) \end{array} \right\}$$

With these wages, no agent has incentives to deviate. Furthermore, actions need not be observable since technology is deterministic, and each deviation from a_i^* will induce a non-optimal production level $x \neq x(a^*)$. Therefore, $a_i^* \forall i$ is an equilibrium of this game. ■

3. (First Order Stochastic Dominance versus Monotone Likelihood Ratio).

- (a) Define monotone likelihood ratio and first-order stochastic dominance.

Solution: Definitions:

$$MLR : \frac{p(x_j | a_i)}{p(x_j | a_k)} \text{ increasing in } x_j \quad \forall k < i$$

$$FOSD : \frac{p(x_j \leq x | a_i)}{p(x_j \leq x | a_k)} \leq 1 \quad \forall x, \forall k < i$$

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(b) Show that with two outcomes, the two notions are equivalent.

Solution: Two outcomes: $x' < x$, two actions $a' < a$. Then

$$\begin{aligned} MLR &\Leftrightarrow \frac{p(x|a)}{p(x|a')} > \frac{p(x'|a)}{p(x'|a')} \\ &\Leftrightarrow \frac{1-p(x'|a)}{1-p(x'|a')} > \frac{p(x'|a)}{p(x'|a')} \\ &\Leftrightarrow p(x'|a') > p(x'|a) \Leftrightarrow FOSD. \end{aligned}$$

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(c) Give an example to show that the equivalence does not hold in general. Show that the monotone likelihood ratio property implies first order stochastic dominance.

Solution: Example: consider the following distributions (with $a' < a$)

$$\begin{array}{ccc} \begin{array}{c} x \rightarrow \\ p \downarrow \end{array} & 0 & \frac{1}{2} & 1 \\ p(x|a) & 0 & \frac{1}{2} & \frac{1}{2} \\ p(x|a') & \frac{1}{2} & 0 & \frac{1}{2} \end{array}$$

The distribution under a' dominates the one under a (the cdfs coincide after $\frac{1}{2}$, but the former starts out lower) but the likelihood ratios are clearly non monotone in x .

Proof. $MLR \Rightarrow FOSD$: Let again $a' < a$. Let $x \in \{x_1, \dots, x_n\}$. Then $MLR \Rightarrow \frac{p(x_1|a)}{p(x_1|a')} < 1$ (since this ratio is monotone, if it were otherwise the two distributions could not both integrate to 1). However, there must exist an $x : \frac{p(x|a)}{p(x|a')} \geq 1$. Let \hat{x} be the first such x . For all $x < \hat{x}$, it must be $p(x_j \leq x | a) < p(x_j \leq x | a')$ by definition of $LR < 1$. For all $x \geq \hat{x}$ it must also be the case that $p(x_j \leq x | a) \leq p(x_j \leq x | a')$, else there would have to exist $x_k \in [\hat{x}, x_n]$ for which $p(x_k | a) < p(x_k | a')$ (otherwise the two distributions couldn't both integrate to 1). This would however violate MLR , since $LR \geq 1$ at $\hat{x} < x_k$. ■

4. (**Moving Support.**) Consider a standard agency model with a risk and effort averse agent. The stochastic output of his action is given by

$$x = a + \varepsilon$$

where the support of ε is given by $[-k, +k]$, where $k > 0$. Suppose for simplicity that ε is uniform on this interval.

(a) Show that with this specification of the production technology, the agency problem can be resolved at the first-best: a^* and w^* .

Solution: The idea behind implementing the first best is that with positive probability the principal is able to detect any deviation by the agent. Let the agent's utility be separable and given by

$$U(w, a) = u(w) - c(a)$$

and normalize his outside option to zero. That said, the principal can offer the following contract

$$\begin{aligned} w(a) &= u^{-1}(c(a)) \text{ if } x \in [a - k, a + k] \\ w(a) &= -\infty \text{ otherwise} \end{aligned}$$

then the agent will exert effort level a and be perfectly insured. Assuming the principal is risk neutral and fully appropriates the agent's output, she will pick

$$\begin{aligned} a^* &= \arg \max E(x(a) - w(a)) \\ &= \arg \max [a - u^{-1}(c(a))] \end{aligned}$$

therefore a^* solves

$$1 - \frac{c'(a)}{u'(u^{-1}(c(a^*)))} = 0$$
$$\Rightarrow c'(a^*) = u'(w(a^*))$$

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- (b) Can you give a general condition in terms of the likelihood ratios when the first-best can be achieved.

Solution: A sufficient condition to implement the 1st best action a^* is that $\exists \hat{x} : \frac{p(x|a^*)}{p(x|a)} = 0 \forall x \leq \hat{x}, \forall a < a^*$. This is of course for a model in which actions are costly to the agent and output provides utility to the principal. In this setting, any production technology with bounded support will do.

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