# Econ 501b. Problem Set 7: Suggested Solutions.

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## 1 A Simple Bayesian Game

(a) Draw the extensive-form tree of the above game.
hmmm.... not too easy to do in latex. You could think of drawing two trees, in either case the information sets should show that player 1 knows what Nature chose, buy not what player 2 has chose. Player 2 does not know anything.

(b) Write down the set of pure strategies.
A pure strategy for player 1 is a mapping from the choice of nature in to an action - that is $s_1 : \{1,2\} \rightarrow \{T, B\}$. A pure strategy for player 2 is just a choice from the set $\{L, R\}$.

(c) Find the pure strategy Bayesian Nash Equilibria. The payoffs in the Bayesian Game are given by the normal form:

| | $L$ | $R$ |
|---|---|---|
| $TT$ | $\frac{1}{2}, \frac{1}{2}$ | $0, 0$ |
| $BT$ | $0, 0$ | $0, 0$ |
| $TB$ | $\frac{1}{2}, 0$ | $1, 1$ |
| $BB$ | $0, 0$ | $1, 1$ |

and there are therefore three pure equilibria $\{(TT, L), (TB, R), (BB, R)\}$.

## 2 Public Good Provision with Private Costs

(a) Define the Notion of a mixed strategy for this game and define the notion of a bayesian equilibrium for this game.

A mixed strategy for player $i$ is a mapping $\sigma_i : [0, 2] \rightarrow \Delta\{C, D\}$.

A strategy profile $\{s_1^*, s_2^*\}$ is a Bayesian Equilibrium if fo rall $i$, $c_i \in [0,1]$ and $a_i \in A_i$

$$\int_0^2 u_i((s_i^*(c_i), s_{-i}^*(c_{-i}), c_i, c_{-i})p(c_{-i}|c_i)dc_{-i} \geq \int_0^2 u_i((a_i, s_{-i}^*(c_{-i}), c_i, c_{-i})p(c_{-i}|c_i)dc_{-i}.$$

(b) Compute a Bayesian Nash Equilibrium of this game in pure strategies:

Lets look for an equilibrium in symmetric cutoff strategies. Thus we have strategies of the form:

$$s_i(c_i) = \begin{cases} C & c_i < x \\ D & c_i \geq x \end{cases}$$

Given the strategy of the other player, player $i$s strategy is a best response so long as

$$(1 - c_i) \geq F(x) \text{ if } c_i \leq x$$

$$(1 - c_i) < F(x) \text{ if } c_i > x$$

$$(1 - c_i) = F(x) \text{ if } c_i = x.$$

Given that the LHS is strictly decreasing in $c_i$ these three inequalities will all hold so long as the last one does. Therefore we can use this equality to solve for $x$ which gives us $x = \frac{2}{3}$ as an equilibrium. Consequently, in equilibrium the players contribute with a probability of $\frac{1}{3}$.

(c) The General Case.

From the foregoing discussion we conclude that an equilibrium exists so long as the following two equations can be satisfied

$$(1 - c_1^*) = F(c_2^*)$$

$$(1 - c_2^*) = F(c_1^*).$$

We will show that these can always be solved where $c_1^* = c_2^*$. This requires only that there exists $c$ such that

$$(1 - c) = F(c).$$

Intuitively it is obvious that there is always a solution, but formally define $f(c) = 1 - F(c)$, we are looking for a fixed point in the function $f$. If $c$ is chosen from a compact set then we can apply Brower's theorem as $f$ is continuous, the domain convex and we can simply expand the range to be equal to the domain. If $c$ is not chosen from a compact set, we cannot apply Brower's theorem. However, we know that $f$ is continuous, strictly decreasing and that $f(c) \to 0$ as $x \to \infty$. Therefore, for all $0 < \delta < 1$ there exists some $c$ such that $f(c) = \delta$. Fix some small $\delta$ and consider the function $f$ restricted to the set $[0, f^{-1}(\delta)]$. This function is defined on a bounded interval and the intermediate value theorem then implies the existence of a fixed point so long as $\delta$ is sufficiently small.

## 3 Battle of the Sexes

See solution next week

## 4 Equivalence of Ex Ante and Interim Definition of BNE[1]

We need to show that the strategy profile $s^*$ is a (pure strategy) Nash equilibrium of this game if, and only if, for all $i = 1, ..., I$, for all $a_i \in A_i$, and almost all $t_i \in [0,1]$,

$$\int_{t_{-i}\in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i^*\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \geq \int_{t_{-i}\in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(a_i, s_{-i}^*\left(t_{-i}\right)\right), t\right). \quad (1)$$

For each player $i$, let $u_i(s) \equiv \int_{t\in[0,1]^I} p(t)\, g_i(s(t), t)$ represent the ex-ante payoff of strategy profile $s$.

Assume first that (1) holds for all $i = 1, ..., I$, almost all $t_i \in [0,1]$, and all $a_i \in A_i$. Now note that, since $A_i$ is finite, $s_i^{-1}(\cdot)$ defines a finite partition of measurable sets $\left\{s_i^{-1}(a_i) : a_i \in A_i\right\}$ of $[0,1]$ for any $s_i \in S_i$. Therefore, for any $i$ and arbitrary $s_i \in S_i$, we have

$$\begin{aligned}
u_i\left(s_i, s_{-i}^*\right) &= \int_T p(t)\, g_i\left(\left(s_i(t_i), s_{-i}^*(t_{-i})\right), t\right) \\
&= \sum_{a_i\in A_i} \int_{s_i^{-1}(a_i)\times T_{-i}} p(t)\, g_i\left(\left(a_i, s_{-i}^*(t_{-i})\right), t\right) \\
&= \sum_{a_i\in A_i} \int_{s_i^{-1}(a_i)} p(t_i) \int_{T_{-i}} p(t_j|t_i)\, g_i\left(\left(a_i, s_{-i}^*(t_{-i})\right), t\right) \\
&= \sum_{a_i\in A_i} p\left(s_i^{-1}(a_i)\right) \int_{T_{-i}} p(t_j|t_i)\, g_i\left(\left(a_i, s_{-i}^*(t_{-i})\right), t\right) \\
&\leq \sum_{a_i\in A_i} p\left(s_i^{-1}(a_i)\right) \int_{T_{-i}} p(t_j|t_i)\, g_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), t\right)
\end{aligned}$$

However, we have

$$\begin{aligned}
u_i\left(s_i^*, s_{-i}^*\right) &= \int_T p(t)\, g_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), t\right) \\
&= \sum_{a_i\in A_i} \int_{s_i^{-1}(a_i)\times T_{-i}} p(t) \int_{T_{-i}} p(t_j|t_i)\, g_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), t\right) \\
&= \sum_{a_i\in A_i} p\left(s_i^{-1}(a_i)\right) \int_{T_{-i}} p(t_j|t_i)\, g_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), t\right)
\end{aligned}$$

[1] Answer provided by Alessandro Bonatti.

Therefore, $u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$. Since $s_i$ and $i$ were arbitrary, $s^*$ is a (Bayesian) Nash equilibrium.

Conversely, assume that (1) does not hold; that is, for some $i \in \{1, ..., I\}$ and some subset $T_i' \subseteq [0,1]$ such that $p\left(T_i'\right) > 0$, there exists an $a_i' \in A_i$ such that, for all $t_i \in T_i'$,

$$\int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i^*\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) < \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(a_i, s_{-i}^*\left(t_{-i}\right)\right), t\right). \quad (2)$$

Define $s_i' \in A_i^{[0,1]}$ by $s_i'\left(t_i\right) = \begin{cases} a_i' & \text{if } t_i \in T_i' \\ s_i^*\left(t_i\right) & \text{if } t_i \notin T_i' \end{cases}$.Then we have

$$\begin{aligned}
u_i\left(s_i', s_{-i}^*\right) &= \int_{t \in T} p(t)\, g_i\left(\left(s_i'\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&= \int_{t_i \in T_i} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i'\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&= \int_{t_i \in T_i'} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i'\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&\quad + \int_{t_i \notin T_i'} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i'\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&= \int_{t_i \in T_i'} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(a_i', s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&\quad + \int_{t_i \notin T_i'} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i^*\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&> \int_{t_i \in T_i'} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i^*, s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&\quad + \int_{t_i \notin T_i'} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i^*\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&= \int_{t_i \in T_i} p\left(t_i\right) \int_{t_{-i} \in T_{-i}} p\left(t_{-i}|t_i\right) g_i\left(\left(s_i^*\left(t_i\right), s_{-i}^*\left(t_{-i}\right)\right), t\right) \\
&= \int_{t \in T} p(t)\, g_i\left(s^*(t), t\right) = u_i\left(s^*\right),
\end{aligned}$$

where the inequality follows from by hypothesis. Therefore, $s^*$ cannot be a Nash equilibrium.

# 5 The complicated Lemons problem

(a) The seller will sell if $\theta v_s \leq p$.

The buyer will buy if $v_b E(\theta|\theta v_s \leq p) \geq p$. Given the sellers rule, the expectation can be written

$$\int_{v_s-\epsilon}^{v_s+\epsilon} \int_0^{\frac{p}{v_s}} \theta \frac{v_s}{p} \frac{1}{2\epsilon} d\theta dv_s. \tag{3}$$

Solving the integral gives us

$$\frac{p}{4\epsilon} \ln(\frac{v_s+\epsilon}{v_s-\epsilon}),$$

which implies that the buyer will buy iff

$$v_b \frac{p}{4\epsilon} \ln(\frac{v_s+\epsilon}{v_s-\epsilon}) \geq p.$$

As in the original setting the $p$ conveniently cancels and we get

$$\frac{v_b}{4\epsilon} \ln(\frac{v_s+\epsilon}{v_s-\epsilon}) \geq 1.$$

At this stage it is worth confirming that, as $\epsilon \to 0$ we recover the solution we had from the standard model in class.

(b) Is trade more or less likely in this case than in the standard case?

If you plot the LHS of the condition above you will see that it is strictly increasing in $\epsilon > 0$. Thus, we see that the amount of trade is increasing in $\epsilon$. Why is this? One intuition is that as $\epsilon$ increases, the fact that the seller is willing to sell gives the buyer less information, therefore the information revelation effect of price is diminished and consequently we get close to the first best outcome.