

# Economics 501, Microeconomics, Spring 2008

Homework 8  
Due April 3, 2008

## 8.1 First Price Auction

- (a) Given the result in class where the valuation is  $t_i$ , a reasonable guess is that  $b_i(t_i) = \frac{1}{2} + \frac{1}{2}t_i$ . We now show that this is in fact an equilibrium. If player  $j$  is bidding according to schedule  $b_j(t_j) = \frac{1}{2} + \frac{1}{2}t_j$  then the expected utility of player  $i$  from bidding  $b_i$  is

$$(t_i + \frac{1}{2} - b_i) \Pr(b_j < b_i) = (t_i + \frac{1}{2} - b_i) \Pr(t_j < 2b_i - 1) = \begin{cases} (t_i + \frac{1}{2} - b_i)(2b_i - 1), & \text{if } b_i \leq 1 \\ (t_i + \frac{1}{2} - b_i), & \text{if } b_i > 1 \end{cases}$$

Notice that since  $b_j$  is bounded above by 1, all bids above 1 are strictly dominated by 1 so the objective is to maximize  $(t_i + \frac{1}{2} - b_i)(2b_i - 1)$ . Taking FOC gives

$$2t_i + 1 - 4b_i + 1 = 0$$

which then implies that

$$b_i(t_i) = \frac{1}{2} + \frac{1}{2}t_i$$

so we have verified that the given strategies constitute a Nash Equilibrium.

- (b) A bidder with valuation  $t_i$  will win the object with probability  $t_i$  and receive expected surplus  $\frac{1}{2}t_i$  if they win. So the expected surplus for the bidder is  $\frac{1}{2}t_i^2$  which is exactly what it was in situation considered in class. This is because increasing everyone's valuations by  $\frac{1}{2}$  just increases each player's bid by  $\frac{1}{2}$  leaving payoffs unchanged.

## 8.2 Common Values Auction

- (a) We hypothesize that  $b_i(t_i) = \alpha + \beta t_i$ . Since those who value the object more highly should be willing to bid more we would expect that  $\beta > 0$ , so the bidder with the higher  $t$  should win the object. Hence conditional on winning the object the expected value to the bidder is

no higher than  $2t_i$  which goes to 0 as  $t_i$  goes to 0. So we would expect that  $\alpha = 0$ . So we begin with the conjecture that  $b_i(t_i) = \beta t_i$ . Given that player  $j$  is bidding according to this strategy the expected payoff to player  $i$  from bidding  $b_i$  is

$$\begin{aligned}
 & E\{(t_1 + t_2 - b_i)1(b_j < b_i)\} \\
 &= E\{(t_1 + t_2 - b_i)1(t_j < \frac{b_i}{\beta})\} \\
 &= E[(t_1 + t_2 - b_i)|t_j < \frac{b_i}{\beta}] \Pr(t_j < \frac{b_i}{\beta}) \\
 &= (t_i + \frac{b_i}{2\beta} - b_i) \frac{b_i}{\beta}
 \end{aligned}$$

Taking FOC gives

$$t_i + \frac{b_i}{\beta} - 2b_i = 0$$

which implies that

$$b_i = \frac{\beta}{2\beta - 1} t_i$$

So we have an equilibrium if  $\beta = \frac{\beta}{2\beta - 1}$ . That is, if  $\beta = 1$ .

Therefore, the bidding strategies  $b_i(t_i) = t_i$  for  $i = 1, 2$  constitute a BNE.

- (b) The equilibrium bid,  $t_i$ , is lower than the bid in question 1, which was  $t_i + \frac{1}{2}$ . This is because, while the unconditional expected value of the object is  $t_i + \frac{1}{2}$  the expected value of the object *conditional* on bidder  $i$  winning the object is lower than  $t_i + \frac{1}{2}$ . In the symmetric Nash Equilibrium with increasing bidding functions the player who observes the higher signal will win the object. Hence, if player  $i$  wins the object they know that  $t_j < t_i$ , and the expected value of the object is then  $t_i + E[t_j|t_j < t_i] = t_i + \frac{1}{2}t_i < t_i + \frac{1}{2}$  whenever  $t_i < 1$ . This is an example of the "winner's curse" in common value auctions: winning the object reveals negative information about the values the other players place on the object, so in equilibrium the bidders must hedge their bids down.

### 8.3 Ex-post Equilibrium

- (a) The bidding strategies  $(b_1(t_1), \dots, b_I(t_I))$  form an ex-post Bayesian Nash Equilibrium if for all  $i, t_i$ ,

$$u_i(b_i(t_i), t_i, t_{-i}, b_{-i}(t_{-i})) \geq u_i(b_i, t_i, t_{-i}, b_{-i}(t_{-i})) \text{ for all } b_i, t_{-i}.$$

That is, we have an ex-post equilibrium if, after all other bidders' types have been revealed no player would want to change their action.

- (b) Taking expectation over  $t_{-i}$  in an ex-post BNE we have that for all  $i, t_i$

$$E_{t_{-i}}[u_i(b_i(t_i), t_i, t_{-i}, b_{-i}(t_{-i}))] \geq E_{t_{-i}}[u_i(b_i, t_i, t_{-i}, b_{-i}(t_{-i}))] \text{ for all } b_i$$

so an ex-post equilibrium is also an interim equilibrium. We established on the previous problem set that interim and ex-ante BNE are equivalent.

- (c) First note that since we have private values and an equilibrium in weakly dominated strategies, the equilibrium  $b_i(t_i) = t_i$  is an ex-post equilibrium. However, the first price auction does not have an ex-post equilibrium. To see this, consider any type profile where one player, WLOG player 1, has a strictly higher valuation than any other player. In any equilibrium any player who wins the object with positive probability must be bidding no higher than their valuation. We cannot have an ex-post equilibrium where player 1 wins the object with probability less than 1, because then player 1 could increase their bid to  $\varepsilon$  above the winning bid and win the object with probability 1. Similarly, we cannot have an equilibrium where player 1 wins the object with probability 1 since then they could reduce their bid by  $\varepsilon$ , still win the object with probability 1 and increase their payoff. So there cannot exist an ex-post equilibrium in the first price auction.

#### 8.4 MWG 23.E.6

- (a) The socially efficient outcome is for both units of the good to be allocated to the agent with the higher value of  $\theta$ .
- (b) Consider the bidding strategy  $b_i = \alpha\theta_i$ . Suppose player  $j$  is playing this strategy. Then by bidding  $b_i$  the expected net gains from

trade to player  $i$  is

$$\begin{aligned}
 & (\theta_i - b_i) \Pr(b_j < b_i) + \Pr(b_i < b_j) E[b_j - \theta_i | b_j > b_i] \\
 &= (\theta_i - b_i) \frac{b_i}{\alpha} + (1 - \frac{b_i}{\alpha}) (\alpha E[\theta_j | \theta_j > \frac{b_i}{\alpha}]) \\
 &= (\theta_i - b_i) \frac{b_i}{\alpha} + (1 - \frac{b_i}{\alpha}) (\frac{\alpha + b_i}{2} - \theta_i)
 \end{aligned}$$

Taking FOC gives

$$(\theta_i - 2b_i) \frac{1}{\alpha} + \frac{1}{2} - \frac{1}{2} - \frac{b_i}{\alpha} + \frac{\theta_i}{\alpha} = \frac{-3b_i + 2\theta_i}{\alpha} = 0$$

so then  $b_i = \frac{2}{3}\theta_i$ .

So we have that both bidders bidding  $b_i = \frac{2}{3}\theta_i$  is a BNE.

NOTE: The wording of the question is unclear. This answer applies to the case where the player who submits the higher bid buys the other player's unit at the price of the higher bid.

- (c) In the above mechanism the both goods are allocated to the agent with the higher valuation, so the ex-post efficient allocation is reached in the Nash Equilibrium. Participation in the mechanism is clearly incentive compatible since the expected payoff from the strategy  $b_i = \theta_i$  is certainly non-negative for all  $\theta_i$  so the payoffs from the equilibrium strategy must be non-negative. In the M-S set-up one player is the Seller and the other is the Buyer, so the seller has an incentive to over-represent their value and similarly the buyer has an incentive to under-represent. So when the values are close together the reports will indicate that trade is inefficient. Here, however, both agents are identical in that they are both buyers and sellers and so will hedge their bids in the same way (in this case, each bidding  $\frac{2}{3}$  of the valuation) so the agent with the higher valuation will bid more and so win both objects, which is the efficient allocation.

## 8.5

- (a) It should be clear that the game has three equilibria: Two pure equilibria (*Opera, Opera*) and (*Baseball, Baseball*) as well as a mixed equilibria  $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$  where the row player goes to the Opera with probability  $\frac{2}{3}$  and the Baseball game with probability  $\frac{1}{3}$  and the column player goes to the Opera w.p.  $\frac{1}{3}$  and the Baseball game w.p.  $\frac{2}{3}$ .

- (b) We can get both pure strategies from the perturbed game immediately since Opera(Baseball) is the strict best response to Opera(Baseball) for any  $\varepsilon_1, \varepsilon_2$  when  $\delta < 1$ . Now we show that we can obtain the mixed equilibrium from the pure strategies of the  $\delta$  perturbed game. We can characterize the pure strategy equilibrium of the  $\delta$  perturbed game as a cut-point equilibrium where player 1 and 2 play strategies

$$s_i(\varepsilon_i) = \left\{ \begin{array}{l} O \text{ if } \varepsilon_i > \varepsilon_i^* \\ B \text{ otherwise} \end{array} \right\}.$$

It remains to find  $\varepsilon_1^*, \varepsilon_2^*$ .

Under the prescribed strategies player 2 choose  $O$  with probability  $\frac{1-\varepsilon_2^*}{2}$ , so the expected payoff from  $O$  to player 1 is

$$(2 + \delta\varepsilon_1)\frac{1 - \varepsilon_2^*}{2} + \delta\varepsilon_1\frac{\varepsilon_2^* + 1}{2} = 1 - \varepsilon_2^* + \delta\varepsilon_1$$

and from playing  $B$  is

$$\frac{1 + \varepsilon_2^*}{2}$$

so if we take  $\varepsilon_1^*$  to be the type that is indifferent between  $O$  and  $B$  then

$$\varepsilon_1^* = \frac{(1 - 3\varepsilon_2^*)}{2\delta}$$

Similarly, the payoff to player 2 from choosing  $O$  is

$$(1 + \delta\varepsilon_2)\frac{1 - \varepsilon_1^*}{2} + \delta\varepsilon_2\frac{1 + \varepsilon_1^*}{2} = \frac{1 - \varepsilon_1^*}{2} + \delta\varepsilon_2$$

and the payoff from  $B$  is

$$2\frac{1 + \varepsilon_1^*}{2} = 1 + \varepsilon_1^*$$

so  $\varepsilon_2^*$  must solve

$$\varepsilon_2^* = -\frac{(1 + 3\varepsilon_1^*)}{2\delta}$$

This gives two equations in two unknowns, and plugging in the value for  $\varepsilon_2^*$  gives

$$\varepsilon_1^* = \frac{(1 - 3\varepsilon_2^*)}{2\delta} = \frac{(1 + 3\frac{(1+3\varepsilon_1^*)}{2\delta})}{2\delta} = \frac{2\delta + 3 + 9\varepsilon_1^*}{4\delta^2}$$

and so

$$\varepsilon_1^* = -\frac{2\delta + 3}{9 - 4\delta^2}$$

and then

$$\varepsilon_2^* = -\frac{(1 + 3\varepsilon_1^*)}{2\delta} = \varepsilon_2^* = -\frac{(1 - 3\frac{2\delta+3}{9-4\delta})}{2\delta} = \frac{6\delta + 4\delta^2}{2\delta(9 - 4\delta^2)} = \frac{3 + 2\delta}{9 - 4\delta^2}$$

Finally, taking  $\delta$  to 0 gives

$$\begin{aligned}\lim_{\delta \rightarrow 0} \varepsilon_1^* &= -\frac{1}{3} \\ \lim_{\delta \rightarrow 0} \varepsilon_2^* &= \frac{1}{3}\end{aligned}$$

And so the fraction of time that Players 1 and 2 chooses  $O$  must converge to  $\frac{1-\varepsilon_1^*}{2} = \frac{2}{3}$  and  $\frac{1-\varepsilon_2^*}{2} = \frac{1}{3}$ . So we get that the mixed equilibrium is the limit of the pure equilibrium of the perturbed game as desired.