Microeconomic Theory 501b
Problem Set 9: Suggested Solutions

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1 Bilateral Trading

(a) Trade is ex post efficient if $v \geq c$. The surplus is given by the expression

$$E[v - c|v > c]$$

which can be calculated from

$$\begin{aligned} E[v - c|v > c] &= \int_0^1 \int_0^v (v - c) dc dv \\ &= \int_0^1 \frac{1}{2} v^2 dv \\ &= \frac{1}{6} \end{aligned}$$

(b) The Bayesian Game is given by:

Players: $\{B, S\}$.
Types: $t_B = v \in [0, 1]$, $t_S = c \in [0, 1]$.
Actions: $p_i \in \mathbb{R}_+$ $i = B, S$.
Prior: $Pr(v \leq x) = x, Pr(c \leq x) = x$ independent.
Payoffs:

$$g_B(v, p_B, p_S) = \begin{cases} v - \frac{1}{2}(p_B + p_S) & p_B \geq p_S \\ 0 & \text{otherwise} \end{cases}$$

$$g_S(c, p_B, p_S) = \begin{cases} \frac{1}{2}(p_B + p_S) - c & p_B \geq p_S \\ 0 & \text{otherwise.} \end{cases}$$

A Bayesian Nash Equilibrium is a strategy $s_i : [0, 1] \to \mathbb{R}_+$ for each player B, S such that, for all i, $t_i \in [0, 1]$ and $a_i \in \mathbb{R}_+$

$$E_{t_{-i}}[g_i(t_i, s_i(t_i), s_{-i}(t_{-i}))] \geq E_{t_{-i}}[g_i(t_i, a_i, s_{-i}(t_{-i}))]$$

(c) We need to show that each player is playing a best response given the strategies of the other player. Consider first the buyer, given the sellers strategy the buyers payoff function is

$$u_B(p_B, v) = \begin{cases} 0 & p_B < p \\ p(v - \frac{p_B+p}{2}) & p_B \geq p. \end{cases}$$

Consider the term $p(v - \frac{p_B+p}{2})$. It is easy to see that for $v < p$ this term is always (weakly) negative (for $p_B \geq p$) and consequently *a* best response is to play $p_B = 0$. Further, for $v \geq p$ the term is always weakly greater than zero, but is decreasing in p_B so long as $p_B \geq p$. Consequently a best response is to play $p_B = p$. Therefore, we have shown that the strategy of B is a best response to the strategy of S. Similar logic implies that the strategy of S is a best response to the strategy of B and consequently we have a BNE.

Note that this works for all values of p in $[0, 1]$ and clearly this is not ex post efficient.

(d) Let

$$p_B(v) = \alpha v + \beta$$
$$p_S(c) = \gamma c + \delta$$

Expected revenues for the buyer are

$$E(\pi) = E[v - \frac{1}{2}(p_B + \gamma c + \delta)|c \leq \frac{p_B - \delta}{\gamma}]p(c \leq \frac{p_B - \delta}{\gamma}) \quad (1)$$

$$= \frac{2v - p_B - \delta - \gamma E(c|c \leq \frac{p_B-\delta}{\gamma})}{2}\frac{p_B - \delta}{\gamma} \quad (2)$$

$$= \frac{2v - p_B - \delta - \gamma\frac{p_B-\delta}{2\gamma}}{2}\frac{p_B - \delta}{\gamma} \quad (3)$$

taking the first order condition gives us

$$p_B = \frac{2}{3}v + \frac{1}{3}\delta$$

Similarly for the seller we get

$$p_S = \frac{2}{3}c + \frac{1}{3}\alpha + \frac{1}{3}\beta.$$

Therefore we know that $\alpha = \frac{2}{3}$ and $\gamma = \frac{2}{3}$ and that leaves us to solve

$$\beta = \frac{1}{3}\delta$$
$$\delta = \frac{2}{9} + \frac{1}{3}\beta$$

which gives $\delta = \frac{1}{4}$ and $\beta = \frac{1}{12}$.

2 Gibbons 4.3

(a) In a separating equilibrium there is no updating of beliefs on the equilibrium path. Therefore, if the receiver observes R she will play d. Now, we must specify beliefs after L to ensure that neither type 1 nor type 2 of the sender wish to deviate (we can specify of equilibrium path beliefs arbitrarily given the definition of PBE). Given d type 1 strictly prefers to play R and therefore the actions of player 2 at L are irrelevant. However, type 2 will only wish to play R if L will result in u with probability greater than $\frac{1}{3}$. Player 2 will only play u with a probability greater than $\frac{1}{3}$ if her beliefs $\mu(t_1|L) \geq \frac{1}{3}$. Therefore our Bayes Nash Equilibrium is $\{a_S(t_1) = R, a_S(t_2) = R, a_R(R) = d, a_R(L) = u, \mu(t_1|R) = 0.5, \mu(t_1|L) \geq \frac{1}{3}\}$.

(b) Again, on the equilibrium path beliefs are given by the prior and consequently the receiver will play u following L. Given this action, L is a best response for both t_1 and t_2, regardless of the actions that the receiver will take if R is observed. For t_3, however, we require that $a_R(R) = u$. This will be the case so long as $\mu(t_3|R) \leq \frac{1}{2}$. Therefore the BNE is $\{a_S(t_1) = L, a_S(t_2) = L, a_S(t_3) = L, a_R(L) = u, a_R(R) = u, \mu(t_1|L) = 0.33, \mu(t_2|L) = 0.33, \mu(t_3|L) \geq \frac{1}{2}\}$.

3 Gibbons 4.5

(a) First, let us check that there is no pooling equilibrium on L. On the equilibrium path, the receiver will play u. Given this, t_2 will always prefer to play R. Therefore there is only one pooling equiibrium, which is as above.

Next suppose $a_S(t_1) = R, a_S(t_2) = L$. If this is the case the receiver will play $a_R(R) = u, a_R(L) = d$. Given these choices t_1 is not playing a best response and therefore this is not an equilibrium.

Finally suppose $a_S(t_1) = L, a_S(t_2) = R$. Then $a_R(R) = d, a_R(L) = u$. Given this type t_1 is not playing a best response and therefore this i not an equilibrium.

Therefore, there is only one equilibrium - the pooling equilibrium.

(b) Here, we cannot have a complete separating equilibrium, only a partially separating equilibrium. There are six of these possible equilibria to consider. However, we can rule out a few because t_1 will not play R because it is strictly dominated by L. Therefore $a_S(t_1) = L$. Therefore, we need only consider three separating equilibrium.

First, suppose $a_S(t_2) = L, a_S(t_3) = R$. Then the receiver plays $a_R(L) = u$ and $a_R(R) = d$. Given this, all types are playing a best response and we therefore have an equilibrium.

Second, suppose $a_S(t_2) = R, a_S(t_3) = R$. We know that $a_R(R) = u$ and it follows that the best response for t_2 is L regardless of the action of the receiver at R. Therefore this is not an equilibrium.

Finally, suppose $a_S(t_2) = R, a_S(t_3) = L$. Then the same argument above holds and this is not an equilibrium.

Therefore there is only one separating equilibrium.

4 Gibbons 4.14

If there is a cutoff d^* such that the defendant will settle if and only if $d > d^*$, then the plaintiff can infer that the amount of damages will be less than d^*. Therefore, the plaintiff expects the value of the damages to be $\frac{d^*}{2}$ so the plaintiff goes to trial if

$$d^* > 2c. \tag{4}$$

Given an offer s and a probability p the defendant accepts if

$$pd \geq s \Rightarrow d^* = \frac{s}{p}. \tag{5}$$

Given $s > 2c$, we need to check all possible values for p. Suppose $p = 1$ then we have $d^* = s$ from (5). Therefore we have that the plaintiff will go trial if $s > 2c$ (from 4) and therefore we have an equilibrium with $p = 1$ and $d^* = s$. Now suppose that $p \in (0, 1)$, this requires that the plaintiff be indifferent between going to trial and not going to trial which implies that $d^* = 2c$, but this then implies that $p = \frac{s}{2c} > 1$ a contradiction.

Given $s \leq 2c$. Suppose $p = 1$ then from above we know that $d^* = s$ which implies that the plaintiff will only go to trial if $s > 2c$ which implies that $p = 0$ a contradiction. Suppose $p = 0$ then the defendant will never accept the offer and consequently $d^* = 1$, which implies that the plaintiff will go to trial for all $c < \frac{1}{2}$ which implies $p = 1$ a contradiction. Suppose $p \in (0, 1)$, then $d^* = \frac{s}{p}$ and $d^* = 2c$ which implies that $p = \frac{s}{2c} \leq 1$. Therefore we have a PBE with $p = \frac{s}{2c}$ and $d^* = 2c$.

For the whole game. Suppose that $c < \frac{1}{3}$, suppose that the plaintiff makes an offer $s > 2c$ then the defendant will reject the offer if $d < s$ and the plaintiff expected payoff will be $\frac{s}{2}(s - c) + s(1 - s) - c$. This is maximized by setting $s = 1 - c$. This is only possible if $1 - c > 2c$ which implies that $c < \frac{1}{3}$ which is OK in our case. Therefore this strategy yields a payoff of $\frac{(1-c)^2}{2}$. For $s \leq 2c$ the payoff is determined entirely by chance and given by $\frac{s}{2c}(1 - \frac{s}{p}) = (1 - 2c)s$ which is maximized by setting $s = 2c$ which gives payoff $(1 - 2c)2c = 2c - 4c^2$. You can show that it is always better to choose $s = 1 - c$ in this case.

Suppose now that $c > \frac{1}{3}$ then, as we saw above for $s > 2c$ it is not possible to reach the maximum, payoff, however, setting $s = 2c$ yields a payoff of $2c - 4c^2$ which is the best you can do. Therefore this is the first move in the PBE.

5 Many Types

(a) i. In a separating equilibrium, the firms knows the agents type in equilibrium and will therefore pay a wage equal to the ability of that type. Let the types be indexed by $k = 0, \dots K$ so that the ability of type k is $\frac{K+k}{K}$ The IC constraints for the worker them imply

$$a_k - \frac{e_k}{a_k} \geq a_j - \frac{e_j}{a_k}, \ \forall j \neq k$$

which can be simplified to

$$a_k(a_k - a_j) \geq e_k - e_j.$$

We can rewrite this equation for higher and lower types

$$e_k - e_j \leq a_k \left(\frac{k-j}{K}\right), j < k$$

$$e_j - e_k \geq a_k \left(\frac{j-k}{K}\right), j > k.$$

Now, our aim is to work out the 'gap', $e_j - e_k$, that is required between any two effort levels in equilibrium. Form the two equations above, for $j > k$ we require that

$$a_j \left(\frac{j-k}{K}\right) \geq e_j - e_k \geq a_k \left(\frac{j-k}{K}\right). \tag{6}$$

The LHS inequality implies that the type j does not wish to mimic the type k and the RHS implies that k does not wish to mimic j.

The aim, therefore, is to find effort levels e_i for $i = 0, \dots, K$ to minimize

$$\sum_{k=0}^{K} p_k e_k$$

subject to (6). The problem is going to be difficult if we cannot simplify the constraint set. Fortunately it turns out that all the constraints in (6) are satisfied if the local upward IC holds with equality.[1] To see this suppose that the local upward IC holds for some $j > k$, that is

$$e_k - e_{k-1} = a_{k-1}\left(\frac{1}{K}\right), \ \forall k > 0 \tag{7}$$

[1] Here I mean the IC that implies you do not wish to pretend to be the type higher than yourself. This should come as no surprise given the two player case.

Then consider the LHS inequality in (6), using (7) we get

$$\begin{aligned} e_j - e_k &= (e_j - e_j - 1) + \ldots + (e_{k+1} - e_k) \\ &= \frac{1}{K}(a_{j-1} + a_{j-2} + \ldots + a_k) > a_k \frac{j-k}{K}. \end{aligned}$$

Next consider the RHS of the inequality, again using (7) we get

$$\begin{aligned} e_j - e_k &= (e_j - e_j - 1) + \ldots + (e_{k+1} - e_k) \\ &= \frac{1}{K}(a_{j-1} + a_{j-2} + \ldots + a_k) < a_j \frac{j-k}{K}. \end{aligned}$$

Consequently we conclude that if the local upward IC holds then they all do.

Next we need to show that the local upward ICs do indeed bind at the optimum. That is, we need to show that we could decreases the objective function if (7) is a strict inequality. Suppose that for some k

$$e_k - e_{k-1} > a_{k-1} \frac{1}{K}$$

then there exists some small ϵ such that

$$e_k - \epsilon - e_{k-1} > a_{k-1} \frac{1}{K}$$

and, further, if we decrease education levels for agent $j > k$ to be $e_j - \epsilon$ all IC will still hold. However, it is clear that the objective function has decreased.

Consequently we conclude that the solution to our problem is given when the complete set of local upward ICs hold with equality and therefore our solution is given by a set of linear equalities. Finally, to solve the equalities we impose that $e_0 = 0$ which will have to old in equilibrium and we can solve

$$e_j = \frac{1}{K} \sum_{k=0}^{j-1} a_k$$

to back out all the education levels.

ii. If the first l types are pooling at effort level e^* then the firm must pay them there expected productivity, or

$$w(e^*) = \frac{\sum_{k=0}^{l} p_k a_k}{\sum_{k=0}^{l} p_k}.$$

Now set the off equilibrium beliefs to be $p(a = 1 | e \neq e^*) = 1$ then it must be the case that the lowest type is the most likely to deviate and therefore we have

$$\frac{\sum_{k=0}^{l} p_k a_k}{\sum_{k=0}^{l} p_k} - e^* \geq 1. \tag{8}$$

This condition ensures that all individuals in $k \leq l$ are happy to remain in the pool rather than playing some e for which there does not exist a such that $e^*(a) = e$. Next we must ensure that the pooled do not wish to mimic the separated. This will be the case if the highest pooled type does not wish to mimic the lowest separated type, therefore

$$w(e^*) - \frac{e^*}{a_l} \geq a_{l+1} - \frac{e^*(l+1)}{a_l}. \tag{9}$$

Finally, we must ensure that the lowest separating type does not wish to pool, therefore

$$a_{l+1} - \frac{e^*(l+1)}{a_{l+1}} \geq w(e^*) - \frac{e^*}{a_{l+1}}.$$

Now, let this final inequality hold with equality (which we can do because we are looking for *an* equilibrium) we get

$$a_{l+1}(a_{l+1} - w(e^*)) = E^*(l+1) - e^*, \tag{10}$$

which in turn implies that (9) is satisfied.

Therefore pick any $e^* \in \left[0, \frac{\sum_{k=0}^{l} p_k a_k}{\sum_{k=0}^{l} p_k} - 1\right]$. This determines $e^*(l+1)$ from (10). Then we can choose the separating levels as in the question above,.

iii. A separating equilibrium will consist of a function $e(a)$ and a wage funcotin $w(e(a))$ hy which workers are identified and get payed their productivity. Therefore we know that $w(e(a)) = a$. IC requires that each agent a finds it optimal to attain eduction $e(a)$. Therefore it must be the case that

$$a = \arg\max_{a'} \in A \left[a' - \frac{e(a')}{a}\right].$$

The FOC from this gives us

$$1 - \frac{e'(a)}{a} = 0 \Rightarrow e'(a) = a \ \forall a.$$

Integrating both sides gives us

$$e(a) = \frac{1}{2}a^2 + C.$$

Finally, we have one boundary condition which is that the lowest type has no incentive to become educated and so $e(1) = 0$ and tehrefore $C = -\frac{1}{2}$. Therefore we have

$$e(a) = \frac{1}{2}a^2 - \frac{1}{2}$$