This is a closed-book exam. The exam lasts for 180 minutes. Please write clearly and legibly. Be especially careful in the definition of the game, the payoff function and the equilibrium notions. The allocated points are also a good indicator for your time budget. Please record the answer for each question in a separate bluebook.
1. (10 minutes) Suppose the government (through its agency, the Internal Revenue Service—IRS) wants to ensure compliance with the tax code. A taxpayer, in filling out a tax return, has two choices, be truthful (T) or lie (L). The IRS can audit the return (A) or not (N) to determine whether the taxpayer was truthful or had lied on his return. The taxpayer’s payoffs are given by $u_t(T, A) = 2$, $u_t(T, N) = 3$, $u_t(L, A) = -2$, and $u_t(L, N) = 4$. The government’s payoffs are given by $u_g(T, A) = -1$, $u_g(T, N) = 1$, $u_g(L, A) = 0$, and $u_g(L, N) = -2$. Suppose the IRS makes the audit decision after the taxpayer has submitted his return (for simplicity, suppose the IRS makes the decision before reading the return).

(a) What is the normal form of this game?
(b) What are the Nash equilibria of this game?

2. (30 minutes) Suppose the IRS in question 1 can commit to a probability of auditing the return before the taxpayer fills out his return, and that the taxpayer observes this probability. (This commitment may involve the early hiring of IRS auditors, for example.)

(a) What is the normal form of this game?
(b) What is the backward induction solution of this game? Is it unique?
(c) Is there a Nash equilibrium in which the IRS audits for sure? If so, describe it. If not, why not.
(d) If the government has a choice between committing early to the probability of audit and not committing, will it commit early? Why or why not.
(e) Will an announcement by the IRS of intended auditing probabilities serve the same function as the commitment?

3. (20 minutes) Suppose the game from question 1 is infinitely repeated. Moreover, suppose that information becomes available at the end of the period that independently tells the IRS whether the taxpayer was truthful in filling out his return (too late to punish the taxpayer in that period). That is, the IRS and the taxpayer repeatedly play the game described in the previous question, with the result of period $\tau$ play (including the truthfulness of the taxpayer) observed before the period $\tau + 1$’s play. Give bounds on the discount factor $\delta$ so that there is a subgame perfect equilibrium in which the taxpayer is truthful and the IRS does not audit. Be sure to fully describe the strategy profile (you can be informal as long as there is no ambiguity).

4. (20 minutes) Consider the following optimal pricing problem with quality-differentiated products. There is a continuum of consumers whose preferences are given by

$$u = \theta v(q) - t(q)$$
where
\[ v(q) = \frac{1 - (1 - q)^2}{2}. \]

The proportion of consumers with high valuation \( \theta_h \) is given by \( \lambda \) and the proportion of consumers with low valuation \( \theta_l \) is given by \( 1 - \lambda \). The monopolist has a constant marginal cost equal to \( c \) of producing quality \( q \), with \( 0 < c < \theta_l < \theta_h \).

(a) Derive the monopolist’s optimal menu subject to the participation constraint of the buyers, assuming for the moment that he can observe the type \( \theta \) of the buyer.

(b) Suppose now that \( \theta \) is not observable. Derive the monopolist’s optimal solution and carefully discuss which new constraints arise for the seller.

(c) Compare your answers from parts (a) and (b).

5. (30 minutes) Consider a first-price sealed-bid auction of an object between two risk-neutral bidders. Each bidder \( i \) (for \( i = 1, 2 \)) simultaneously submits a bid \( b_i \geq 0 \). The bidder who submits the highest bid receives the object and pays his bid; both bidders win with equal probability in case they submit the same bid. Before the auction takes place, each bidder \( i \) privately observes the realization of a random variable \( t_i \) that is drawn independently from a uniform distribution over the interval \([0, 1]\). The actual valuation of the object to bidder \( i \) is equal to \( t_i + 0.5 \). Therefore, the payoff of bidder \( i \) is given by

\[
u_i = \begin{cases} 
    t_i + 0.5 - b_i & \text{if } b_i > b_j \\
    \frac{1}{2} (t_i + 0.5 - b_i) & \text{if } b_i = b_j \\
    0 & \text{if } b_i < b_j
\end{cases}
\]

(a) Define the notion of a pure strategy and a pure strategy Bayesian Nash equilibrium for this bidding game.

(b) Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form \( b_i = \alpha t_i + \beta \)).

(c) What is the conditionally expected payoff of bidder \( i \) with type \( t_i \) in this equilibrium?

6. (40 minutes) Consider the first-price auction of (5), except that the actual valuation of the object to bidder \( i \) is now equal to \( t_i + t_j \ (j \neq i) \) and therefore the payoff of bidder \( i \) now becomes

\[
u_i = \begin{cases} 
    t_i + t_j - b_i & \text{if } b_i > b_j \\
    \frac{1}{2} (t_i + t_j - b_i) & \text{if } b_i = b_j \\
    0 & \text{if } b_i < b_j
\end{cases}
\]
(a) Given his own private type $t_i$, compute the expected value of the object for agent $i$. How does it compare to the expected value in (5)?

(b) Given that the other agent $j$ is using a linear bidding strategy $b_j = \alpha t_j + \beta$, compute the expected value of the object for agent $i$ given his type $t_i$ and given the bid $b_j = \alpha t_j + \beta$ of agent $j$.

(c) Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form $b_i = \alpha t_i + \beta$) using the conditional expected value you computed in (6b).

(d) Compare the equilibrium bid for any given type $t_i$ in this problem to that in problem (5). Interpret your result.

7. **(30 minutes)** Consider the following signalling game with two agents. Player 1 can either be weak or strong and this is private information to player 1. Player 1 and player 2 are then engaged in an extensive form game in which player 1 moves first and player 2 moves second. The extensive form is depicted below.

(a) Define the notion of perfect bayesian equilibrium for this game.

(b) Find all the perfect bayesian equilibria of this game.

(c) Define the notion of equilibrium domination (as an equilibrium refinement notion) for this game.

(d) Do all equilibria identified in (7b) pass the refinement test?