1. (30 minutes) Suppose the government (through its agency, the Internal Revenue Service—IRS) wants to ensure compliance with the tax code. A taxpayer, in filling out a tax return, has two choices, be truthful (T) or lie (L). The IRS can audit the return (A) or not (N) to determine whether the taxpayer was truthful or had lied on his return. The taxpayer’s payoffs are given by $u_t(T, A) = 2$, $u_t(T, N) = 3$, $u_t(L, A) = -2$, and $u_t(L, N) = 4$. The government’s payoffs are given by $u_g(T, A) = -1$, $u_g(T, N) = 1$, $u_g(L, A) = 0$, and $u_g(L, N) = -2$. Suppose the IRS makes the audit decision after the taxpayer has submitted his return (for simplicity, suppose the IRS makes the decision before reading the return).

(a) What is the normal form of this game?

(b) What are the Nash equilibria of this game?

(c) Suppose the game is infinitely repeated. Moreover, suppose that information becomes available at the end of the period that independently tells the IRS whether the taxpayer was truthful in filling out his return (too late to punish the taxpayer in that period). That is, the IRS and the taxpayer repeatedly play the game described in the previous question, with the result of period $\tau$ play (including the truthfulness of the taxpayer) observed before the period $\tau + 1$’s play. Give bounds on the discount factor $\delta$ so that there is a subgame perfect equilibrium in which the taxpayer is truthful and the IRS does not audit. Be sure to fully describe the strategy profile (you can be informal as long as there is no ambiguity).

Soln:

(a) The normal form is

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>2, -1</td>
<td>3, 1</td>
</tr>
<tr>
<td>$L$</td>
<td>-2, 0</td>
<td>4, -2</td>
</tr>
</tbody>
</table>
(b) The game has no pure strategy equilibria. To calculate the mixed strategy equilibrium, let \( q \) be the probability the taxpayer tells the truth (\( T \)) and \( p \) be the probability that the government audits (\( A \)). The taxpayer is indifferent if

\[
2p + 3(1 - p) = -2p + 4(1 - p)
\]

\[
\iff 3 - p = 4 - 6p
\]

\[
\iff p = 1/5.
\]

The government is indifferent if

\[
2q = q - 2(1 - q)
\]

\[
\iff 4q = 2
\]

\[
\iff q = 1/2.
\]

Hence, the unique Nash equilibrium is \( \sigma^* = (\frac{1}{2} \circ T + \frac{1}{2} \circ L, \frac{1}{5} \circ A + \frac{4}{5} \circ N) \).

(c) There are two natural profiles to consider. The first is Nash reversion. The payoff to the taxpayer of the stage game Nash equilibrium is \( 2/5 + 3 \times 4/5 = 14/5 \). In the Nash reversion profile, in each period, \( TN \) is played, with any deviation triggering permanent play of the stage game Nash equilibrium. After a deviation, play is trivially subgame perfect as the stage game Nash equilibrium is played in every period. The profile has a state automaton description, with \( \mathcal{W} = \{ w_{TN}, w^* \} \), initial state \( w^0 = w_{TN} \), output function \( f(w_{TN}) = TN \) and \( f(w^*) = \sigma^* \) defined in the last answer, and transitions

\[
\tau(w, a) = \begin{cases} 
  w_{TN}, & \text{if } w = w_{TN} \text{ and } a = TN \\
  w^*, & \text{otherwise.} 
\end{cases}
\]

It only remains to verify that no player wishes to deviate in state \( w_{TN} \). Since the government is playing a myopic best reply, and the static Nash equilibrium has a payoff of 0, the government has no incentive to deviate for all \( \delta \).

The taxpayer has no incentive to deviate iff

\[
3 \geq 4(1 - \delta) + \frac{\delta 14}{5}
\]

\[
\iff 15 \geq 20 - 20\delta + 14\delta
\]

\[
\iff 6\delta \geq 5 \iff \delta \geq \frac{5}{6}.
\]

Thus Nash reversion supports \( TN \) in every period if \( \delta \geq 5/6 \).

The other natural profile is to use \( LA \) as a punishment. Any deviation results in one period of \( LA \), followed by \( TN \). The automaton
description is \( W = \{ w_{TN}, w_{LA} \} \), initial state \( w^0 = w_{TN} \), output function \( f(w_a) = a \), and transitions

\[
\tau(w, a) = \begin{cases} 
  w_{TN}, & \text{if } w = w_{TN} \text{ and } a = TN \text{ or } w = w_{LA} \text{ and } a = LA \\
  w_{LA}, & \text{otherwise}.
\end{cases}
\]

The values are

\[
\begin{align*}
V_t(w_{TN}) &= 3, \quad V_g(w_{TN}) = 1 \\
V_t(w_{LA}) &= -2(1 - \delta) + 3\delta = 5\delta - 2, \\
\text{and } V_g(w_{LA}) &= 0 \times (1 - \delta) + 1\delta = \delta.
\end{align*}
\]

Since the government is always playing a myopic best reply, it has no incentive to deviate. The taxpayer will not deviate in \( w_{TN} \) if

\[
3 \geq 4(1 - \delta) + \delta(5\delta - 2) = 4 - 6\delta + 5\delta^2
\]

\[
\iff -5\delta^2 + 6\delta - 1 \geq 0
\]

\[
\iff (5\delta - 1)(1 - \delta) \geq 0,
\]

or \( \delta \geq 1/5 \).

We need to also check that the taxpayer will not deviate in \( w_{LA} \), which requires

\[
5\delta - 2 \geq 2(1 - \delta) + \delta(5\delta - 2) = 2 - 4\delta + 5\delta^2
\]

\[
\iff -5\delta^2 + 9\delta - 4 \geq 0
\]

\[
\iff (5\delta - 4)(1 - \delta) \geq 0,
\]

or \( \delta \geq 4/5 \). Thus this profile is a subgame perfect equilibrium for a larger range of \( \delta \) than Nash reversion.

---

2. (30 minutes) Suppose the IRS in question 1 can commit to a probability of auditing the return before the taxpayer fills out his return, and that the taxpayer observes this probability. (This commitment may involve the early hiring of IRS auditors, for example.) We are concerned here only with one-shot play of this game, and not the infinitely repeated version.

(a) What is the normal form of this game?

(b) What is the backward induction solution of this game? Is it unique?

(c) Is there a Nash equilibrium in which the IRS audits for sure? If so, describe it. If not, why not.

(d) If the government has a choice between committing early to the probability of audit and not committing, will it commit early? Why or why not.
(e) Will an announcement by the IRS of intended auditing probabilities serve the same function as the commitment?

**Soln:**

(a) The government commits to \( p \in [0, 1] \), the probability of \( A \), audit. The taxpayer chooses whether to cheat as a function of the probability committed to, i.e., \( s : [0, 1] \rightarrow \{T, L\} \). Hence, the strategy space for the government is \( S_g = [0, 1] \) and the strategy space for the taxpayer is the set of all (measurable) functions \( s : [0, 1] \rightarrow \{T, L\} \). The payoffs in the normal form are, for the taxpayer,

\[
    u_t(s, p) = pu_t(s(p), A) + (1 - p)u_t(s(p), N)
\]

and for the taxpayer,

\[
    u_g(s, p) = pu_g(s(p), A) + (1 - p)u_g(s(p), N).
\]

(b) The taxpayer has \( T \) as a best reply iff

\[
    2p + 3(1 - p) \geq -2p + 4(1 - p) \\
    \iff 3 - p \geq 4 - 6p \\
    \iff p \geq 1/5,
\]

and has \( L \) as a best reply iff \( p \leq 1/5 \). Let \( s^* \) denote any strategy for the taxpayer satisfying

\[
    s^*(p) = \begin{cases} 
    T, & \text{if } p > 1/5, \\
    L, & \text{if } p < 1/5. 
    \end{cases}
\]

Backward inducting, the government’s payoff is

\[
    u_g(s^*, p) = \begin{cases} 
    -2(1 - p), & \text{if } p < 1/5, \\
    1 - 2p, & \text{if } p > 1/5. 
    \end{cases}
\]

The government’s payoff is increasing in \( p \) for \( p < 1/5 \) and decreasing in \( p \) for \( p > 1/5 \), and is strictly larger in a neighborhood of \( 1/5 \) for \( p > 1/5 \) (since the taxpayer is truthful). While the taxpayer is indifferent between \( T \) and \( L \) when \( p = 1/5 \), the government strictly prefers the taxpayer to be truthful, which can be induced by any probability strictly larger than \( 1/5 \). Consequently, the unique backward induction solution is \((1/5, \hat{s})\), where \( \hat{s}(p) = s^*(p) \) for all \( p \neq 1/5 \) and \( \hat{s}(1/5) = T \).

(c) No, there is no such equilibrium: In such a putative equilibrium, the best response of the taxpayer must be to be truthful. In order to provide incentives for the government to audit for sure, its payoff must be lower if it audited with probability \( 1 - \varepsilon \). But, for small \( \varepsilon > 0 \), there
is no “threat” since $L$ leads to an increase in payoffs. (If the payoffs for the government were 1 after $TA$ and 3 after $TN$, then other calculations are unchanged, but now there is a Nash equilibrium in which the government audits for sure, supported by the specification that the taxpayer lies if the probability of audit is strictly less than 1.)

(d) The government will commit early because while its audit probability is the same, the taxpayer is truthful for sure with commitment (giving a payoff of $3/5$), and is randomizing without (giving a payoff of $-1/2 < 3/5$).

(e) An announcement cannot serve the same function, because the taxpayer understands that, if the taxpayer were to believe any announcement, the government has no incentive to carry out the announcement.

3. (40 minutes) Consider a first-price sealed-bid auction of an object between two risk-neutral bidders. Each bidder $i$ (for $i = 1, 2$) simultaneously submits a bid $b_i \geq 0$. The bidder who submits the highest bid receives the object and pays his bid; both bidders win with equal probability in case they submit the same bid. Before the auction takes place, each bidder $i$ privately observes the realization of a random variable $t_i$ that is drawn independently from a uniform distribution over the interval $[0, 1]$. The actual valuation of the object to bidder $i$ is equal to $t_i + 0.5$. Therefore, the payoff of bidder $i$ is given by

$$u_i = \begin{cases} t_i + 0.5 - b_i & \text{if } b_i > b_j \\ \frac{1}{2} (t_i + 0.5 - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

(a) Define the notion of a pure strategy and a pure strategy Bayesian Nash equilibrium for this bidding game.

(b) Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form $b_i = \alpha t_i + \beta$).

(c) What is the conditionally expected payoff of bidder $i$ with type $t_i$ in this equilibrium?

aa. The strategy for each $i$ is the bid function:

$$b_i : [0, 1] \rightarrow R_+,$$

and the payoffs are:

$$u_i(b_1, b_2; t_i) = \begin{cases} t_i + \frac{1}{2} - b_i & \text{if } b_i > b_j \\ \frac{1}{2} (t_i + \frac{1}{2} - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}.$$
A Bayesian Nash equilibrium is \( \{b^*_i(\cdot), b^*_j(\cdot)\} \) such that for \( i = 1, 2 \)

\[
\mathbb{E}_t[u_i(b^*_i(t_i), b^*_j(t_j) ; t_i)] \geq \mathbb{E}_t[u_i(b_i, b^*_j(t_j) ; t_i)].
\]

for all \( b_i \in \mathbb{R}_+ \) and all \( t_i \). The expectation is with respect to \( t_j \) which is distributed according to the uniform distribution.

bb. Let us conjecture that the bid function is of the form:

\[
b_i(t_i) = \alpha_i + \beta_i t_i.
\]

Player \( i \) with stype \( t_i \) chooses his bid to maximize his expected payoff:

\[
b_i(t_i) = \arg \max_b \int_0^{b - \alpha_j} (t_i + \frac{1}{2} - b) dt_j = \max_b (t_i + \frac{1}{2} - b) \frac{b - \alpha_j}{\beta_j},
\]

and the first order conditions are

\[
(t_i + \frac{1}{2} - b) \frac{1}{\beta_j} - \frac{b - \alpha_j}{\beta_j} = 0.
\]

Rearranging we get:

\[
b = \frac{1}{2}(t_i + \frac{1}{2} + \alpha_j).
\]

Assuming symmetry we substitute \( b_i(t) = \alpha + \beta t \):

\[
\alpha + \beta t_i = \frac{1}{2}(t_i + \frac{1}{2} + \alpha).
\]

Equating the coefficients we obtain

\[
b_i(t) = \frac{1}{2} + \frac{1}{2} t_i.
\]

cc. The expected net payoff for a type \( t \) is:

\[
\int_0^t [t + \frac{1}{2} - (\frac{1}{2} + \frac{1}{2} t)] dt_j = \frac{t^2}{2}.
\]

4. (40 minutes) Consider the first-price auction of (3), except that the actual valuation of the object to bidder \( i \) is now equal to \( t_i + t_j \) \((j \neq i)\) and therefore the payoff of bidder \( i \) now becomes

\[
u_i = \begin{cases} 
  t_i + t_j - b_i & \text{if } b_i > b_j \\
  \frac{1}{2} (t_i + t_j - b_i) & \text{if } b_i = b_j \\
  0 & \text{if } b_i < b_j 
\end{cases}
\]
(a) Given his own private type \( t_i \), compute the expected value of the object for agent \( i \). How does it compare to the expected value in (3)?

(b) Given that the other agent \( j \) is using a linear bidding strategy \( b_j = \alpha t_j + \beta \), compute the expected value of the object for agent \( i \) given his type \( t_i \) and given the bid \( b_j = \alpha t_j + \beta \) of agent \( j \).

(c) Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form \( b_i = \alpha t_i + \beta \)) using the conditional expected value you computed in (4b).

(d) Compare the equilibrium bid for any given type \( t_i \) in this problem to that in problem (3). Interpret your result.

aa. The strategy for each \( i \) is the bid function:

\[
b_i : [0, 1] \rightarrow R_+.
\]

and the payoffs are:

\[
u_i(b_1, b_2; t_i, t_j) = \begin{cases} t_i + t_j - b_i & b_i > b_j \\ \frac{1}{2}(t_i + t_j - b_i) & b_i = b_j \\ 0 & b_i < b_j \end{cases}
\]

The expected value is

\[
E_{t_j}[t_i + t_j] = t_i + \frac{1}{2},
\]

and the unconditional gross expected value is the same as before.

bb. The expected gross payoff is:

\[
t_i + \frac{b_j - \alpha_i}{\beta_j}.
\]

cc. The expected net payoff from bidding \( b \) is given by:

\[
\max_b \left\{ \int_0^{\frac{b}{\alpha}} [t_i + t_j - b]dt_j \right\} = \max_b \left\{ (t_i - b) \frac{b - \alpha}{\beta} + \frac{1}{2} \left( \frac{b - \alpha}{\beta} \right)^2 \right\}
\]

and the first order conditions are

\[
(t_i - b) \frac{1}{\beta} - \frac{b - \alpha}{\beta} + \left( \frac{b - \alpha}{\beta} \right) \frac{1}{\beta} = 0.
\]

We rearrange:

\[
b = \frac{\alpha(1 - \beta) - \beta t_i}{1 - 2\beta}.
\]

By symmetry \( b = \alpha + \beta t_i \). Compare coefficients to obtain \( b_i = t_i \).
dd. The equilibrium conditional expected payoff to bidder $i$ is:

$$\int_0^{t_i} (t_i + t_j - t_i) dt_j = \frac{1}{2} t_i^2.$$ 

The bid in the common value is lower since $\frac{1}{2} t_i + \frac{1}{2} > t_i$ for all $t_i < 1$. The bids are equal for $t_i = 1$. The intuition is that in the common value auction, player $i$'s expected payoff depends on player $j$’s signal $t_j$. While the expected value $E_j(t_i + t_j) = t_i + \frac{1}{2}$ in both auctions, player $i$ knows that if he wins, then player $j$’s bid was lower than his. This would mean that player $j$’s signal was lower than his so conditional on winning the value is lower than $t_i + \frac{1}{2}$. This drives down the bids compared to the private value auction. This situation is sometimes referred to as the "winner’s curse".

5. (40 minutes) Consider the following signalling game with two agents. Player 1 can either be weak or strong and this is private information to player 1. Player 1 and player 2 are then engaged in an extensive form game in which player 1 moves first and player 2 moves second. The extensive form is depicted below.

(a) Define the notion of perfect bayesian equilibrium for this game.

(b) Find all the perfect bayesian equilibria of this game.

(c) Define the notion of equilibrium domination (as an equilibrium refinement notion) for this game.

(d) Do all equilibria identified in (5b) pass the refinement test?

aa. A PBE is $\{a_1^1(t_1), a_2^1(a_1), \mu_2^1(t_1)\}$ such that sequential optimality is satisfied for all agents and the belief of player 2 satisfies Bayes’ law on the equilibrium path.

bb. There do not exist separating PBE in the Beer-Quiche example. There are two classes of pooling PBE:

$$\{Q, Q; d|B, n|Q; \mu(w|Q) = 0.1, \mu(w|B) \geq 0.5\}$$

and

$$\{B, B; n|B, d|Q; \mu(w|B) = 0.1, \mu(w|Q) \geq 0.5\}$$

cc. The relevant definition follows

**Definition 1 (Equilibrium Domination)** Given a PBE, the message $m$ is equilibrium-dominated for type $t$, if for all possible assessments $p(t|m)$:

$$u(m^*(t), t) > u(m, t).$$
Definition 2 (Intuitive Criterion)  If the information set following \( m \) is off the equilibrium path and \( m \) is equilibrium dominated for type \( t \), then

\[ p(t|m) = 0. \]

1. dd. It is immediate to see that only the pooling equilibria

\[ \{B, B; n | B, d | Q ; \mu (w | B) = 0.1, \mu (w | Q) \geq 0.5\} \]

satisfies the equilibrium domination test because the action \( B \) is equilibrium dominated by \( Q \) for the weak agent and hence the beliefs of the equilibrium path should be:

\[ \mu (w | B) = 0. \]