1. **Nonlinear Pricing.** Consider the problem of an optimal menu when \( v(\theta, q) = \theta \sqrt{q} \) and \( c(q) = q \) and the distribution is given by the uniform distribution on the unit interval.

(a) Compute the revenue maximizing direct mechanism, which associates to every reported type \( \theta \) a pair \((q(\theta), t(\theta))\) of quantities \(q(\theta)\) and prices \(t(\theta)\), or \( \theta \mapsto (q(\theta), t(\theta)) \)

Answer

The problem of the seller is (as we saw in class)

\[
\max_{q(\theta)} \int_0^\theta \left\{ \sqrt{q(\theta)} \left[ \theta - \frac{(1 - F(\theta))}{f(\theta)} \right] - c(q(\theta)) \right\} dF(\theta)
\]

Which, in this case, implies

\[
\max_{q(\theta)} \int_0^\theta \left\{ \sqrt{q(\theta)} [2\theta - 1] - q(\theta) \right\} dF(\theta)
\]

First, note that \( q(\theta) > 0 \) if and only if \( \left[ \theta - \frac{(1 - F(\theta))}{f(\theta)} \right] = 2\theta - 1 > 0 \iff \theta > 1/2 \). We can conclude immediately that \( q(\theta) = t(\theta) = 0 \) if \( \theta \leq 1/2 \).

Now, if \( \theta > 1/2 \), then, maximizing the above expression pointwise, we get:

\[
\frac{1}{2} q(\theta)^{-1/2} [2\theta - 1] - 1 = 0
\]

\[
q(\theta) = \frac{(2\theta - 1)^2}{4}
\]

Now, remember that the transfer \( t(\theta) \) must satisfy the (IC) restriction:
\( t'(\theta) = \frac{\partial v(\theta, q)}{\partial q} . q'(\theta) \). Therefore if \( \theta > 1/2 \), then,

\[
t'(\theta) = \theta \frac{1}{2 \sqrt{q(\theta)}} q'(\theta)
\]

\[
= \theta
\]

\[
\implies t(\theta) = \frac{\theta^2}{2} - \frac{1}{8}
\]

where the last line comes from integrating \( t'(\cdot) \) from \( 1/2 \) to \( \theta \). Therefore the direct mechanism that maximizes expected profit is
\[(q(\theta), t(\theta)) = \left(\frac{(2\theta - 1)^2}{4}, \frac{\theta^2}{2} - \frac{1}{8}\right) \cdot \text{1[}\theta > 1/2]\]

where \(1[.]\) is the indicator function.

(b) Translate the direct mechanism into an indirect mechanism, in particular to a nonlinear pricing mechanism \((q, t(q))\) which associates to every \(q\) a \(t(q)\), or \(q, \rightarrow t(q)\).

Answer First of all, let’s invert \(q(\theta)\). If \(\theta > 1/2\), then \(q = \frac{(2\theta - 1)^2}{4} \implies \theta = \sqrt{q} + \frac{1}{2}\). Therefore, we must have that \(t(q) = \frac{(\sqrt{q} + \frac{1}{2})^2}{2} - \frac{1}{8}\). Hence,

\[t(q) = \frac{q + \sqrt{q}}{2}\]

and the direct mechanism is given by

\[(q, t(q)) = \left(q, \frac{q + \sqrt{q}}{2}\right), \text{ where } q \in [0, \frac{1}{4}]\]

(c) What can you say about \(t(q)/q\), i.e. the price per quantity as the quantity increases.

Answer We have that

\[\frac{t(q)}{q} = \frac{q + \sqrt{q}}{2q} = \frac{1}{2} + \frac{1}{2\sqrt{q}}\]

which is decreasing in \(q\).

2. **Competition with Nonlinear Pricing.** Suppose now that there are a large (infinite) number of sellers who can offer nonlinear pricing contracts \((q, t(q))\) to the customer. The contracts are offered simultaneously and each firm seeks to maximize the expected revenue from its offering. Describe the symmetric equilibrium contract and compare it with the nonlinear pricing result with the monopolist analyzed in (1).

Answer Because there are a large number of sellers, we must have zero profit in equilibrium, otherwise one firm would have a profitable deviation offering a contract with strictly positive profit. Therefore, \(t = c(q) =\)
Now let’s look at the customer’s problem. She chooses the contract that maximizes her utility:

\[
\max_q \theta \sqrt{q} - t = \max_q \theta \sqrt{q} - q \\
\implies \frac{\theta}{2\sqrt{q}} - 1 = 0 \\
\implies t = q = \frac{\theta^2}{4}
\]

Therefore the equilibrium contracts when there is competition with nonlinear pricing is \((q, t) = (\frac{\theta^2}{T}, \frac{\theta^2}{T})\). Notice that, contrary to the equilibrium observed in exercise (1), we have that:

- Every customer is able to buy the good (there is no rationing) and obtain a utility of \(\theta \sqrt{q} - t = \theta \sqrt{\frac{\theta^2}{T}} - \frac{\theta^2}{T} = \frac{\theta^2}{T} > 0\).
- Even those we were able to buy the good before \((\theta > 1/2)\) are better off now since

\[
\theta \sqrt{\left(\frac{2\theta - 1}{4}\right)^2} - \left(\frac{\theta^2}{2} - \frac{1}{8}\right) < \frac{\theta^2}{4} \text{ if } \theta > 1/2
\]

- Actually, this equilibrium is efficient because the social welfare (in this quasi-linear world) is given by \(\int (\theta \sqrt{q} - t + t - q) \, dF(\theta)\) and, so, it is maximized in the competitive case (as you can see from the agent problem). Moreover, all the surplus goes to the agent.

Interestingly, in the competitive equilibrium the first best is achieved regardless of the adverse selection. This is known as the Irrelevance Result (see Salanie’s book).

Remark Note that if you try to make the expected profit equal zero, i.e., find \(q(\theta)\) such that \(\int \left\{ \sqrt{q(\theta)} [2\theta - 1] - q(\theta) \right\} dF(\theta) = 0\), and then try to find the corresponding \(t(\theta)\), then you may run into the problem that for some \(\theta\) this contract does not give an ex-post zero profit for the firm. So the firms would have an incentive to deviate offering a different menu of contracts. That is why we need the condition that \(t(\theta) = c(q(\theta))\) for all \(\theta\).
3. **Nonlinear Pricing.** Now generalize the analysis to \( \theta v(q) \) with \( v'(q) > 0 \) and \( v''(q) < 0 \) and \( F(\theta) \), maintain the linear cost model and assume that

\[
\theta - \frac{(1 - F(\theta))}{f(\theta)}
\]

is increasing in \( \theta \).

(a) Compute the revenue maximizing direct mechanism.

Answer Let’s follow the steps from the exercise (1.a). The revenue maximizing direct mechanism is such that

\[
\max_{q(\theta)} \int_0^{\theta} \left\{ v(q(\theta)) \left[ \theta - \frac{(1 - F(\theta))}{f(\theta)} \right] - q(\theta) \right\} dF(\theta) \tag{1}
\]

Note that

- \( \theta - \frac{(1 - F(\theta))}{f(\theta)} \) evaluated at zero is \( -\frac{1}{f(\theta)} < 0 \);
- \( 1 - \frac{(1 - F(1))}{f(1)} = 1 \); and
- by assumption \( \theta - \frac{(1 - F(\theta))}{f(\theta)} \) is increasing in \( \theta \).

Then, by the intermediate value theorem, there exists a unique \( \theta^* \) (provided \( \theta - \frac{(1 - F(\theta))}{f(\theta)} \) is continuous) such that \( \theta^* - \frac{(1 - F(\theta^*))}{f(\theta^*)} = 0 \). Consequently, the revenue maximizing direct mechanism must put \( q(\theta) = 0 \) for all \( \theta \leq \theta^* \). Of course the corresponding \( t(\theta) \) must be zero for the same set of \( \theta \)'s.

On the other hand, if \( \theta > \theta^* \) then we can maximize \( (??) \) pointwise. The FOC is

\[
v'(q^*(\theta)) \left[ \theta - \frac{(1 - F(\theta))}{f(\theta)} \right] - 1 = 0
\]

Hence,

\[
q^*(\theta) = \begin{cases} 
0 & \text{if } \theta \leq \theta^* \\
(\theta - \frac{(1 - F(\theta))}{f(\theta)}) & \text{if } \theta > \theta^*.
\end{cases}
\]
Notice that $q^*(\theta)$ is increasing in $\theta$ because $\theta - \frac{(1-F(\theta))}{F(\theta)}$ is increasing in $\theta$ and because $v(.)$ is strictly concave (i.e., $v'(.)$ is decreasing).

Now, to obtain $t^*(\theta)$ for $\theta > \theta^*$, one can use either the (IC) restriction: $t^*(\theta) = \theta v'(q(\theta))q'(\theta)$, or the indirect utility of the agent: $t(\theta) = \theta v(q(\theta)) - \pi(\theta)$. (where $\pi(\theta)$ is the indirect utility, as we saw in class). Using the fact that $\pi(\theta) = \pi(0) + \int_0^\theta v(q(s))ds$, then we have

$$t^*(\theta) = \begin{cases} 0 & \text{if } \theta \leq \theta^* \\ \theta v(q^*(\theta)) - \int_0^\theta v(q^*(s))ds - \pi(0) & \text{if } \theta > \theta^*. \end{cases}$$

(there is no closed form solution at this level of generality)

(b) Establish that $t(\theta)/q$ is decreasing in $q$.

**Answer** Note that $t(\theta) \equiv t(q(\theta)) \equiv t(q)$. Then, $t'(\theta) = t'(q(\theta))q'(\theta)$ implying $t'(q) = t'(\theta)\frac{q'(\theta)}{q'(\theta)}$ (provided $q'(\theta) \neq 0$, which holds for all $\theta > \theta^*$, i.e., for all customers that trade in equilibrium). Moreover, from the incentive compatibility (IC) restriction, $t'(\theta) = \theta \frac{\partial v(q)}{\partial q} q'(\theta)$. Therefore,

$$t'(q) = \frac{t'(\theta)}{q'(\theta)} = \theta \frac{\partial v(q)}{\partial q} q'(\theta) = \theta \frac{\partial v(q)}{\partial q}$$

Which implies

$$t''(q) = \theta \frac{\partial^2 v(q)}{\partial q^2} < 0$$

Therefore $t(q)$ is concave and, so, $t(q)/q$ is decreasing in $q$.

(c) Establish that the revenue maximizing direct mechanism could also be implemented by a menu of two part-tariffs, $(T(\theta), p(\theta))$, where $T(\theta)$ is the fixed fee and $p(\theta)$ is the price per unit (i.e. if a customer chooses a particular two-part tariff then he has to pay $T(\theta)$ independent of the quantity he purchase, but can then buy as many units as he likes at the price $p(\theta)$ per unit. (The concavity of $t(q)$ might be useful.)

**Answer** Consider the revenue maximizing direct mechanism $(q(\theta), t(\theta))$. Denote the two-part tariff direct mechanism by $(\tilde{q}(\theta), \tilde{t}(\theta))$, where $\tilde{t}(\theta) = T(\theta) + p(\theta)\tilde{q}(\theta)$. The menu of two-part tariffs direct mechanism will be able to implement the maximizing direct mechanism if (i) $q(\theta) = \tilde{q}(\theta)$ and if (ii) $t(\theta) = T(\theta) + p(\theta)\tilde{q}(\theta)$. So, let’s fix $\tilde{q}(\theta) = q(\theta)$ and check if it is possible to obtain condition (ii).
First, notice that $t(\theta)$ is such that $t'(q) = \frac{v'(q)}{q(\theta)} = \theta v'(q(\theta))$ if $\theta > \theta^*$ (as we saw in the previous exercise). Moreover, $\bar{t}(\theta) = T(\theta) + p(\theta)q(\theta)$, which implies $\bar{t}'(q) = p(\theta)$. Hence we must have $p(\theta) = \theta v'(q(\theta))$.

Furthermore, from the previous item (a), we must have $p(\theta) = v(1 - F(q)) f(q)$ otherwise the FOC of the seller will not be satisfied in the direct mechanism.

Now let’s find the corresponding $T(\theta)$. We want that

$$t(\theta) = T(\theta) + p(\theta)q(\theta)$$

which implies

$$t'(\theta) = T'(\theta) + p'(\theta)q(\theta) + p(\theta)q'(\theta)$$

Moreover, for $\theta > \theta^*$, we have that $t'(\theta) = \theta v'(q(\theta))q'(\theta)$. So, we want that

$$\theta v'(q(\theta))q'(\theta) = T'(\theta) + p'(\theta)q(\theta) + p(\theta)q'(\theta)$$

But, for $\theta > \theta^*$ the two-part tariff must be such that $p(\theta) = \theta v'(q(\theta))$, implying

$$\theta v'(q(\theta))q'(\theta) = T'(\theta) + p'(\theta)q(\theta) + \theta v'(q(\theta))q'(\theta)$$

$$T'(\theta) = -p'(\theta)q(\theta)$$

So, noticing that we can put $T(0) = 0$, as explained below, then

$$T(\theta) = -\int_0^\theta p'(s)q(s)ds$$

It remains to check the price $p(\theta)$ when $\theta \leq \theta^*$. Actually, it is enough to put $p(\theta) = 0$. Notice that $T(\theta) = 0$, for all $\theta \leq \theta^*$, because then $p'(\theta) = 0$ and $q(\theta) = 0$, for all $\theta \leq \theta^*$, so that $\bar{t}(\theta) = t(\theta) = 0$ in this region.
Therefore the two-part tariff direct mechanism by \((\bar{q}(\theta), \bar{t}(\theta))\) implements the maximizing direct mechanism if \(\bar{q}(\theta) = q(\theta)\) and \(\bar{t}(\theta) = T(\theta) + p(\theta)\tilde{q}(\theta)\) with \(T(\theta) = -\int_0^\theta p'(s)q(s)ds\) and \(p(\theta) = \frac{\theta}{\sigma_0(\sigma_0)}\) \(1[\theta > \theta^*]\).

4. **Insurance with Homogenous Agents.** Consider an economy with a continuum of identical agents \(i\) with \(i \in [0,1]\) Each one has an endowment of \(w\) with probability \(1 - \pi\) and an endowment \(w - L\) with probability \(\pi\). The risk is idiosyncratic to each agent. \(L > 0\) can be interpreted as the loss from an accident. The von Neumann Morgenstern utility function is \(u(\cdot) > 0\) and \(u'(\cdot) < 0\). (Hint: For question 4 - 6, a graphical description of the consumer behavior in terms of indifference curves and budget lines across the two states, accident and no accident are very informative and illustrative.)

(a) Consider an egalitarian social welfare function which maximizes the sum (or the integral of the agents) with equal weight to each of the agents. Describe the Pareto optimal allocation, using the fact that with a large number (a continuum of agents) a fraction of the agents always have an accident.

**Answer** Suppose there is a transfer of \(\tau\) to agents who have an accident and \(t\) to agents who don’t. The utility of society is

\[
\pi u(w - L + \tau) + (1 - \pi)u(w + t)
\]

By budget balancedness we must have that

\[
\pi \tau + (1 - \pi)t \leq 0
\]

since the utility function is increasing this must bind so the objective is to choose \(\tau\) to maximize

\[
\pi u(w - L + \tau) + (1 - \pi)u(w - \frac{\pi}{1 - \pi})
\]

which occurs when

\[
\tau - L = -\frac{\pi}{1 - \pi}
\]

and so \(\tau = (1 - \pi)L\), which corresponds to full insurance.
(b) Consider now an insurance company which offers an actuarially fair insurance such that it charges a premium for every $1 compensation in case of an accident. The premium has to be paid independent of the occurrence of an accident or not. Formulate the decision of the representative household in terms of quantity of insurance purchase given the price \( p = \pi \) of insurance and describe the nature of the optimal household solution. What happens if the price \( p \) increases beyond \( \pi \)?

Answer

Suppose the price of a dollar of insurance is \( p \). Then by buying \( D \) units of insurance the utility of the household is

\[
\pi u(w - L - pD + D) + (1 - \pi)u(w - pD).
\]

The household wishes to choose the \( D \) that maximizes their utility. Taking FOC we get that the optimal level of \( D \), given \( p \) is determined by

\[
\pi u'(w - L - pD^* + D^*)(1 - p) - (1 - \pi)u'(w - pD^*)p = 0.
\]

Suppose \( p = \pi \). Then this condition reduces to

\[
\pi(1 - \pi)(u'(w - L - pD^* + D^*) - u'(w - pD^*)) = 0
\]

and since \( u \) is strictly concave, this implies

\[
w - L - pD^* + D^* = w - pD^*
\]

and so

\[D^* = L.\]

Now suppose \( p > \pi \). Then the FOC implies that

\[
1 < \frac{p(1 - \pi)}{\pi(1 - p)} = \frac{u'(w - L - pD^* + D^*)}{u'(w - pD^*)}
\]

which implies that

\[
w - L - pD^* + D^* < w - pD^*
\]

and so

\[D^* < L.\]
5. **Insurance with Heterogeneous Agents.** Consider an economy as in (4), but now assume that there are two different kind of consumers. A fraction $\alpha$ of consumers behaves carefully and a fraction $1 - \alpha$ of consumers behaves risky, and correspondingly the two groups have different accident rates, $0 < \pi_c < \pi_r < 1$.

(a) Now analyze the behavior of the two different households for an common insurance premium $p$. How does the purchase behavior vary as $p$ varies in the interval $[\pi_c, \pi_r]$.

Answer  From our answers to question 2 we can see that for $i = c, r$

$$\frac{p(1 - \pi_i)}{\pi_i(1 - p)} = \frac{u'(w - L - pD_i^* + D_i^*)}{u'(w - pD_i^*)}$$

So $D_i(p)$ is decreasing in $p$, with

$$D_r(p) = \begin{cases} L, & \text{if } p = \pi_r \\ > L, & \text{if } p < \pi_r \end{cases}$$

$$D_c(p) = \begin{cases} L, & \text{if } p = \pi_c \\ < L, & \text{if } p > \pi_c \end{cases}$$

(b) For an arbitrary price $p \in [\pi_c, \pi_r]$ and given the optimal purchase behavior of the consumers, does the insurance company receives zero-net profit in expectations?

Answer  Note, that by the implicit function theorem $D_i(p)$ is continuous function of $p$. The profits of the firm from offering price $p$ is then

$$\gamma(p) = \alpha(p - \pi_c)D_c(p) + (1 - \alpha)(p - \pi_r)D_r(p)$$

which is a continuous function of $p$. Note that

$$\gamma(\pi_c) = (1 - \alpha)(\pi_c - \pi_r)D_r(\pi_c) < 0$$

so it is not necessarily true that the firm would earn 0 profits in expectations.

(c) Does there exists a price $p \in [\pi_c, \pi_r]$ such that given the optimal purchase behavior of the consumers, the insurance company receives zero-net profit in expectations?
Answer Notice that $\gamma(p)$ is continuous with $\gamma(\pi_c) < 0$ and $\gamma(\pi_r) = \alpha(\pi_r - \pi_c)D_c(\pi_r) \geq 0$ so there must be a $p \in [\pi_c, \pi_r]$ with $\gamma(p) = 0$.

6. **Competition in Contracts.** We now allow for a richer set of insurance contracts (beyond linear price insurance contracts) to be offered by the insurance companies, namely price-quantity contracts. Consider a large number of competing insurance companies who can each offer one or multiple insurance contracts for a given premium and a given coverage level. Each contract $j$ is restricted to ask for a premium $P_j$ for a net coverage $C_j$. In other words, the unit price of contract $j$ is $p_j = P_j/(P_j + C_j)$ for a quantity of coverage $q_j = (P_j + C_j)$ and the net coverage is $(1 - p_j)q_j = C_j$.

(a) We now look for a perfect Bayesian equilibrium of this competition game in which the competing firms simultaneously make offers to the customers, and then given the offers the customers accept or reject at most one insurance contract. Since there are only two types in the economy, we can restrict attention to pooling and separating contracts in equilibrium.

i. First argue that there can be no pooling equilibrium in this competitive insurance market.

Answer Suppose there is a pooling contract. In order for the firm to earn 0 profits it must be that

$$p = \frac{P}{P + C} = \alpha \pi_c + (1 - \alpha)\pi_r.$$  

In order for NO firm to be able to enter, it must be that $P + C$ is the amount of insurance the cautious consumer would purchase at price $p$ because

- if it were lower a firm could offer more insurance at price slightly above $p$ and all consumers would buy; and
- if it were higher a firm could offer less insurance at the same rate and only the low risk consumers would buy resulting in positive profits.

Note that the risky consumers would prefer more insurance at this rate. At this point, their indifference curve is flatter than the cautious consumer (as you can check looking at the marginal rate of substitution and/or drawing the indifference curves). So it must be possible to find
a price arbitrarily close to $p$ and a lower level of coverage such that the cautious consumer would buy the insurance but the risky consumer would not. A firm could offer this contract and earn positive profits, breaking the pooling equilibrium.

ii. Describe the nature of a contract equilibrium which displays separation across types. What can you say about the insurance coverage offered in these contracts to different types of the agents.

Answer Suppose we have a separating contract: $(P_c, C_c), (P_r, C_r)$. Consider the contract for the risky agents first. Since we are in a competitive environment we must have that the firms earn 0 profits, so

$$p_r = \frac{P_r}{P_r + C_r} = \pi_r$$

We must also have $P_r + C_r = L$ since otherwise a firm could offer contract $L$ at per unit price slightly above $\pi_r$ (since the agent would strictly prefer more insurance at $p_r = \pi_r$). So we must have that $(P_r, C_r) = (\pi_r L, (1 - \pi_r)L)$. This results in utility to the risky agent of

$$u(w - \pi_r L).$$

Now consider the contract to the cautious agent. In order to get separation it must be that the utility from choosing $(P_c, C_c)$ is no higher then from choosing $(P_r, C_r)$ (i.e., it must satisfy the IC restriction). That is,

$$\pi_r u(w - \pi_r L) \geq \pi_r u(w - L + C_c) + (1 - \pi_r)u(w - P_c)$$

Notice three things:

1. Zero profit condition guarantees that $\frac{P_c}{P_c + C_c} = \pi_c$ so $P_c = \frac{\pi_c C_c}{1 - \pi_c}$.

2. Must have $P_c + C_c < L$ or the risky agents would select the contract $(P_c, C_c)$.

3. The incentive compatibility constraint (IC) above must bind because otherwise a firm could offer the cautious consumers slightly more insurance at a slightly higher price and earn positive profits without violating the IC restriction (and the risky consumers still would not buy this new contract).

Notice that if $\pi_r$ is much larger than $\pi_c$ the insurance for the caution consumer is far away from the full insurance level.
iii. As a function of \( \alpha \), the fraction of low risk agents, what can you say about the existence of perfect Bayesian equilibrium in this contract game.

**Answer** Suppose the fraction of risky consumers is small. As the fraction of risky goes to 0 the price of the pooling contract must approach \( \pi_c \) and so the utility to the cautious consumer must go to the utility from full insurance at the actuarially fair rate. So if the fraction of risky customers is small the cautious consumers would prefer the pooling contract to the separating contract. So a firm could offer a pooling contract, at a price slightly above the actuarial fair price, and earn positive profits. So the separating equilibrium breaks down. But we have already established that there cannot be a pooling equilibrium so there cannot exist an equilibrium when the fraction of risky consumers is small.

7. Compare the nature of the competitive market and its impact for the social efficiency of the market outcome in the model of nonlinear pricing and the insurance model. Are there any differences between the two markets and if so, what explains the nature of the differences.

**Answer** The crucial difference between the nonlinear pricing and the insurance model is that the former is a private values model while the second is a common values model. The profit of the Principal in the first model only depend on the actions of the agents, but in the second model the Principal’s profit (the insurer) depends also directly on the type (risky class) of the agents.

The consequences of this difference are:

- In the nonlinear pricing model, perfect competition makes the adverse selection irrelevant. The first best (i.e., the allocation under perfect information that maximizes social welfare) is achieved despite the presence of incomplete information.

- In the insurance model, however, the first best cannot be achieved even in the perfect competition case. The reason is that in the perfect information competitive equilibrium each agent obtains full insurance. But that violates the incentive compatibility constraint and, so, cannot be sustained in equilibrium in the presence of incomplete information. Moreover, the Bayesian Equilibrium is much more fragile in the presence of common values, since the Principal must react differently
depending on who buys the contract, and, as a result, the equilibrium may not even exist.