1. First Price Auction. Consider the first price auction in a symmetric environment with binary valuations, i.e. the value of bidder \( i \) is given by \( v_i \in \{v_l, v_h\} \) with \( 0 \leq v_l < v_h < 1 \). The prior probability is given by \( \Pr(v_i = v_h) = \alpha \) for all \( i \).

a. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

Answer Consider only the case with 2 bidders and focus on symmetric interim BNE. First of all, note that for any type \( \{v_l, v_h\} \), it is irrational to bid above her own valuation, i.e., \( b(v_i) \leq v_i \), for all \( i \). Given this, any bidder of the high type, \( v_h \), will always bid above \( v_l \), since there is no point in bidding below \( v_l \) and incur the risk of loosing against a low type when she could easily win and make positive profit. So, in any BNE it must be the case that \( v_l \leq b(v_h) \leq v_h \). Now let’s focus on each type separately.

- **Low Type:** in any BNE, we must have \( b(v_l) = v_l \), because for any strategy such that \( b(v_l) < v_l \), the low type bidder has an incentive to deviate and bid slightly more than \( b \), win with positive probability (when the other bidder is a low type) and, so, obtain a positive expected payoff. (Of course she always get a payoff of zero when playing against a high type).

- **High Type:** once that \( b(v_l) = v_l \) and \( v_l \leq b(v_h) \leq v_h \) in any BNE, we see that the high type prefers to bid as close as possible to \( v_l \) when playing against a low type (since then she wins the auction and pays the lowest price possible) but as large as \( v_h \) when against a high type (since for any given strategy profile such that \( b(v_h) < v_h \), she would have an incentive to deviate and bid slightly above \( b(v_h) \), and obtain positive payoff for sure). In any case, if (a) \( b_l < b_j < v_h \), then \( i \) would have a profitable deviation; if (b) \( v_l \leq b_j < b_i < v_h \), then again \( i \) would have a profitable deviation; and if (c) \( b_j \leq b_i = v_h \), once more \( i \) would have a profitable deviation since this action guarantees a zero payoff while bidding less than \( v_h \) gives a positive expected payoff (since high type always win against low type and she would pay less than her own valuation). Because of this discontinuity in the expected payoff, there is no pure strategy best reply to strategies such that \( b(v_l) = v_l \) and \( v_l \leq b(v_h) \leq v_h \). I.e., there is no pure strategy BNE here.
So, let’s look for a mixed strategy BNE. From the discussion above, the only type that is willing to mix is the high type. So let’s propose a BNE such that \( b(v_l) = v_l \) and the high type mixes with the distribution \( F(b) \) over some support \([b, \tilde{b}]\).

Of course, from above, we must have \( v_l \leq b \leq \tilde{b} \leq v_h \). Moreover, the lower bound must be \( b = v_l \), because if \( b \) was strictly greater than \( v_l \) then bids in the interval \((v_l, b)\) would be preferred to \( b \). A subtlety here is that although \( b = v_l \), we must have the support to be open at the lower bound, ie, \([v_l, \tilde{b}]\). The reason is that when the high type bids \( v_l \) there is a tie whenever the opponent is a low type and they get the good with probability 1/2, by the tie-breaking rule. This situation gives a smaller payoff for the high type than the expected payoff obtained when bidding slightly above \( v_l \). However, if the mixed strategy \( F(b) \) puts mass zero at this left-end point, then this event occurs with zero probability and we do not need to care about it. In this case we could use the support as \([v_l, \tilde{b}]\).\(^1\)

Actually, the distribution \( F(b) \) has to be continuous because if it is not (at some point \( b \)), the other player could put more mass right above \( b \) and get higher expected payoff. So from now on, suppose \( F(b) \) is continuous and the support is \([v_l, \tilde{b}]\). Now we have to find both \( \tilde{b} \), \( F(b) \) and we have to show that there is no incentive for the high type to deviate from this strategy.

- **Find \( \tilde{b} \):** when \( b = v_l \), the expected payoff of the high type is \((1 - \alpha)(v_h - v_l)\). At any other \( b \) in the support, the expected payoff is \((1 - \alpha)(v_h - b) + \alpha(v_h - b)F(b)\). Therefore, \( \tilde{b} \) must be such that \((1 - \alpha)(v_h - v_l) = (1 - \alpha)(v_h - \tilde{b}) + \alpha(v_h - \tilde{b})\), since \( F(\tilde{b}) = 1 \). Implying \( \tilde{b} = \alpha v_h + (1 - \alpha)v_l \). Note, as (should be) expected, that \( \tilde{b} < v_h \).

- **Find \( F(b) \):** from the previous reasoning it is easy to deduce that for all \( b \in [v_l, \alpha v_h + (1 - \alpha)v_l] \), we have that

\[
(1 - \alpha)(v_h - v_l) = (1 - \alpha)(v_h - b) + \alpha(v_h - b)F(b)
\]

\[
\Rightarrow F(b) = \frac{1 - \alpha}{\alpha} \left[ \frac{(v_h - v_l)}{(v_h - b)} - 1 \right]
\]

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\(^1\) Another solution would be to impose a tie-breaking rule that gives the object to the highest type whenever a tie occurs.
• Check this is a BNE: the high type is indifferent, by construction, over the support \([v_l, \alpha v_h + (1 - \alpha)v_l]\). Moreover, she does not want to deviate since (i) any \(b < v_l\) gives a payoff of zero, which is smaller than the payoff \((1 - \alpha)(v_h - v_l)\) obtained when using the mixed strategy; and (ii) any \(b \in (\alpha v_h + (1 - \alpha)v_l, v_h]\) gives a payoff smaller than \(v_h - \{\alpha v_h + (1 - \alpha)v_l\} = (1 - \alpha)(v_h - v_l)\). Hence, bids within the support are better than bids outside the support. Consequently, the high type does not deviate from the proposed mixed strategy. Since the low type gives a best response \(b(v_l) = v_l\), we found a symmetric BNE of this game.

b. Does the revenue equivalence result between the first and the second price auction still hold with the binary payoff types?

Answer Consider the Second Price Auction first. The equilibrium in dominant strategies is, as usual, \(b(v_i) = v_i\) for both types. The expected revenue therefore is \(R_{(SPA)} = (1 - \alpha)^2 v_l + 2\alpha(1 - \alpha)v_l + \alpha^2 v_h\), where the first term is the probability of having two bidders with low type times the corresponding payment; the second term reflects the cases where one bidder is low type and the other is a high type and so on.

In the First Price Auction, the expected revenue is \(R_{(FPA)} = (1 - \alpha)^2 v_l + 2\alpha(1 - \alpha)E(b_h) + \alpha^2 E(b_{(1)})\), where \(E(b_h)\) is the expected bid of the high type when she mixes (since that is the expected payment when one bidder is a low type and the other is a high type); and \(E(b_{(1)}) = E(\max\{b_1, b_2\})\) (since that is the expected payment when the two bidders are of a high type).

Now we have to solve some tedious algebra. First note that \(R_{(SPA)} = R_{(FPA)}\) if and only if the last two terms of each coincide, i.e., iff \(2\alpha(1 - \alpha)v_l + \alpha^2 v_h = 2\alpha(1 - \alpha)E(b_h) + \alpha^2 E(b_{(1)})\).

Now, notice that, by integration by parts,

\[
E(b_h) = \int_{v_l}^{v_h} b \cdot dF(b) = b - \int_{v_l}^{v_h} \frac{1 - \alpha}{\alpha} \left[ \frac{(v_h - v_l)}{(v_h - b)} - 1 \right] db
\]

therefore,

\[
2\alpha(1 - \alpha)E(b_h) = 2\alpha(1 - \alpha)b - 2(1 - \alpha)^2 \int_{v_l}^{v_h} \left[ \frac{(v_h - v_l)}{(v_h - b)} - 1 \right] db \quad (1)
\]

Also, by integration by parts again and noticing that \(b_{(1)} \sim [F(b)]^2\)
\[ E(b(1)) = b - \int_{v_l}^{\bar{b}} \left( \frac{1 - \alpha}{\alpha} \right)^2 \left[ \frac{(v_h - v_l)}{(v_h - b)} - 1 \right]^2 db \]

implying

\[ \alpha^2 E(b(1)) = \alpha^2 b - (1 - \alpha)^2 \int_{v_l}^{\bar{b}} \left[ \frac{(v_h - v_l)}{(v_h - b)} - 1 \right]^2 db \]

\[ = \alpha^2 b - (1 - \alpha)^2 \int_{v_l}^{\bar{b}} \left( \frac{v_h - v_l}{v_h - b} \right)^2 db + 2(1 - \alpha)^2 \int_{v_l}^{\bar{b}} \left( \frac{v_h - v_l}{v_h - b} \right) db - (1 - \alpha)^2 \int_{v_l}^{\bar{b}} 1 dB \tag{2} \]

Notice the last term of \(2\alpha(1 - \alpha)E(b_h)\), in (1), and the two last terms of \(\alpha^2 E(b(1))\), in (2). Therefore, we have that

\[ [2\alpha(1 - \alpha)E(b_h)] + [\alpha^2 E(b(1))] = \]

\[ = \left[ 2\alpha(1 - \alpha)\bar{b} + (1 - \alpha)^2 \int_{v_l}^{\bar{b}} 1 dB \right] + \left[ \alpha^2 \bar{b} - (1 - \alpha)^2 \int_{v_l}^{\bar{b}} \left( \frac{v_h - v_l}{v_h - b} \right)^2 db \right] \]

\[ = 2\alpha(1 - \alpha)\bar{b} + (1 - \alpha)^2(\bar{b} - v_l) + \left[ \alpha^2 \bar{b} - (1 - \alpha)^2(v_h - v_l)^2 \left[ \frac{1}{v_h - b} \right]_{v_l}^{\bar{b}} \right] \]

\[ = 2\alpha(1 - \alpha)\bar{b} + (1 - \alpha)^2(\bar{b} - v_l) + \alpha^2 \bar{b} - (1 - \alpha)^2(v_h - v_l)^2 \left[ \frac{1}{(1 - \alpha)(v_h - v_l) - (v_h - v_l)} \right] \]

\[ = 2\alpha(1 - \alpha)\bar{b} + (1 - \alpha)^2(\bar{b} - v_l) + \alpha^2 \bar{b} - (1 - \alpha)(v_h - v_l) + (1 - \alpha)^2(v_h - v_l) \]

After some more algebra and using \(\bar{b} = \alpha v_h + (1 - \alpha)v_l\), we obtain

\[ = 2\alpha(1 - \alpha)v_l + \alpha^2 v_h \]

Hence \(2\alpha(1 - \alpha)E(b_h)] + [\alpha^2 E(b(1))] = 2\alpha(1 - \alpha)v_l + \alpha^2 v_h\), and, so, the revenue equivalence holds here.

2. **All Pay Auction.** The rules of the all pay auction are: (i) the highest bid receives the object, (ii) each bidder pays his bid, independent of whether he wins or lose the object.
a. Characterize the equilibrium of the all-pay auction in the symmetric environment with a uniform distribution on the unit interval for $N$ bidders. (Hint: Guess that the equilibrium bidding function is an increasing and quadratic function.)

Answer Let there be $N$ bidders with uniformly distributed valuations. Remember that in this incomplete information game, a player’s type is her valuation and a strategy is a function mapping valuations into bids.

Look for a symmetric equilibrium in which players use a strictly increasing (identical) bidding function $b_i(v_i) = b(v) \ \forall i$. If all players $j \neq i$ follow a strategy $\beta_j(v_j)$, then player $i$’s payoff is given by:

$$u_i(v_i, b_i, \beta_j) = v_i \prod_{j \neq i} \Pr(b_i > \beta_j) - b_i$$

$$= v_i \prod_{j \neq i} \Pr(\beta_j^{-1}(b_i) > v_j) - b_i$$

$$= v_i \mathbb{P}^{N-1}(\beta_j^{-1}(b_i)) - b_i$$

$$= v_i \left[\beta_j^{-1}(b_i)\right]^{N-1} - b_i.$$

Differentiating with respect to $b_i$ one obtains

$$v_i (N - 1) \left[\beta_j^{-1}(b_i)\right]^{N-2} \frac{1}{\beta_j'(\beta_j^{-1}(b_i))} - 1 = 0$$

Imposing symmetry ($\beta \equiv b, \beta_j^{-1}(b) = v$):

$$v (N - 1) v^{N-2} \frac{1}{b'(v)} - 1 = 0$$

$$b'(v) = (N - 1) v^{N-1}$$

$$b(v) = \frac{N - 1}{N} v^N$$

From this solution you can verify that for $N = 2$ the bidding function is quadratic.

b. Characterize the equilibrium of the all-pay auction in the symmetric environment with a continuously differentiable distribution function.
Answer Follow the previous steps, suppose a symmetric equilibrium in which players use a strictly increasing (identical) bidding function \( b_i(v_i) = b(v) \) \( \forall i \). If all players \( j \neq i \) follow a strategy \( \beta_j(v_j) \), then player \( i \)'s payoff is given by:

\[
  u_i(v_i, b_i, \beta_j) = v_i \prod_{j \neq i} \Pr(b_i > \beta_j) - b_i \\
  = v_i \prod_{j \neq i} \Pr(\beta^{-1}_j(b_i) > v_j) - b_i \\
  = v_i F^{N-1}(\beta^{-1}_j(b_i)) - b_i
\]

Differentiating with respect to \( b_i \) one obtains

\[
  v_i (N - 1) \left[ F(\beta^{-1}_j(b_i)) \right]^{N-2} \frac{f(\beta^{-1}_j(b_i))}{\beta'_j(\beta^{-1}_j(b_i))} - 1 = 0
\]

Imposing symmetry \( (\beta \equiv b, \beta^{-1}(b) = v) \):

\[
  v(N - 1) F(v)^{N-2} \frac{f(v)}{b'(v)} - 1 = 0 \\
  b'(v) = v(N - 1) F(v)^{N-2} f(v) \\
  b(v) = (N - 1) \int_0^v yF(y)^{N-2} f(y) dy
\]

Notice that if \( F(y) \sim Uniform[0, 1] \), we obtain the result of part (a).

3. **Bank Run and Bayesian Updating.** In class we discussed a model of bank run with incomplete information. We used an improper prior, i.e. a prior with an infinite mass, to have convenient expressions for the expected value and the conditional probability. The following variation asks you to consider the same stage game

\[
  \begin{array}{c|cc}
    & l & w \\
    \hline
    l & \theta, \theta & \theta - 1, 0 \\
    w & 0, \theta - 1 & 0, 0 \\
  \end{array}
\]
with a different model of uncertainty, i.e. a model with proper priors. We assume that the state of the world is given by:

$$\theta \sim \mathcal{U}[-x, +x]$$

and that each agent observes

$$t_i = \theta + \varepsilon_i$$

with

$$\varepsilon_i \sim \mathcal{U}[-z, +z]$$

We consider $0 < z < 1 \ll x < \infty$. Derive the threshold equilibrium of the bank run model. As part of the equilibrium you will need to have to compute the conditional density $f(\theta|t_i)$ and $f(t_j|t_i)$. In particular, you may want to ask yourself what happens to the conditional density $f(t_j|t_i)$ for fixed $z$ as $x \to \infty$. (This limit result was the reason why the improper prior was useful in class.

Answer For both players, the payoff from playing $w$ is zero. The expected payoff of player $i$ from playing $l$ given the player’s type, $t_i$, is

$$E(\theta | t_i) - \Pr(\text{player } j \text{ plays } w | t_i)$$

Because this expected payoff is increasing in $\theta$, let’s propose a threshold strategy that plays $l$ when the signal is large. Ie, let’s propose the following strategy profile as the symmetric BNE:

$$s(t_i) = \begin{cases} 
    l & \text{if } t_i \geq k \\
    w & \text{if } t_i \leq k 
\end{cases}$$

When player $j$ uses this strategy, player $i$ plays $l$ only if

$$E(\theta|t_i) - \Pr(t_j \leq k|t_i) \geq 0$$

Let’s look at $E(\theta | t_i)$ first. Of course $E(\theta | t_i) = E(t_i - \varepsilon_i | t_i) = t_i - E(\varepsilon_i | t_i) = t_i$, since $\varepsilon_i$ is a random variable with distribution $\mathcal{U}[-z, +z]$ independent of $t_i$ and, so, $E(\varepsilon_i | t_i) = E(\varepsilon_i) = 0$ (note that this comes from the primitives of the problem, but you could have noticed that $t$ does not help predict $\varepsilon$ since every $\varepsilon$ is equally possible
for a given $t$ because of the distribution of $\theta$). Therefore, player $i$ plays $l$ only if

$$t_i - \Pr(t_j \leq k \mid t_i) \geq 0$$

Now we have to find $\Pr(t_j \leq k \mid t_i)$. We know that $\text{Support}(t_i) = [-x, +x]$. But when the signal lies on the extremes it may be informative about $\theta$. More specifically, if $t_i \in [-x, -z)$ or if $t_i \in (+z, +x]$, then the player knows that $t_i = \theta$. Let’s put these cases aside for a moment and focus now on the case where $t_i \in [-z, +z].$

When $t_i \in [-z, +z]$, we have that $\theta + \varepsilon_i \in [-z, +z]$, and so,

$$\Pr(t_j \leq k \mid t_i) = \Pr(t_j - t_i \leq k - t_i \mid t_i) = \Pr(\varepsilon_j - \varepsilon_i \leq k - t_i \mid t_i)$$

and, since the density of $\varepsilon_i$ is $f(\varepsilon_i) = 1/(2z)$ and $f(\varepsilon_j, \varepsilon_i) = f(\varepsilon_j) f(\varepsilon_i) = 1/(4z^2)$, then

$$\Pr(\varepsilon_j - \varepsilon_i \leq a) = \int_{-z}^{z} \int_{-z}^{a+\varepsilon_i} \frac{1}{4z^2} d\varepsilon_j d\varepsilon_i$$

$$= \frac{1}{4z^2} \int_{-z}^{z} [a + \varepsilon_i + z] d\varepsilon_i$$

$$= \frac{1}{4z^2} [2az + 2z^2]$$

$$= \frac{a}{2z} + \frac{1}{2}$$

Therefore, when $t_i \in [-z, +z]$, $i$ plays $l$ if

$$t_i - \left( \frac{(k - t_i)}{2z} + \frac{1}{2} \right) \geq 0$$

And, therefore, the indifferent type is given by

$$k - \left( \frac{(k - k)}{2z} + \frac{1}{2} \right) = 0 \iff k = \frac{1}{2}$$

Let’s assume we have $k = 1/2 \in [-z, +z]$ to make this case interesting. (otherwise the proposed strategy would prescribe playing $w$ whenever $t_i \in [-z, +z]$).
Now, let’s return to the cases where $t_i$ is informative about $\theta$, i.e., the cases $t_i \in [-x, -z)$ and $t_i \in (+z, +x]$. Once the restrictions $0 (<k)< z < 1\iff x < \infty$ are imposed, we have that:

(i) If $t_i \in [-x, -z)$, then $t_i = \theta < -z < 0$. But this implies $E(\theta \mid t_i) = t_i < 0$ while $0 \leq \Pr(j \text{ plays } w \mid t_i) \leq 1$. Therefore, we have that $E(\theta \mid t_i) - \Pr(j \text{ plays } w \mid t_i) < 0$. So, for any type $t_i$ in this region, $w$ (withdrawing) is a dominant strategy. Hence, whenever $t_i \leq k = 1/2$, the best action is $w$ (and this is the best reply even when $(-x) \rightarrow -\infty$).

(ii) If $t_i \in (+z, +x]$, then $t_i = \theta$ again. Now we have two possibilities. If $\theta > 1$, then $E(\theta \mid t_i) - \Pr(j \text{ plays } w \mid t_i) = \theta - \Pr(j \text{ plays } w \mid t_i) > 0$ and, so, it is a dominant strategy to leave the money, i.e., to play $l$ (and this is the best reply even when $(x) \rightarrow -\infty$).

On the other hand, if $z < t_i = \theta \leq 1$, then $\theta - \Pr(j \text{ plays } w \mid t_i)$ could be, in principle, smaller or greater than zero depending on which number is greater $\theta$ or $\Pr(j \text{ plays } w \mid t_i)$. Note however that this is the situation where player $i$ knows $\theta$ and that $\theta$ is greater then the cutoff point. This suggests for him that it is unlikely that the other player, $j$, is going to play $w$ (i.e., $j$ is probably receiving a high signal and, so, is probably leaving the money). So, the best reply for $i$ will be to leave the money. The math confirm this intuition:

\[
\Pr(t_j \leq k \mid t_i = \theta) = \Pr(\theta + \varepsilon_j \leq k \mid \theta)
\]
\[
= \Pr(\varepsilon_j \leq \frac{1}{2} - \theta)
\]
\[
\leq \Pr(\varepsilon_j \leq \frac{1}{2} - z) \quad (\text{since } z < \theta)
\]
\[
\leq \Pr(\varepsilon_j \leq 0) \quad (\text{because } z > \frac{1}{2})
\]
\[
\leq \frac{1}{2}
\]

Hence $\theta - \Pr(j \text{ plays } w \mid t_i) > (1/2) - (1/2) > 0$. And, so, $i$ prefers to play $l$ in this situation.

The conclusion from all these cases is that the threshold strategy with $k = 1/2$, is indeed a BNE.