Question 11.1 Optimal Taxation [Salanié 3.1]

Note: the question actually refers to an old model in the first edition of the text. When Salanie updated his book to the second edition in 2005, he did not update the exercises, hence 3.1(a) has this strange $U'(C())$ in it. Modifying 3.1a slightly, we’re able to arrive at the conclusions of 3.1b.

11.1a

IC1 on page 49 requires that

$$C'(\theta) = v' \left( \frac{Q(\theta)}{\theta} \right) \frac{Q'(\theta)}{\theta}.$$  

Using this, we can easily establish

$$T'(\theta) = Q'(\theta) - C'(\theta) = Q'(\theta) - v' \left( \frac{Q(\theta)}{\theta} \right) \frac{Q'(\theta)}{\theta} = \left( \theta - v' \left( \frac{Q(\theta)}{\theta} \right) \right) \frac{Q'(\theta)}{\theta}.$$  

11.1b

Now we can write $T'(\theta)$, using the first IC condition, as:

$$T'(\theta) = Q'(\theta) - C'(\theta) = Q'(\theta) - v' \left( \frac{Q(\theta)}{\theta} \right) \frac{Q'(\theta)}{\theta} = \left( \theta - v' \left( \frac{Q(\theta)}{\theta} \right) \right) \frac{Q'(\theta)}{\theta}.$$  

Substitute in the equation

$$\theta - v' (L(\theta)) = \frac{G(\theta) - F(\theta)}{\theta f(\theta)} (v''(L(\theta)) L(\theta) + v'(L(\theta))),$$

found on page 50 of Salanié (which maximizes the virtual surplus pointwise), we get

$$T'(\theta) = \frac{G(\theta) - F(\theta)}{\theta f(\theta)} (v''(L(\theta)) L(\theta) + v'(L(\theta))) \frac{Q'(\theta)}{\theta}. \quad (1)$$

Since at $\theta = \bar{\theta}$ or $\underline{\theta}$, $G(\theta) = F(\theta)$, we therefore must have

$$T'(\bar{\theta}) = T'(\underline{\theta}) = 0.$$

To establish $T'(\theta) \geq 0$, we notice that

1. By assumption (page 48, Salanié), $G \geq F$, so $\frac{G(\theta) - F(\theta)}{\theta f(\theta)} \geq 0$. And if we assume $G(\theta) > F(\theta)$, $\forall \theta \in (\bar{\theta}, \underline{\theta})$, then $\frac{G(\theta) - F(\theta)}{\theta f(\theta)} > 0$, $\forall \theta \in (\bar{\theta}, \underline{\theta})$.

2. By assumption $v'' > 0$, $v' > 0$ and because labor $L(\theta) \geq 0$, the term in the bracket in the middle of (1) is strictly positive.

3. IC 2 (page 49, Salanié) requires that $Q'(\theta) \geq 0$. Moreover, for a fully separating equilibrium, we need $Q'(\theta) > 0$. And since $\theta > 0$ (productivity must be positive) we have $\frac{Q'(\theta)}{\theta} > 0$.

Together, the three points above show that $T'(\theta) > 0$, for all $\theta \in (\bar{\theta}, \underline{\theta})$ as long as $G(\theta) > F(\theta)$, $\forall \theta \in (\bar{\theta}, \underline{\theta})$. 

1
**Question 11.2 Payoff Equivalence**

We want to show \( \forall \theta, \theta' \in \Theta, \)

\[
U(\theta) \geq \theta q(\theta') - t(\theta')
\]

\( \Leftrightarrow U(\theta) \geq \theta q(\theta') - t(\theta') + \theta' q(\theta') - \theta' q(\theta') \)

\( \Leftrightarrow U(\theta) \geq \theta q(\theta') + U(\theta') - \theta' q(\theta') \)

\( \Leftrightarrow U(\theta) - U(\theta') \geq (\theta - \theta')q(\theta'). \tag{2} \)

Now using (1), we get

\[
U(\theta) - U(\theta') = \int_{0}^{\theta} q(s) \, ds - \int_{0}^{\theta'} q(s) \, ds = \int_{\theta'}^{\theta} q(s) \, ds.
\]

If \( \theta \geq \theta' \), then using (2) we know \( \forall s \in [\theta', \theta], q(s) \geq q(\theta') \), implying:

\[
\int_{\theta'}^{\theta} q(s) \, ds \geq \int_{\theta'}^{\theta'} q(\theta') \, ds = (\theta - \theta')q(\theta').
\]

On the other hand, if \( \theta < \theta' \), then using (2) we know \( \forall s \in [\theta, \theta'], q(s) \leq q(\theta') \) or \( -q(s) \geq -q(\theta') \), implying

\[
\int_{\theta}^{\theta'} q(s) \, ds = \int_{\theta}^{\theta'} -q(s) \, ds \geq \int_{\theta'}^{\theta} -q(\theta') = (\theta' - \theta) (-q(\theta')) = (\theta - \theta')q(\theta').
\]

Together, they establish (2).

**Question 11.3 Revelation Principle**

11.3a pure strategy equilibrium in dominant strategies

(Revelation Principle for Dominant Strategy Mechanisms) Suppose a mechanism \( \Gamma = (q, M) \) and a strategy combination \( \sigma : \Theta \rightarrow M \) for \( \Gamma \) are such that for each type \( \theta_i \) of each buyer \( i \) the strategy \( \sigma_i(\theta_i) \) is a dominant strategy in \( \Gamma \). Then there exists a direct mechanism \( \Gamma' = (f, \Theta) \) and a strategy combination \( \sigma' : \Theta \rightarrow \Theta \) of \( \Gamma' \) such that for every type \( \theta_i \) of each buyer \( i \) the strategy \( \sigma'_i(\theta_i) \) is a dominant strategy in \( \Gamma' \), and:

(i) The strategy vector \( \sigma' \) satisfies for every \( i \) and every \( \theta_i \):

\[
\sigma'_i(\theta_i) = \theta_i,
\]

that is, \( \sigma' \) prescribes telling the truth.

(ii) For every vector \( \theta \) of types the distribution over allocations and the expected payments that result in \( \Gamma \) if the agents play \( \sigma \) is the same as the distribution over allocations and the expected payments that result in \( \Gamma' \) if the agents play \( \sigma' \).

**Proof:**

First, Construct \( \Gamma' \) from \( \Gamma \) as required by part (ii) of the Proposition.

Next, we show (i) is satisfied by demonstrating that truth telling will be a dominant strategy in the constructed direct mechanism \( \Gamma' \). To see this suppose it were not. Then under \( \Gamma' \), there exists a player \( i \) of type \( \theta_i \) who prefers to report that her type is \( \theta'_i \) for some type vector of the other agents \( \theta_{-i} \). However, this means that the same player \( i \) of type \( \theta_i \) would have preferred to deviate from \( \sigma_i \), and to play the strategy that \( \sigma_i \) prescribes for \( \sigma'_i \) in \( \Gamma \), for the strategy combination that the types \( \theta_{-i} \) play in \( \Gamma' \). Hence \( \sigma_i \) would not be a dominant strategy under \( \Gamma \), a contradiction.
11.3b pure strategy BNE

(revelation Principle for BNE) For every mechanism \( \Gamma \) and Bayesian Nash equilibrium \( \sigma \) of \( \Gamma \) there exists a direct mechanism \( \Gamma' \) and a Bayesian Nash equilibrium \( \sigma' \) of \( \Gamma' \) such that:

(i) The strategy vector \( \sigma' \) satisfies for every \( i \) and every \( \theta_i \):

\[
\sigma'_i(\theta_i) = \theta_i,
\]

that is, \( \sigma' \) prescribes telling the truth;

(ii) For every vector \( \theta \) of types the distribution over outcomes that results in \( \Gamma' \) if the agents play \( \sigma' \) is the same as the distribution over outcomes that results in \( \Gamma' \) if the agents play \( \sigma' \), and the expected value of the transfer payments that result in \( \Gamma' \) if the agents play \( \sigma' \) is the same as the transfer payments that results in \( \Gamma' \) if the agents play \( \sigma' \).

Proof (exactly the same as before):

Construct \( \Gamma' \) by defining the allocation and transfer functions as required by item (ii) in the Proposition. We can prove the result by showing that truth telling is a Bayesian equilibrium of the game. Suppose it were not. If type \( \theta_i \) prefers to report that her type is \( \theta'_i \), then the same type \( \theta_i \) prefers to deviate from \( \sigma \), and to play the strategy that \( \sigma \) prescribes for \( \theta'_i \) in \( \Gamma \). Hence \( \sigma \) is not a Bayesian equilibrium of \( \Gamma \).

11.3c Differences

(c) The structure of both proofs is identical. The only difference is the notion of equilibrium used.

11.4 Bilateral Trading

11.4a

The ex post efficient trading rule will have all the highest valuation agents (both buyers and sellers) owning a unit of the good. That is, we may have only buyers, only sellers or a mixture of both ending up with the good, as long as there is some \( \theta \) such that all agents with \( \theta \geq \tilde{\theta} \) have one unit of the good and all others do not. The total amount of good is given by the continuum \([\tilde{\theta}_1, \tilde{\theta}_1]\).

11.4b

Define the following “competitive” social choice function: let \( q_S \) and \( q_D \) denote market supply and demand, and be defined as:

\[
q_D = \begin{cases} 
\bar{\theta}_2 - \theta_2 & \text{for } \theta_2 \leq p \leq \bar{\theta}_2 \\
(\bar{\theta}_2 - \theta_2) \int_p^{\bar{\theta}_2} d\Phi_2(\theta) & \text{for } \theta_2 \leq p \leq \bar{\theta}_2 \\
0 & \text{for } p \geq \bar{\theta}_2
\end{cases}
\]

\[
q_S = \begin{cases} 
0 & \text{for } p \leq \bar{\theta}_1 \\
(\bar{\theta}_1 - \theta_1) \int_p^{\bar{\theta}_1} d\Phi_1(\theta) & \text{for } \theta_1 \leq p \leq \bar{\theta}_1 \\
\bar{\theta}_1 - \theta_1 & \text{for } p \geq \bar{\theta}_1
\end{cases}
\]

The market equilibrium price \( p^* \equiv \min\{p : q_D(p) \leq q_S(p)\} \), which is a well defined object, will cause efficient trade, and will lead to the efficient outcome described in (a) above.

Trivially, incentive compatibility holds since there is no need for announcements. It is individually rational since a buyer will buy if and only if \( p \leq \theta_2 \) and a seller will sell if and only if \( p \geq \theta_1 \).

This example shows that for a continuum of buyers and sellers the Myerson-Satterthwaite theorem (nonexistence of an individual rational, incentive compatible and ex post efficient direct mechanism) does not hold.

11.5 Single Unit Auction

11.5a

Let \( \{q(\theta), t(\theta)\} \) be a direct mechanism where \( q_i(\theta) \) is the probability that \( i \) gets the object, and \( t_i(\theta) \) the transfer from \( i \) to the seller.
Efficiency requires that the object goes to the player with the higher valuation (note that \( u_i \geq u_j \Leftrightarrow \theta_i \geq \theta_j \) since \( \gamma \in (0, 1) \)). Therefore, an efficient mechanism\(^1\) should have for all \( i \neq j \)

\[
q_i(\theta) = q_i(\theta_i, \theta_j) = 1\{\theta_i > \theta_j\}.
\]

With quasilinear utilities we have that player \( i \)'s payoff is then

\[
U_i(\theta) = q_i(\theta_i + \gamma \theta_{-i}) - t_i.
\]

Consider the following transfer rule

\[
t_i(\theta) = (1 + \gamma)\theta_j q_i(\theta)
\]

where \( q_i \) is the efficient allocation rule defined above.

Let’s check in this direct mechanism truth-telling is an ex post equilibrium and there is efficient allocation of the good. Truth-telling is an ex post equilibrium iff \( \forall i, \forall \theta' 
\]

\[
q_i(\theta)[(\theta_i + \gamma \theta_j) - (1 + \gamma)\theta_j] \geq q_i(\theta_i', \theta_j)[(\theta_i + \gamma \theta_j) - (1 + \gamma)\theta_j] \quad \forall \theta_i'
\]

(3)

For arbitrary \( \theta_i \) and \( \theta_j \), from (3) we need to check \( \forall \theta_i' \)

\[
1\{\theta_i > \theta_j\}[\theta_i - \theta_j] \geq 1\{\theta_i' > \theta_j\}[\theta_i - \theta_j]
\]

(4)

If \( \theta_i > \theta_j \) (4) implies

\[
1 \geq 1\{\theta_i' > \theta_j\}
\]

which holds \( \forall \theta_i' \), and if \( \theta_i < \theta_j \) (4) implies

\[
0 \leq 1\{\theta_i' > \theta_j\}
\]

which again holds \( \forall \theta_i' \).

11.5b

Truth-telling is an equilibrium in (weakly) dominant strategies iff\(^2\) \( \forall i, \forall \theta, \forall \theta_i' 
\]

\[
q_i(\theta_i, \theta_j'')[\theta_i + \gamma \theta_j] - (1 + \gamma)\theta_j] \geq q_i(\theta_i', \theta_j'')[\theta_i + \gamma \theta_j] - (1 + \gamma)\theta_j] \quad \forall \theta_j'
\]

(5)

and

\[
q_i(\theta_i, \theta_j'')[\theta_i + \gamma \theta_j] - (1 + \gamma)\theta_j] > q_i(\theta_i', \theta_j'')[\theta_i + \gamma \theta_j] - (1 + \gamma)\theta_j] \quad \text{for some} \ \theta_j'.
\]

(6)

To show that truth-telling is not an equilibrium in dominant strategies we just need to show an example that violates (5) and (6). Say \( j \) has valuation \( \theta_j \) but reports \( \theta_j' \) with \( \theta_j' > \theta_i > \theta_j \). If \( i \) reports his true valuation, \( \theta_i \), he loses and gets a payoff of 0. On the other hand if he reports \( \theta_i' > \theta_j \), he wins and gets a payoff of \( \theta_i - \theta_j > 0 \). Hence, truth-telling is no longer optimal given the other player had deviated.

---

\(^1\) Assuming the prior distributions are such that ties are zero-probability events.

\(^2\) A similar definition can be given in an interim version without changing the results.