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Microeconomic Theory (521b)

Problem Set 2. Robust Mechanism and Dynamic Allocation Indices

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This problem set is due Thursday, 4/10/08.

1. **Dynamic Allocation Indices.** Consider the deterministic allocation model we discussed in class:

$$x_i : \mathbb{N} \rightarrow \mathbb{R}_+$$

for $i = 1, \dots, I$. We defined the index of each alternative as

$$v_i(s_i) = \max_{\tau_i} \frac{\sum_{t=0}^{\tau_i} \delta^t x_i(s_i + t)}{\sum_{t=0}^{\tau_i} \delta^t}, \quad (1)$$

where $\tau_i^*(s_i)$ is a stopping time which is a solution to (1).

- (a) Show that for all $s_i, \tau_i^*(s_i)$ and $0 < \tau \leq \tau_i^*(s_i)$ we have

$$\frac{\sum_{t=\tau}^{\tau_i^*(s_i)} \delta^t x_i(s_i + t)}{\sum_{t=\tau}^{\tau_i^*(s_i)} \delta^t} \geq v_i(s_i).$$

- (b) In class we suggested a different method of computing the index v_i . For alternative i alone, consider the following optimal stopping problem

$$V_i(s_i, m_i) = \max \left\{ x_i(s_i) + \delta V_i(s_{i+1}, m_i), \frac{m_i}{1 - \delta} \right\}$$

for a given $m_i \in \mathbb{R}$. Define

$$\alpha_i(s_i) \triangleq \sup \left\{ m_i \mid V_i(s_i, m_i) > \frac{m_i}{1 - \delta} \right\}.$$

Prove that $\alpha_i(s_i) = v_i(s_i)$ as given by (1).

- (c) Finally prove formally that an allocation policy across I alternatives is an optimal policy in the sense of maximizing the discounted value if and only if it is an index policy as defined above.

2. Dynamic Allocation Indices for a Simple Stochastic Model.

- (a) Consider now a random return process. The true return process is either

$$x_i = \{1$$

or

$$x_i = \{0 .$$

The initial probability is β_i . Compute the Gittins index (1) by taking into account that the stopping times can now depend on the realization of the sample path and hence can be contingent on the realization in the past period.

- (b) Consider the following simple extension of the above return process, where the return is either governed by

$$x_i = \begin{cases} 1 & \text{with probability } \lambda \\ 0 & \text{with probability } 1 - \lambda \end{cases}$$

with probability β_i or

$$x_i = \{0 .$$

with probability $1 - \beta_i$. Compute the Gittins index (1) by taking into account that the stopping times can now depend on the realization of the sample path and hence can be contingent on the realization in the past period. Which sample path are you taking into account?

- 3. Revenue Maximizing and Type Spaces.** Consider the following model of an optimal auction with two agents and two types for each agent. Consider the following distributions over values for the two agents given by a common prior

	θ_1	θ_2	
θ_1	$\frac{1}{3}$	$\frac{1}{6}$	(2)
θ_2	$\frac{1}{6}$	$\frac{1}{3}$	

and suppose that $\theta_1 = 1$ and $\theta_2 = 2$. The seller has two units of the good and each buyer wants to buy at most one unit of the good. It is thus a multi-unit auction.

- (a) Compute the expected social surplus in this multi-unit auction given the common prior.
- (b) Derive the optimal (= revenue maximizing) mechanism in a Bayes Nash equilibrium for the "naive type space" given by (2). What is the expected revenue in this mechanism?
- (c) Derive the optimal (= revenue maximizing) mechanism in an ex post equilibrium for the "naive type space" given by (2). What is the expected revenue in this mechanism?

4. Consider now the following larger type space for this multi-unit auction, where the t_i at the beginning of the row (column) labels the type and θ_i and the end of the row (column) identifies the payoff type of the agent with type t_i :

	t_2^1	t_2^2	
t_1^1	$\frac{1}{30}$	$\frac{1}{20}$	θ_1
t_1^2	$\frac{1}{10}$	$\frac{1}{15}$	θ_1
t_1^3	$\frac{1}{5}$	$\frac{1}{20}$	θ_1
t_1^4	$\frac{1}{20}$	$\frac{1}{5}$	θ_2
t_1^5	$\frac{1}{15}$	$\frac{1}{10}$	θ_2
t_1^6	$\frac{1}{20}$	$\frac{1}{30}$	θ_2
	θ_1	θ_2	

(3)

- (a) Compute the distribution over payoff types implied by (3) alone. How does it compare to (2)? What can you say about the expected social surplus in this situation?
- (b) Derive the optimal (= revenue maximizing) mechanism in a Bayes Nash equilibrium for the "large type space" given by (3). What is the expected revenue in this mechanism?
- (c) Consider the following final variation of a "large type space":

	t_2^1	t_2^2	t_2^3	t_2^4	
t_1^1	$\frac{1}{6}$	0	0	0	θ_1
t_1^2	0	$\frac{1}{6}$	$\frac{1}{6}$	0	θ_1
t_1^3	0	$\frac{1}{6}$	$\frac{1}{6}$	0	θ_2
t_1^4	0	0	0	$\frac{1}{6}$	θ_2
	θ_1	θ_1	θ_2	θ_2	

(4)

and derive the optimal (= revenue maximizing) mechanism in a Bayes Nash equilibrium for the "large type space" given by (4). What do you conclude?