Econ 521b

Problem Set 3

Suggested Solutions

1) (a) and (b). We first suppose first that θ_1 is observable. We write $U\left(\theta_0, \hat{\theta}_0\right)$ for the expected utility of the type θ_0 buyer reporting valuation $\hat{\theta}_0$:

$$U\left(\theta_0, \hat{\theta}_0\right) = \int_0^1 \left[P\left(\hat{\theta}_0, \theta_1\right)\left[\alpha\theta_0 + (1-\alpha)\theta_1)\right] - t\left(\hat{\theta}_0, \theta_1\right)\right] d\theta_1 \tag{1}$$

where P is the probability of receiving the object while t is the transfer made.

Using the envelope theorem we have:

$$\begin{aligned} U'\left(\theta_0\right) &= \alpha \int_0^1 P\left(\theta_0, \theta_1\right) d\theta_1 \\ &\Rightarrow \; U\left(\theta_0\right) = \alpha \int_0^{\theta_0} \int_0^1 P\left(s, z\right) dz ds \end{aligned}$$

Therefore, we should maximize:

$$\int_0^1 \left[\int_0^1 P\left(\theta_0, \theta_1\right)\left[\alpha\theta_0 + (1-\alpha\theta_1)\right] d\theta_1 - \alpha \int_0^{\theta_0} \int_0^1 P\left(s, z\right) dz\right] d\theta_0 \tag{2}$$

After some algebra we have the following maximization problem:

$$\max_{P \in [0,1]} \int_0^1 \int_0^1 P\left(\theta_0, z\right)\left[\alpha[2\theta_0 - 1] + (1-\alpha)z\right] dz d\theta_0 \tag{3}$$

Pointwise maximization implies:

$$P^*\left(\theta_0, z\right) = \begin{cases} 1 \text{ if } \alpha[2\theta_0 - 1] + (1-\alpha)z > 0 \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

(4) deserves a couple comments. First, notice that for any $\alpha \in (0,1)$ if $\theta_0 > \frac{1}{2}$ we have $P^* (\theta_0, z) = 1$. Consider the case in which $\theta_0 > \frac{1}{2}$, $\alpha \in (0,1)$ and write $\Phi(\theta_0, z, \alpha) = \alpha[2\theta_0 - 1] + (1-\alpha)z$. We have

$$\frac{\partial \Phi(\theta_0, z, \alpha)}{\partial \alpha} = [2\theta_0 - 1 - z] < 0$$

that is, an increase in the ex-ante asymmetry of information increase informational rents and therefore, induce to a more inneficient allocation through the exclusion of more types in the optimal mechanism. The maximum exclusion occurs when $\alpha = 1$. On the other hand, when $\alpha = 0$ no exclusion occurs because the agents do not have *ex-ante* private information

Notice that the buyer will obtain the object with some probability iff:

$$\begin{aligned} \alpha[2\theta_0^* - 1] + (1-\alpha) &\geq 0 \\ \theta_0^* &\geq \frac{1-2\alpha}{2\alpha} \end{aligned} \tag{5}$$

We have to consider two cases:

a) $\theta_0^* \leq 0$ iff $\alpha \leq \frac{1}{2}$. In this case the first period information is not very important, thus all types θ_0 will obtain the object with positive probability.

b) $\theta_0^* > 0$ iff $\alpha > \frac{1}{2}$. In this case the first period information important, thus efficient surplus extraction imply that some types will never obtain the object.

We will solve (a). The same procedure can be applied to (b), but restricting attention to types $\theta_0 \in [\theta_0^*, 1]$.

Our approach consists in designing a mechanism $\{p(\theta), t(\theta)\}$ in which the buyer pays a fee $t(\theta)$ to participate and then faces a price $p(\theta)$. We will construct $\{p(\theta), t(\theta)\}$ in such a way that the seller obtains the same expected revenue under both mechanism. Finally, the proof is completed by showing that the buyer has the same incentives in both mechanisms.

Notice that from (5) we need $p_1(\theta_0) = \alpha(1-\theta_0)$ Thus for $\theta_0 = 0$ we have $p_1(0) = \alpha$.

Since the $\theta_0 = 0$ buyer will have rent equal to zero we can calculate , $t(0)$ by:

$$-t(0) + \int_0^1 \max\{(1-\alpha)s - \alpha, 0\} \, ds = 0$$

thus:

$$-t(0) + \int_{\frac{\alpha}{1-\alpha}}^1 [(1-\alpha)s - \alpha] \, ds = 0$$

with some algebra we have:

$$t(0) = \frac{1}{2}\left(\frac{(1-2\alpha)^2}{1-\alpha}\right) \tag{6}$$

We still have to find $t(\theta_0)$ for all other types.

If a θ_0 buyer announces type $\hat{\theta}_0$ his utility is given by:

$$\begin{aligned}
U(\theta_0, \hat{\theta}_0) &= -t(\hat{\theta}_0) + \int_0^1 \max\left\{\alpha\theta_0 + (1-\alpha)\, z - p(\hat{\theta}_0), 0\right\} dz \\
&= -t(\hat{\theta}_0) + \int_0^1 \max\left\{\alpha\theta_0 + (1-\alpha)\, z - \alpha(1-\hat{\theta}_0), 0\right\} dz \\
&= -t(\hat{\theta}_0) + \int_{\frac{\alpha(1-\hat{\theta}_0-\theta_0)}{(1-\alpha)}}^1 \left[\alpha\theta_0 + (1-\alpha)\, z - \alpha(1-\hat{\theta}_0)\right] dz
\end{aligned}$$

Therefore, the following condition is necessary for truthfully reporting:

$$\frac{\partial U(\theta_0, \hat{\theta}_0)}{\partial \hat{\theta}_0} \mid_{\hat{\theta}_0=\theta_0} = -t'(\theta_0) + \alpha \left[\frac{1-2\alpha(1-\theta_0)}{1-\alpha}\right] = 0$$

Therefore using (6) we have:

$$t(\theta_0) = \frac{1}{2}\left(\frac{(1-2\alpha)^2}{1-\alpha}\right) + \int_0^{\theta_0} \left(\frac{1-2\alpha(1-s)}{(1-\alpha)}\right) \alpha ds \tag{7}$$

Finally, notice that:

$$\begin{aligned}
U'(\theta_0) &= 2\alpha \int_{\frac{\alpha(1--2\theta_0)}{(1-\alpha)}}^1 dz \\
&= 2\alpha \int_0^1 P^*(\theta_0, z)\, dv
\end{aligned}$$

Therefore both mechanisms yield same rents. Finally, it is straightforward to see that a global deviation is profitable in the mechanism $\{p(\theta), t(\theta)\}$ iff it is profitable in the previous (optimal) mechanism (you can adapt the proof provided in **2**).

2) We assume that θ_l is uniformly distributed over $[-1,1]$.

We write $V_0(\theta_0, \hat{\theta}_0)$ for the expected utility of an agent with type θ_0 who announces type $\hat{\theta}_0$ in period 0. We write P for the probability of receiving the good, t for the transfer and V_z for his continuation payoff from period z on. We write $U(\theta_0)$ for the expected utility at $t = 0$. Thus:

$$V(\theta_0, \hat{\theta}_0) = \theta_0 P_1\left(\hat{\theta}_0\right) - t_0(\hat{\theta}_0) + \delta \int_{-1}^{1} V_1\left(\theta_0, \hat{\theta}_0, s\right) ds \tag{8}$$

By the Envelope Theorem we have:

$$U'(\theta_0) = \frac{\partial V(\theta_0, \hat{\theta}_0)}{\partial \theta_0} \mid_{\hat{\theta}_0 = \theta_0} = P_1(\theta_0) + \delta \int_{-1}^{1} \frac{\partial V_1\left(\theta_0, \hat{\theta}_0, s\right)}{\partial \theta_0} \mid_{\hat{\theta}_0 = \theta_0} ds$$

Thus since

$$\begin{aligned} & V_z\left(\theta_k, \hat{\theta}_{k-1}\right) \\ & \max_{\hat{\theta} \in [-1,1]} V_z\left(\theta_k, \hat{\theta}_{k-1,}\hat{\theta}\right) \\ = \quad & \max_{\hat{\theta} \in [-1,1]} \left[\sum_{l=1}^{k} \theta_l P_1\left(\hat{\theta}_k\right) - t_z(\hat{\theta}_k) + \delta \int_{-1}^{1} V_{z+1}\left(\theta_0, \hat{\theta}_0, s\right) ds\right] \end{aligned}$$

Sucessive applications of the envelope theorem yield:

$$U'(\theta_0) = E\left[\sum_{t=0}^{\infty} \delta^t P_t(\theta \mid \theta_0)\right] \tag{9}$$

Therefore the expected rent of an agent type θ_0 is given by:

$$U(\theta_0) = \int_{-1}^{\theta_0} E\left[\sum_{t=0}^{\infty} \delta^t P_t(\theta \mid s)\right] ds \tag{10}$$

Thus our problem can be written as:

$$\max_{P_t} \int_{-1}^{1} \left(\frac{E\left[\sum_{t=0}^{\infty} \delta^t v_t(\theta) P_t(\theta \mid s)\right] - \int_{-1}^{\theta_0} E\left[\sum_{t=0}^{\infty} \delta^t P_t(\theta \mid s)\right] ds}{2}\right) d\theta \tag{11}$$

For instance, if the mechanism prescribes that the buyer never consumes after t' we have: $P_z = 0$ if $z > t'$

Integrating (11) by parts we obtain the following expression for the principal profits:

$$
\begin{aligned}
\Pi^* &= \left(\frac{1}{2}\right)\int_{-1}^{1}\Pi^*(\theta_0)d\theta_0 \\
&= \left(\frac{1}{2}\right)\int_{-1}^{1}E\left(\sum_{t=0}^{\infty}\delta^t P_t^*(\theta \mid s)\left[v_t(\theta)-(1-\theta_0)\right]\right)d\theta_0 \qquad (12)
\end{aligned}
$$

(12) has a recursive formulation given by:

$$
\begin{aligned}
\Pi^*(\theta_0) &= \max\left\{0, \theta_0-(1-\theta_0)+\delta\int_{-1}^{1}\left(\frac{\Pi^*(\theta_0,s)}{2}ds\right)\right\} \qquad (13) \\
\Pi^*(\theta_0,....,\theta_t) &= \max\left\{0, 2\theta_0-1+\sum_{j=1}^{t}\theta_j+\delta\int_{-1}^{1}\left(\frac{\Pi^*(\theta_0,....,\theta_t,s)}{2}ds\right)\right\}
\end{aligned}
$$

Therefore the optimal allocation is a stopping time $T^*(\theta)$ solving (13).

Now we have to find the optimal mechanism $\{(t(\theta_0), t(\theta_0,\theta_1), ...\}$ that leads to $T^*(\theta)$. By the Bellman equation above we can make $t(\theta_0,\theta_1) = t(\theta_0,\theta_1,\theta_2) = t(\theta_0,...,\theta_t) = (1-\theta_0)$.

Therefore, we just have to find $t(\theta_0)$. Since the lowest type should have zero rent we impose:

$$
t(-1) = -E\left[\sum_{t=0}^{T^*(\theta|\theta_0=-1)}\delta^t v_t(\theta \mid \theta_0=-1)\right] \qquad (14)
$$

From (8) we need:

$$
\frac{\partial V_0\left(\theta,\hat{\theta}\right)}{\partial\hat{\theta}} \mid_{\hat{\theta}=\theta} = 0
$$

what leads to:

$$
\begin{aligned}
\frac{\partial V_0\left(\theta,\hat{\theta}\right)}{\partial\hat{\theta}} &\mid_{\hat{\theta}=\theta} = 0 \\
&\Leftrightarrow -t_0'(\theta)+\frac{\partial V_0\left(\theta,\hat{\theta}\right)}{\partial\hat{\theta}} \mid_{\hat{\theta}=\theta} = 0
\end{aligned}
$$

From (9) we have:

$$
t_0'(\theta_0) = E\left[\sum_{t=0}^{T(\theta|\theta_0)}\delta^t\right]
$$

what leads to:

$$t_0(\theta_0) = \int_{-1}^{\theta} E\left[\sum_{t=0}^{T(\theta|s)} \delta^t\right] ds$$

We worked with the relaxed problem in which we were just using the envelope theorem.

In order to see sufficiency take $\theta^1 > \theta^2$ and $\hat{\theta}^1 > \hat{\theta}^2$, and let $T\left(\theta, \hat{\theta}\right)$ be the induced stopping time by an agent who is type θ but announces $\hat{\theta}$. We have:

$$\frac{\partial V_0\left(\theta, \hat{\theta}\right)}{\partial \theta} = E\left[\sum_{t=0}^{T\left(\theta, \hat{\theta}\right)} \delta^t\right]$$

Now notice that the stopping time $T\left(\theta, \hat{\theta}\right)$ is increasing in θ. Thus if $\theta^{'} > \theta^{''}$ we have:

$$\frac{\partial V\left(\theta, \hat{\theta}\right)}{\partial \theta} |_{\theta=\theta'} - \frac{\partial V\left(\theta, \hat{\theta}\right)}{\partial \theta} |_{\theta=\theta''} = E\left[\sum_{t=0}^{T\left(\theta', \hat{\theta}\right)} \delta^t\right] - E\left[\sum_{t=0}^{T\left(\theta'', \hat{\theta}\right)} \delta^t\right] \geq 0$$

Thus:

$$\frac{\partial^2 V\left(\theta, \hat{\theta}\right)}{\partial \theta \partial \hat{\theta}} \geq 0 \text{ (a.e.)} \tag{15}$$

(15) guarantees that no type has a profitable global deviation.