Mechanism Design with Common Values

In this lecture, we cover the case where the players’ care about each others’ types.

- Market for lemons.
- Hiring a worker.
- Financial markets.
- Auctions of oil tracts.
- Etc.

Simplest example: Wallet game. An indivisible object is auctioned to one of two potential bidders. Bidder $i$ has type $\theta_i$. The bidders have pure common values:

$$v_i = \theta_1 + \theta_2$$

Assume for simplicity that the $\theta_i$ are i.i.d. random variables with a continuous density on $[0, 1]$.

How would you bid in a first price auction?
How would you bid in a second price auction?
Does the second price auction have dominant strategy equilibria?
Does the game have other equilibria that are similar to dominant strategy equilibria in some sense?

Analysis of the first price auction: Find a symmetric equilibrium in strictly increasing strategies. Implication: Player $i$ wins if $\theta_i > \theta_j$. Given the continuous distribution, ties can be ignored. Therefore the expected payoff to player $i$ with type $\theta_i$ from bidding $b(\theta'_i)$ is:

$$U(\theta_i, \theta'_i) = (\theta_i + \mathbb{E}[\theta_j \mid \theta'_i > \theta_j] - b(\theta'_i))F(\theta'_i).$$

In equilibrium, we must have:

$$\theta'_i = \theta_i.$$

Let

$$U(\theta_i, \theta_i) \equiv U(\theta_i).$$

Therefore:

$$U'(\theta_i) = F(\theta_i),$$
and we conclude that revenue equivalence from the earlier lectures holds.

We have:

$$\int_0^{\theta_i} F(s) ds = (\theta_i + \mathbb{E}[\theta_j \mid \theta_i > \theta_j] - b(\theta_j)) F(\theta_i).$$

Therefore we can solve:

$$b(\theta_i) = (\theta_i + \mathbb{E}[\theta_j \mid \theta_i > \theta_j]) - \frac{\int_0^{\theta_i} F(s) ds}{F(\theta_i)}.$$

Consider for example a uniform distribution. Then we have:

$$b(\theta_i) = \theta_i + \frac{1}{2} \theta_i - \frac{1}{2} \theta_i = \theta_i.$$

We could perform the same steps for second price auction to get to the stage:

$$\int_0^{\theta_i} F(s) ds = (\theta_i + \mathbb{E}[\theta_j - b(\theta_j) \mid \theta_i > \theta_j]) F(\theta_i).$$

Since both sides are equal for all values of $\theta_i$, the derivatives of the two sides of the equality must coincide. Therefore:

$$F(\theta_i) = F(\theta_i) + \theta_i f(\theta_i) + \theta_i f(\theta_i) - b(\theta_i) f(\theta_i).$$

Solving for $b(\theta_i)$ gives:

$$b(\theta_i) = 2\theta_i.$$

Is there a direct argument for the result?

You want to win if and only if the payoff exceeds your payment. If you bid twice and win your value, you pay $2 \min[\theta_i, \theta_j]$ whereas your value is $\theta_i + \theta_j > 2 \min[\theta_i, \theta_j]$.

Therefore you win if and only if your value exceeds your payment.

Is this a dominant strategy equilibrium?

Not quite. What if the other player bids a constant amount, say $.5$ in the uniform distribution case? Then you want to win regardless of your own type. (Why?)

Nevertheless the equilibrium seems to have nice properties. First of all, beliefs about the distribution of your opponent’s value do not appear at all in the calculation of the optimal strategy. Furthermore, no bidder has an incentive to change their strategy even
if they learn the other bidders’ types. An equilibrium with this property is called an \emph{ex post} equilibrium.

**Definition 1.** A mechanism \((q(\theta), t(\theta))\) is \emph{ex post incentive compatible} if for all \(\theta_i, \theta_{-i}\)

\[
\theta_i \in \arg\max_{\theta_i \in \Theta_i} u_i(q(\hat{\theta_i}, \theta_{-i}), \theta_i) - t(\hat{\theta_i}, \theta_{-i}).
\]

In an ex post equilibrium, players do not care about each others’ types at all conditional that they report truthfully. It turns out that the property of having a BNE that is independent of the prior beliefs is closely connected to the equilibrium being an ex post equilibrium. This is really Dirk’s stuff under the heading: Robust Mechanism Design.

**Implementability of Efficient Decision Rules**

In the quasi-linear setting with private values, VCG-mechanisms implement the efficient decision rules in dominant strategies. Is this possible with common values. In the Wallet Game example, we saw that dominant strategy equilibria will fail to exist in general for common values settings. What about Bayesian or ex post equilibria? The following example shows that it is not possible to implement the efficient decision rule in common values settings. Recall that in the case of independent types, a necessary condition for implementability is that the allocation rule be monotonic in each player’s type. Efficient rules are clearly monotonic in this sense in the private values case since an increase in the type of one agent leaves the values of other agents unchanged.

**Failure of monotonicity**

Two players. Problem: how to allocate an indivisible object? Only player 1 has private information \(\theta_1 \in [0, 2]\). Both players’ payoffs depend on this information:

\[
v_1 = \theta_1, v_2 = 2\theta_1 - 1.
\]

Efficient allocation of the object (letting \(q(\theta_1)\) be the probability of assigning the good to 1):

\[
\theta_1 > 1 \Rightarrow q(\theta_1) = 0, \quad \theta_1 \leq 1 \Rightarrow q(\theta_1) = 1.
\]

Is there a transfer function that could make this allocation incentive compatible? At most two transfers can be used (transfers must be constant conditional on the allocation). (Why?)
Let \( t(0) \) be the transfer payment when 1 gets the object, and \( t(1) \) the payment that 1 makes when 2 gets the object.

Then we have from incentive compatibility:

\[
\theta_1 - t(0) \geq -t(1) \text{ for all } \theta_1 \leq 1,
\]

and

\[
-t(1) \geq \theta_1 - t(0) \text{ for all } \theta_1 > 1.
\]

But these inequalities are seen to be incompatible by e.g. summing them up.

The efficient decision rule fails to be implementable in this example because monotonicity of the decision rule fails. The reason for this failure can be traced to the fact that player 1’s type has a larger effect on the valuation of player 2 than on the valuation of player 1 herself.

**Single Crossing**

Consider next that case where all players \( i \in 1, \ldots, N \) have private information represented by a single dimensional type \( \theta_i \). Assume for now that the types are independent across players, and they are distributed on an interval \([\underline{\theta}, \overline{\theta}]\). Write the payoff functions of the players as:

\[
u_i(q(\hat{\theta}), \theta) - t(\hat{\theta}).\]

We say that the efficient decision rule \( x(\theta) \) is **monotonic** if for all \( \theta_{-i} \) and all \( \hat{x} \), the set \( S_i(\hat{x}) = \theta_i \mid x(\theta_i, \theta_{-i}) = \hat{x} \) is connected.

Notice that the notion of monotonicity we have here is quite general.

We say that the model satisfies **single crossing** if for all \( i, j \), all \( \theta_i \), and all \( \hat{x} \), we have:

\[
\frac{\partial^2 u_i(\hat{x}, \theta)}{\partial \theta_i \partial x} \geq 0
\]

Monotonicity and single crossing are sufficient for ex post implementation of the efficient decision rule.

**Theorem 2.** Suppose that the efficient decision rule \( x(\theta) \) is monotonic and satisfies single crossing. Then there exists a schedule of transfers \( t(\theta) \) such that \( x(\theta), q(\theta) \) is an ex post incentive compatible mechanism. We call such mechanisms generalized VCG-mechanisms.
Proof. The proof is a slight modification of proof for the VCG-mechanisms in the private values case. Let \( u_{-i}(x, \theta) \) denote the sum of gross utilities to players other than \( i \) from the efficient allocation at \( \theta \). For an arbitrary announcement \( \theta_{-i} \) by other players, let the transfer be defined as:

\[
t_i(\theta_{-i}, \theta_i) = -\int_{\theta_i}^{\theta} \frac{\partial u_{-i}(x, v_i, \theta_{-i})}{\partial x} \frac{\partial x(v_i, \theta_{-i})}{\partial \theta_i} dv_i + h(\theta_{-i}).
\]

With this transfer scheme, it is clear that the agent’s first order condition for truth-telling is satisfied. (Think about the usual envelope type arguments).

Second order condition is satisfied if single crossing holds. \( \square \)

So how exactly did our simple example violate this theorem?

Write the valuation of bidder \( i \) as \( x\theta_i \), where \( x \) is the probability of getting the object. Then the order given by the efficient allocation says that \( x \) and \( \theta_i \) are inversely ordered. But with this ordering, single crossing of \( u_i \) fails. (It holds for the ‘natural ordering’ on the real line for \( x, \theta \).

What does the modified VCG mechanism look like in simple applications?

- Is this the second price auction? Ascending auction?
- Why different transfers?
- Is it possible to construct pivotal mechanisms where the equilibrium payoff of each player coincides with her contribution to the societal welfare?

Common Values and Multi-Dimensional Signals

In general, it is impossible to implement efficient outcomes.

Simplest example: Allocation of an indivisible object between two players.

Only player 1 has private information.

Type is two-dimensional: \( (\theta, s) \).

Valuations for the object:

\[
u_1 = \theta + 2s, \quad u_2 = 2\theta + s.
\]

Efficiency: 1 gets object if \( \theta < s \).

How could this be implemented?

Jehiel and Moldovanu, Econometrica 2001 show that this problem extends to general allocation problems with multi-dimensional signals.
Finally, Jehiel et al. (Econometrica, 2006) have a very negative result about implementation in ex post equilibria when types are multi-dimensional. The result states essentially that only constant functions are implementable in ex post equilibria. This is quite remarkable since it does not even allow for the implementation of ‘dictatorial’ rules, where some agent is given free choice of allocation conditional on her own information. The failure for this to be an ex post equilibrium comes from the fact that in general, the best alternative of for the dictator depends on the information of others. In ex post equilibrium, one cannot simply close one’s eyes and behave based on the prior only (can this be done in Bayesian implementation?).

To sum: Generalizing to multi-dimensional types leads quickly to negative results (in particular for ex post equilibrium). Recent work of Dirk and Stephen Morris on robust mechanism design makes the connection to models with higher order beliefs. Only ex post equilibria are robust to higher order beliefs, and therefore the difficulty of getting ex post equilibria can be interpreted as difficulty of doing mechanism design that does not depend on higher order beliefs.

There are, however, some positive results in the literature:

Mezzetti, Econometrica 2004, shows that ex post implementation of efficient alternatives is possible if one may use transfer schedules that depend on the player’s types and reported utilities from the allocations. The mechanisms are two stage in the sense that in the first stage, the agents report their types and this leads to the choice of an (efficient) allocation. In the second stage, agents report their utilities from the allocation. This second stage gives an extra chance to check the truthfulness of the first stage reports and thus limits the gains from misreporting types.