A General Framework

- A seller with an indivisible item for sale, zero cost
- \( I \) bidders: \( i = 1, \ldots, I \)
- Each bidder \( i \) has private information \( v_i \in V_i \).
- Given the profile \( \nu = (v_i, v_{-i}) \), bidder \( i \)'s valuation is \( u_i (v_i, v_{-i}) \) if he gets the item and zero otherwise.
- The prior distribution over \( V \equiv \times_{i=1}^I V_i \) is \( F (\nu) \). After knowing one’s own \( v_i \), bidder \( i \) forms the posterior distribution of others’ valuation payoff as \( F_i (v_{-i} | v_i) \).
- All bidders and seller have von-Neuman Morgenstern expected and transferable utility function.
\( S_i \): (pure) Strategy space for bidder \( i \) (the amount \( i \) can bid in auction)

Let \( P_i (s_1, \ldots, s_l) : \times_{i=1}^l S_i \rightarrow \mathbb{R}_+^l \) be the probability that bidder \( i \) wins.

For any strategy profile \((s_i, s_{-i})\), \( 0 \leq \sum_{i=1}^l P_i (s_i, s_{-i}) \leq 1 \).

Let \( T_i (s_1, \ldots, s_l) : \times_{i=1}^l S_i \rightarrow \mathbb{R}^l \) be monetary payment that bidder \( i \) transfers to seller (no matter \( i \) wins or not).

\( T_i (s_i, s_{-i}) \) can even be negative.
**Private value** model: if for all $v_i, v'_i$,
\[ v_i = v'_i \Rightarrow u_i (v_i, v_{-i}) = u_i (v'_i, v_{-i}) . \]

**Common value** model: if the above condition is violated.

**Independent value** model: if $v = (v_i, v_{-i})$ is independently drawn. So, $f(v) = \prod_{i=1}^{l} f_i (v_i)$.

**Symmetric** case: if $f_i (v) = f_j (v)$ for any $i$ and $j$.

Throughout the lecture, we will focus on the independent, symmetric and private value model in which all $v_i$s are i.i.d. drawn from a common distribution.

We also assume that all bidders and seller are risk neutral.

Hence, given the strategy profile $(s_i, s_{-i})$, bidder $i$’s payoff is $v_i P_i (s_i, s_{-i}) - T_i (s_i, s_{-i})$. 
Four Standard Auctions

- **First Price Auction (High-bid Auction)**
  - buyers simultaneously submit bids
  - the highest bidder wins (tie broken by flip coin)
  - winner pays bid (losers pay nothing)

\[
P_i(s_i, s_{-i}) = \begin{cases} 
1 & \text{if } s_i > s_j, \forall j \neq i \\
\frac{1}{K} & \text{if } s_i \text{ ties for highest with } K - 1 \text{ others} \\
0 & \text{otherwise}
\end{cases}
\]

\[
T_i(s_i, s_{-i}) = \begin{cases} 
s_i & \text{if } i \text{ wins} \\
0 & \text{otherwise}
\end{cases}
\]
Dutch Auction (Open Descending Auction)

- Auctioneer starts with a high price and continuously lowers it until some buyer agrees to buy at current price
- The highest bidder wins (tie broken by flip coin)

\[
P_i(s_i, s_{-i}) = \begin{cases} 
1 & \text{if } s_i > s_j, \forall j \neq i \\
1/K & \text{if } s_i \text{ ties for highest with } K - 1 \text{ others} \\
0 & \text{otherwise}
\end{cases}
\]

\[
T_i(s_i, s_{-i}) = \begin{cases} 
s_i & \text{if } i \text{ wins} \\
0 & \text{otherwise}
\end{cases}
\]

This is the same as the case in FPA.

Dutch Auction and First Price Auction are \textit{strategically} equivalent.
Second Price Auction (Vickrey Auction)

- same rules as FPA except that winner pays second highest bid
- proposed in 1961 by William Vickrey

\[
P_i (s_i, s_{-i}) = \begin{cases} 
1 & \text{if } s_i > s_j, \forall j \neq i \\
1/K & \text{if } s_i \text{ ties for highest with } K - 1 \text{ others} \\
0 & \text{otherwise}
\end{cases}
\]

\[
T_i (s_i, s_{-i}) = \begin{cases} 
\max_{j \neq i} s_i & \text{if } i \text{ wins} \\
0 & \text{otherwise}
\end{cases}
\]

Recall that in SPA, it is a (weakly) dominant strategy for each bidder to bid truthfully, i.e., \( s_i = v_i \).
English Auction (Ascending Price Auction)
- buyers announce bids, each successive bid higher than previous one
- the last one to bid the item wins at what he bids

As long as the current price $p$ is lower than $v_i$, bidder $i$ has a chance to get positive surplus. He will not drop out until $p$ hits $v_i$.

Only when anyone else drops out before bidder $i$, i.e., $p = \max_{j\neq i} v_j$ can he win by paying $p$, the second highest valuation.

This shows that English Auction and Second Price Auction are equivalent.
Let $v_i$ be agent $i$’s valuation (type).

- $v_i \sim F(\cdot)$ on $[0, 1]$, i.i.d. and symmetric across all bidders.

- Bidder $i$’s payoff is $v_i - t_i$ if he wins the auction and pays $t_i$ and zero otherwise.

- We now discuss the equilibrium features in SPA and FPA.
Order Statistics

- Given a random variable $\tilde{v}_i$, we know that $F(v) = \Pr[\tilde{v}_i \leq v]$.
- Suppose we sample $n$ times i.i.d. and get $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n$.
- Denote the maximum $\tilde{v}_{(1)} = \max\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n\}$.
  - $\Pr[\tilde{v}_{(1)} \leq v] = [F(v)]^n$
Denote the second maximum

\[ \tilde{v}_{(2)} = \max \left\{ \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n \right\} \backslash \left\{ \tilde{v}_{(1)} \right\}. \]

\[ \Pr [\tilde{v}_{(2)} \leq v] = \Pr [\tilde{v}_{(1)} \leq v] + \Pr [\tilde{v}_{(2)} \leq v < \tilde{v}_{(1)}] \]
\[ = [F(v)]^n + n[F(v)]^{n-1}[1 - F(v)] \]

From an individual bidder’s perspective, the probability that his valuation \( v \) is the highest amounts to:

\[ G(v) = \Pr [\text{all the others’ valuations are no more than } v] \]
\[ = [F(v)]^{n-1}. \]
Recall: in SPA, it is a (weakly) dominant strategy for each bidder to bid truthfully, i.e., \( b_i(v_i) = v_i \).

Buyer’s expected payoff is

\[
\pi^{SPA}(v) = \int_0^v (v - x) G(x) \, dx
\]

\[
= (v - x) G(x)|_{x=0}^{x=v} + \int_0^v G(x) \, dx
\]

\[
= \int_0^v G(x) \, dx.
\]
Seller’s expected revenue is

\[
R^{SPA} = \mathbb{E}[\tilde{\nu}(2)] = \int_0^1 x d \left\{ [F(x)]^n + n [F(x)]^{n-1} [1 - F(x)] \right\}
\]

\[
= n(n-1)[F(x)]^{n-2} [1-F(x)] f(x) dx
\]

\[
= n \int_0^1 x [1 - F(x)] (n - 1) [F(x)]^{n-2} f(x) dx
\]

\[
= d[F(x)]^{n-1} = dG(x)
\]

\[
= n \int_0^1 x [1 - F(x)] dG(x) \quad \text{by part}
\]

\[
= n x [1 - F(x)] G(x) \bigg|_{x=0}^{x=1}
\]

\[
- n \int_0^1 [1 - F(x) - xf(x)] G(x) dx
\]

\[
= n \int_0^1 \left[ x - \frac{1 - F(x)}{f(x)} \right] G(x) dF(x)
\]
Recall: in FPA the highest bidder wins and pays his bid
uniform tie breaking rule
If an individual bidder’s valuation is $v_i$ and he bids $b_i$, his expected payoff is:

$$\pi_i (v_i, b_i) = (v_i - b_i) \hat{Q}_i (b_i),$$

where $\hat{Q}_i (b_i)$ is the probability of winning if he bids $b_i$ and each other bidder $j \neq i$ bids according to $b_j (\cdot)$

bid function is a mapping from valuation space into positive real line $b_i : [0, 1] \rightarrow \mathbb{R}_+$

We consider the symmetric and pure strategy where the amount of a bid only depends on one’s valuation.
Conventionally, we use FOC and SOC to solve out the optimal $b_i$ for a bidder, i.e.

$$
FOC : \frac{\partial \pi_i (v_i, b_i)}{\partial b_i} = (v_i - b_i) \hat{Q}_i' (b_i) - \hat{Q}_i (b_i) = 0
$$

This gives a best response for type $v_i$ against $b_j, j \neq i$, an ODE (ordinary differentiable equation) of $Q_i (\cdot)$.

Let’s make some guess first

- Guess A: $b^* (v)$ is monotone.
- Guess B: $b^* (v)$ is differentiable.
Theorem

In the independent, private value, symmetric environment, the unique equilibrium strategy is

\[ b^*(v) = v - \int_0^v \frac{G(x)}{G(v)} \, dx. \]

Observe that

- Underbidding: \( b^*(v) < v \), but the gap \( v - b^*(v) \) decreases as \( n \) increases (Competition effect).
- \( b^*(v) \) is strictly increasing in \( v \):
  \[ b'^*(v) = \frac{g(v)}{G(v)} \int_0^v \left( \frac{G(x)}{G(v)} \right) \, dx > 0 \text{ for any } v > 0. \]
In equilibrium, \( b_i = b(v_i) \), \( v_i = \varphi(b_i) \) and \( \varphi'(b_i) = 1/b'(v_i) \):

\[
(v_i - b(v_i)) \frac{G'(v_i)}{b'(v_i)} = G(v_i).
\]

Rearranging terms,

\[
v_i G'(v_i) = b(v_i) G'(v_i) + b'(v_i) G(v_i) = \frac{d}{dv_i} [b(v_i) G(v_i)].
\]

Integrating, \( \int_0^{v_i} xG'(x) \, dx = b(v_i) G(v_i) \).

LHS can be computed as

\[
\int_0^{v_i} xG'(x) \, dx = \int_0^{v_i} x \, dG(x) = v_i G(v_i) - \int_0^{v_i} G(x) \, dx.
\]

We thus achieve the goal. □
Efficiency: the bidder who values the item most wins the auction.

A bidder’s ex ante payoff is

$$\pi^{FPA} (v) = [v - b^* (v)] \cdot \frac{\text{Prob. of winning}}{G(v)} = \int_0^v \frac{G(x)}{G(v)} dx \cdot G(v)$$

$$= \int_0^v G(x) \, dx.$$  

Bidder’s ex ante payoff in FPA is identical to that in SPA.
Seller’s expected revenue is

\[ \mathbb{E} \left[ b \left( \bar{v}^{(1)} \right) \right] = \int_0^1 \left( v - \int_0^v \frac{G(x)}{G(v)} \, dx \right) \, d \left\{ [F(v)]^n \right\} \]

\[ = n \int_0^1 \left( vG(v) - \int_0^v G(x) \, dx \right) \, dF(v) \]

\[ = n \int_0^1 \left( \int_0^v x \, dG(x) \right) \, dF(v) = n \int_0^1 x \left( 1 - F(x) \right) \, dG(x) \]

\[ = nx \left( 1 - F(x) \right) G(x) \bigg|_{x=0}^{x=1} \]

\[ -n \int_0^1 \left( 1 - F(x) - xf(x) \right) G(x) \, dx \]

\[ = n \int_0^1 \left( x - \frac{1 - F(x)}{f(x)} \right) G(x) \, dF(x) = R_{SPA} \]
- Seller’s expected revenue in FPA is *identical* to that in SPA.
- We thus show the payoff and revenue equivalence between FPA and SPA under the independent, private value setting.
- Such equivalence can also be generalized to the case if two auctions (i) assign the item to the bidder of the highest value and (ii) leave the bidders of the lowest value zero surplus. (See our lecture note on "Optimal Auction".)
So far we have studied bidder’s bidding strategies while taking the auction rule as given, compute and compare the expected revenue for seller and payoff for bidders. You can figure out any other auction rule and solve it from the very beginning.

We don’t know whether a specific auction can extract revenue as much as possible for the seller.

We need to explore the optimal mechanism where the seller’s ex ante revenue is maximized and how to design a specific auction to achieve such goal. This will be covered in our ongoing lectures on "Optimal Auction".