Problem Set 3. Optimal Auctions and Implementation

Suggested Answer

23.D.2. in MGW. Suppose agent 1 is the seller and 2 is the buyer. Trade occurs if $\theta_1 \leq \theta_2$, so the decision rule is

$$k_1 (\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 \leq \theta_2 \\ 0 & \text{if } \theta_1 > \theta_2 \end{cases},$$

and thus:

$$v_1 (k, \theta_1, \theta_2) = (1 - k) \theta_1, v_2 (k, \theta_1, \theta_2) = k \theta_2.$$

Now

$$-h_1 (\theta_2) = E_{\theta_1} [v_1 (k^* (\theta_1, \theta_2), \theta_2)] = E_{\theta_1} [\theta_1 | \theta_1 > \theta_2] = \int_{\theta_2}^{1} \theta_1 d\theta_1 = \frac{1 - \theta_2^2}{2},$$

and

$$-h_2 (\theta_1) = E_{\theta_2} [v_2 (k^* (\theta_1, \theta_2), \theta_2)] = E_{\theta_2} [\theta_2 | \theta_2 \leq \theta_2] = \int_{\theta_1}^{1} \theta_2 d\theta_2 = \frac{1 - \theta_1^2}{2}.$$

Hence, the optimal transfer for agent 1 is the sum of these two terms:

$$\Rightarrow t_1 (\theta_1, \theta_2) = E_{\theta_2} [v_2 (k^* (\theta_1, \theta_2), \theta_2)] + E_{\theta_1} [v_1 (k^* (\theta_1, \theta_2), \theta_2)] = \frac{1 - \theta_1^2}{2} - \frac{1 - \theta_2^2}{2} = \frac{\theta_2^2 - \theta_1^2}{2},$$

and by symmetry we have

$$t_2 (\theta_1, \theta_2) = \frac{\theta_1^2 - \theta_2^2}{2}.$$

(b) For the seller,
\[
\max_{\hat{\theta}_1 \in [0, 1]} EU_1 (\theta_1, \hat{\theta}_1) = E_{\theta_2} \left[ v_1 \left( k^* (\hat{\theta}_1, \theta_2), \theta_1, \theta_2 \right) + \frac{\theta_2^2 - \theta_1^2}{2} \right] \\
= E_{\theta_2} \left[ v_1 \left( k^* (\hat{\theta}_1, \theta_2), \theta_1, \theta_2 \right) \right] + E_{\theta_2} \left[ \frac{\theta_2^2 - \theta_1^2}{2} \right] \\
= E_{\theta_2} \left[ \theta_1 | \hat{\theta}_1 \geq \theta_2 \right] + E_{\theta_2} \left[ \frac{\theta_2^2}{2} \right] - E_{\theta_2} \left[ \frac{\theta_1^2}{2} \right] \\
= \theta_1 \hat{\theta}_1 + E_{\theta_2} \left[ \frac{\theta_2^2}{2} \right] - \frac{\theta_1^2}{2},
\]

which is quadratic in \( \hat{\theta}_1 \).

FOC w.r.t. \( \hat{\theta}_1 \) is enough to ensure the global maximization which yields

\[ \theta_1 - \hat{\theta}_1 = 0, \]

so \( \hat{\theta}_1 = \theta_1 \) maximizes \( EU_1 (\theta_1, \hat{\theta}_1) \), showing that truth-telling is optimal.

For the buyer,

\[
\max_{\hat{\theta}_2 \in [0, 1]} EU_2 (\theta_2, \hat{\theta}_2) = E_{\theta_1} \left[ v_2 \left( k^* (\hat{\theta}_1, \theta_2), \theta_1, \theta_2 \right) + \frac{\theta_1^2 - \theta_2^2}{2} \right] \\
= E_{\theta_1} \left[ v_2 \left( k^* (\hat{\theta}_1, \theta_2), \theta_1, \theta_2 \right) \right] + E_{\theta_1} \left[ \frac{\theta_1^2 - \theta_2^2}{2} \right] \\
= E_{\theta_1} \left[ \theta_2 | \hat{\theta}_2 > \theta_1 \right] + E_{\theta_1} \left[ \frac{\theta_1^2}{2} \right] - E_{\theta_1} \left[ \frac{\theta_2^2}{2} \right] \\
= \theta_2 \hat{\theta}_2 + E_{\theta_1} \left[ \frac{\theta_1^2}{2} \right] - \frac{\theta_2^2}{2},
\]

FOC w.r.t. \( \hat{\theta}_2 \) is enough to ensure the global maximization which yields:

\[ \theta_2 - \hat{\theta}_2 = 0 \]

Therefore, \( \hat{\theta}_2 = \theta_2 \) maximizes \( EU_2 (\theta_2, \hat{\theta}_2) \), establishing optimality in truth-telling.
One can compute that

$$EU_1 \left( \theta_1, \theta_1 \right) \bigg| \theta_1 = \theta_1 = \frac{\theta_1^2}{2} + \frac{1}{6}.$$

The seller is willing to participate the game if and only if \( \frac{\theta_1^2}{2} + \frac{1}{6} > \theta_1 \), or \( \theta_1 < 1 - \sqrt{\frac{\theta_1}{3}} \).

Note that buyer’s reservation utility is zero and his expected utility is always positive: 

$$EU_2 \left( \theta_2, \theta_2 \right) \bigg| \theta_2 = \theta_2 = \frac{\theta_2^2}{2} + \frac{1}{6} > 0.$$

Hence buyer of any type will participate.

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**23.E.6 in MGW.** (a) The efficient trading rule is such that if \( \theta_i > \theta_j \) then \( j \) should sell the good to \( i \) for any price \( p \in [\theta_j, \theta_i] \). We specify that no trade occurs if \( \theta_i = \theta_j \).

(b) Let \( b_i \) denote agent \( i \)'s bid. The ex post utility of agent \( i \) given their bids \( (b_i, b_j) \) and his own type \( \theta_i \) is:

$$U_i (b_i, b_j) = 1 (b_i > b_j) (2 \theta_i - b_i) + 1 (b_i < b_j) b_j + \theta_i 1 (b_i = b_j),$$

where \( 1 (b_i > b_j) \) takes value of 1 if \( b_i > b_j \) holds and 0 otherwise. Note that since the payment is transferred between them, the total surplus is \( 2 \theta_k \) where \( b_k = \max (b_i, b_j) \).

With no loss of generality, we restrict our attention to symmetric equilibrium: \( b_i = b(\theta_i) \). Furthermore, we only consider the case when \( b(\cdot) \) is continuous, differentiable and strictly increasing. Hence, \( 1 (b_i > b_j) = 1 (\theta_i > \theta_j) \) and the case of \( b_i = b_j \) can be ignored. Denote the expected payoff for agent \( i \) with \( \theta_i \) who bids at \( b(\theta_i') \) by \( EU_i (\theta_i, \theta_i') \) :

$$EU_i (\theta_i, \theta_i') = [2 \theta_i - b(\theta_i')] E_{\theta_j} \left[ 1 (\theta_i' > \theta_j) \right] + E_{\theta_j} \left[ 1 (\theta_i' < \theta_j) b(\theta_j) \right]$$

$$= [2 \theta_i - b(\theta_i')] \int_{\theta_i'}^{\theta_i} d\theta_j + \int_{\theta_i'}^{1} b(\theta_j) d\theta_j$$

$$= [2 \theta_i - b(\theta_i')] \theta_i + \int_{\theta_i'}^{1} b(\theta_j) d\theta_j.$$

Incentive compatibility condition requires that \( \theta_i = \arg \max_{\theta_i'} EU_i (\theta_i, \theta_i') \). Using the F.O.C., we have:

$$\frac{\partial EU_i (\theta_i, \theta_i')}{\partial \theta_i'} \bigg|_{\theta_i' = \theta_i} = 2 \theta_i - 2b(\theta_i) - \theta_i b'(\theta_i) = 0.$$
Multiplying $\theta_i$ on both sides and rearranging, we have:

$$2\theta_i^2 = 2\theta_i b(\theta_i) + \theta_i^2 b'(\theta_i) = \frac{d}{d\theta_i} \left[ \theta_i^2 b(\theta_i) \right].$$

Integrating,

$$\theta_i^2 b(\theta_i) = 2 \int_0^{\theta_i} t^2 dt = \frac{2}{3} \theta_i^3 \Rightarrow b(\theta_i) = \frac{2}{3} \theta_i.$$

(c) The social choice function is: if $\theta_i > \theta_j$, $i$ buys the good from $j$ at price $\frac{2}{3} \theta_i$. Hence, the ex-post efficient allocation is reached. The analysis shown in (b) verifies that it is incentive compatible.

Agent $i$’s expected payoff is:

$$EU_i(\theta_i, \theta_j) = [2\theta_i - b(\theta_i)] \theta_i + \int_{\theta_i}^{1} b(\theta_j) d\theta_j = \left[ 2\theta_i - \frac{2}{3} \theta_i \right] \theta_i + \int_{\theta_i}^{1} \frac{2}{3} \theta_j d\theta_j$$

$$= \frac{4}{3} \theta_i^2 + \frac{1}{3} (1 - \theta_i^2) = \theta_i^2 + \frac{1}{3} > \theta_i$$

for $\theta_i \in [0, 1]$.

Hence, it is ex ante individually rational.

In the Myerson-Satterthwaite theorem, one participant can only buy and the other can only sell. Hence, the seller has an incentive to over-represent her value to earn more money and the buyer to under-represent so as to pay less. Hence, when their values are close to each other, no trade occurs which leads to inefficiency.

In this problem, each participant can buy and sell so they over/under represent their value in exactly the same way by symmetry: the agent with higher value will thus bid more, leading to efficient trade.

Endnote

Though we have shown that the proposed mechanism in (b) is ex ante individually rational, it is not ex post individually rational. Compute the ex post utility for agent $i$ in (1) by:

$$U_i(\theta_i, \theta_j) = 1(\theta_i > \theta_j) \left( 2\theta_i - \frac{2}{3} \theta_i \right) + 1(\theta_i < \theta_j) \frac{2}{3} \theta_j + 1(\theta_i = \theta_j) \theta_i$$

$$= \begin{cases} 
\frac{4}{3} \theta_i & \theta_i > \theta_j \\
\theta_i & \text{if } \theta_i = \theta_j \\
\frac{2}{3} \theta_j & \theta_i < \theta_j 
\end{cases}.$$

Note that when $\theta_i < \theta_j$, agent $i$ has to sell the good to agent $j$ only at price $\frac{2}{3} \theta_j$ which might be lower than his own valuation $\theta_i$. Therefore, it is not ex post individually rational. An explicit
assumption in (b) is that if auction starts, no one can reject to sell his good as long as he loses in bidding stage. To overcome this point, we reconsider the mechanism in which everyone still has the right to say no after the bidding stage. If, say, \( b_i > b_j \), agent \( j \) still can keep his good if he says no, i.e., when \( b_i < \theta_j \) happens.

Hence, we augment the initial game as follows. Each one proposes a bid price. The one with the higher bid wins and asks the other to sell his good. If the other one rejects, there is no trade. Otherwise, trade occurs at the price of the winner’s bid.

We claim that \( b_i \in [0, \theta_i] \). Suppose not, i.e., let’s assume that \( b_i > \theta_i \). If agent \( i \) wins, he thus ends up with payoff \( 2\theta_i - b_i \), which is less than \( \theta_i \), his outside option. Therefore, he will never bid like that and we must have \( b_i \leq \theta_i \).

Then (1) should be rewritten as:

\[
U_i (b_i, b_j) = \mathbf{1} (b_i > \theta_j) (2\theta_i - b_i) + \mathbf{1} (b_j > \theta_i) b_j + [1 - \mathbf{1} (b_i > \theta_j) - \mathbf{1} (b_j > \theta_i)] \theta_i. \tag{2}
\]

The first term captures the case when agent \( i \) wins two goods and the second term points to the case when agent \( j \) wins two goods. The last term corresponds to the scenario when both keep their own good.

We still consider the symmetric equilibrium in which \( b (\cdot) \) is continuous, increasing and differentiable. Given that agent \( j \) is on the equilibrium path, the expected payoff for agent \( i \) with \( \theta_i \) who bids at \( b (\theta'_i) \) by \( EU_i (\theta_i, \theta'_i) \):

\[
EU_i (\theta_i, \theta'_i) = \mathbf{E}_{\theta_j} \left[ \mathbf{1} (b (\theta'_i) > \theta_j) \right] (\theta_i - b (\theta'_i)) + \mathbf{E}_{\theta_j} \left\{ \mathbf{1} (b (\theta_j) > \theta_i) [b (\theta_j) - \theta_i] \right\} + \theta_i
\]

\[
= \mathbf{E}_{\theta_j} \left[ \mathbf{1} (\theta_j < b (\theta'_i)) \right] (\theta_i - b (\theta'_i)) + \mathbf{E}_{\theta_j} \left\{ \mathbf{1} (\theta_j > b^{-1} (\theta_i)) [b (\theta_j) - \theta_i] \right\} + \theta_i
\]

\[
= [\theta_i - b (\theta'_i)] \int_0^{b(\theta'_i)} d\theta_j + \int_{b^{-1}(\theta_i)}^{1} [b (\theta_j) - \theta_i] d\theta_j + \theta_i
\]

\[
= [\theta_i - b (\theta'_i)] b (\theta'_i) + \int_{b^{-1}(\theta_i)}^{1} [b (\theta_j) - \theta_i] d\theta_j + \theta_i
\]

Using the F.O.C., we have:

\[
\left. \frac{\partial EU_i (\theta_i, \theta'_i)}{\partial \theta'_i} \right|_{\theta'_i=\theta_i} = \left[ \theta_i - 2b (\theta_i) \right] b' (\theta_i) = 0.
\]
Then we have \( b_i = \theta_i/2 \), which is less aggressive than what we get in (b). However, we can only achieve ex post efficiency when their values are far away from each other, or more precisely, \( \theta_i > 2\theta_j \) or \( \theta_j > 2\theta_i \).

The probability that there is no ex post efficiency equals to

\[
1 - \Pr (b_i > \theta_j) - \Pr (b_j > \theta_i) = 1 - \Pr (2\theta_i > \theta_j) - \Pr (2\theta_j > \theta_i) = \frac{1}{2},
\]

which is pretty large. This is a bad mechanism in terms of efficiency though ex post individual rationality is allowed.

The main reason behind this is that we adopt a first-price auction like game in the bidding stage which prevents them from quoting price aggressively and thus leaves a large space for no trade to occur. An alternative way is to adopt a second-price auction in the bidding stage. Each one proposes a bid price. The one with the higher bid wins and asks the other to sell his good. If the other one rejects, there is no trade. Otherwise, trade occurs at the price of second highest price (here, the other one’s bid). Truthful bidding is one’s (weakly) dominant strategy by using the same argument as in proving the second-price auction. This mechanism is efficient for all types.

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23.E.7 in MGW. (a) Let agent 1 be the seller and agent 2 be the buyer. We want to explore the mechanism which is:

1. Bayesian Incentive Compatible;
2. Interim individually rational and
3. Ex-post efficient.

For the seller, ex post efficiency says that trade occurs iff \( \theta_1 \leq \theta_2 \). Given the announcement profile \( (\hat{\theta}_1, \hat{\theta}_2) \), denote the expected transfer from the buyer to the seller by \( t_1 (\hat{\theta}_1, \hat{\theta}_2) \). Given that the buyer is telling truth, the expected payoff for seller if his type is \( \theta_1 \) but he reports \( \hat{\theta}_1 \) is:

\[
EU_1 (\theta_1, \hat{\theta}_1) = E_{\theta_2} [\theta_1 \mathbb{1} (\hat{\theta}_1 > \theta_2) + t_1 (\hat{\theta}_1, \theta_2)]
= \theta_1 \hat{\theta}_1 + E_{\theta_2} [t_1 (\hat{\theta}_1, \theta_2)].
\]

Differentiating w.r.t. \( \hat{\theta}_1 \) yields
\[
\theta_1 + \frac{dE_{\theta_2} \left[ t_1 \left( \hat{\theta}_1, \theta_2 \right) \right]}{d\theta_1} = 0.
\]

Bayesian incentive compatibility implies that it is optimal to set \( \hat{\theta}_1 = \theta_1 \):

\[
\theta_1 + \frac{dE_{\theta_2} \left[ t_1 \left( \theta_1, \theta_2 \right) \right]}{d\theta_1} = 0.
\]

Integrating w.r.t. \( \theta_1 \) from \( \hat{\theta}_1 \) to 1:

\[
\int_{\hat{\theta}_1}^{1} \theta_1 d\theta_1 + \int_{\hat{\theta}_1}^{1} \frac{dE_{\theta_2} \left[ t_1 \left( \theta_1, \theta_2 \right) \right]}{d\theta_1} d\theta_1 = 0
\]

\[
\Rightarrow \frac{1 - \theta_1^2}{2} + E_{\theta_2} \left[ t_1 \left( 1, \theta_2 \right) \right] - E_{\theta_2} \left[ t_1 \left( \hat{\theta}_1, \theta_2 \right) \right] = 0
\]

\[
\Rightarrow E_{\theta_2} \left[ t_1 \left( 1, \theta_2 \right) \right] = E_{\theta_2} \left[ t_1 \left( \theta_1, \theta_2 \right) \right] - \frac{1 - \theta_1^2}{2}, \text{ where we rewrote } \hat{\theta}_1 \text{ as } \theta_1.
\]

Interim individually rationality implies \( E_{\theta_2} \left[ t_1 \left( 1, \theta_2 \right) \right] \geq 0 \) or otherwise type \( \theta_1 = 1 \) will not participate, thus leading to:

\[
E_{\theta_2} \left[ t_1 \left( \theta_1, \theta_2 \right) \right] - \frac{1 - \theta_1^2}{2} \geq 0 \Rightarrow E_{\theta_2} \left[ t_1 \left( \theta_1, \theta_2 \right) \right] \geq \frac{1 - \theta_1^2}{2}
\]

Hence,

\[
EU_1 \left( \theta_1, \hat{\theta}_1 \right) \bigg|_{\hat{\theta}_1=\theta_1} = \theta_1^2 + E_{\theta_2} \left[ t_1 \left( \theta_1, \theta_2 \right) \right] \geq \theta_1^2 + \frac{1 - \theta_1^2}{2} = \frac{1 + \theta_1^2}{2}.
\]

On the other hand, for the buyer:

\[
EU_2 \left( \hat{\theta}_2, \theta_2 \right) = E_{\theta_1} \left[ \theta_2 \mathbb{1} \left( \theta_1 \leq \hat{\theta}_2 \right) + t_2 \left( \theta_1, \hat{\theta}_2 \right) \right]
\]

\[
= \theta_2 \hat{\theta}_2 + E_{\theta_1} \left[ t_2 \left( \theta_1, \hat{\theta}_2 \right) \right].
\]

Differentiating w.r.t. \( \hat{\theta}_2 \) yields

\[
\theta_2 + \frac{dE_{\theta_1} \left[ t_2 \left( \theta_1, \hat{\theta}_2 \right) \right]}{d\hat{\theta}_2} = 0.
\]

Bayesian incentive compatibility implies that it is optimal to set \( \hat{\theta}_2 = \theta_2 \):
\[ \theta_2 + \frac{dE_{\theta_1}}{d\theta_2} [t_2 (\theta_1, \theta_2)] = 0. \]

Integrating w.r.t. \( \theta_2 \) from 0 to \( \tilde{\theta}_2 \):

\[
\int_0^{\tilde{\theta}_2} \theta_2 d\theta_2 + \int_0^{\tilde{\theta}_2} \frac{dE_{\theta_1}}{d\theta_2} [t_2 (\theta_1, \theta_2)] d\theta_2 = 0.
\]

\[
\Rightarrow \frac{\tilde{\theta}_2^2}{2} + E_{\theta_1} \left[ t_2 \left( \theta_1, \tilde{\theta}_2 \right) \right] - E_{\theta_1} \left[ t_2 \left( \theta_1, 0 \right) \right] = 0
\]

\[
\Rightarrow E_{\theta_1} \left[ t_2 \left( \theta_1, 0 \right) \right] = E_{\theta_1} \left[ t_2 \left( \theta_1, \theta_2 \right) \right] + \frac{\theta_2^2}{2}, \text{ where we rewrote } \tilde{\theta}_2 \text{ as } \theta_2.
\]

Interim individually rationality implies \( E_{\theta_1} \left[ t_2 \left( \theta_1, 0 \right) \right] \geq 0 \) or otherwise type \( \theta_2 = 0 \) will not participate. This leads to:

\[
E_{\theta_1} \left[ t_2 \left( \theta_1, \theta_2 \right) \right] + \frac{\theta_2^2}{2} \geq 0 \Rightarrow E_{\theta_1} \left[ t_2 \left( \theta_1, \theta_2 \right) \right] \geq -\frac{\theta_2^2}{2}
\]

Hence,

\[
EU_2 \left( \hat{\theta}_2 = \theta_2, \theta_2 \right) = \theta_2^2 + E_{\theta_1} \left[ t_2 \left( \theta_1, \theta_2 \right) \right] \geq \theta_2^2 - \frac{\theta_2^2}{2} = \frac{\theta_2^2}{2}. \quad (4)
\]

(3) and (4) lead to

\[
\text{Expected social surplus} \geq \int_0^1 \int_0^1 \left( \frac{1 + \theta_1^2}{2} + \frac{\theta_2^2}{2} \right) d\theta_1 d\theta_2 = \frac{5}{6}.
\]

(b) Suppose the social planner can observe both \( \theta_1 \) and \( \theta_2 \) and allocates the good according to the ex post efficient outcome. This then leads to the maximum expected total utility:

\[
E_{\theta_1, \theta_2} \max \left( \theta_1, \theta_2 \right) = E_{\theta_1, \theta_2} \left[ \theta_1 \mid \theta_1 > \theta_2 \right] + E_{\theta_1, \theta_2} \left[ \theta_2 \mid \theta_1 \leq \theta_2 \right]
\]

\[
= \int_0^1 \int_0^1 \theta_1 d\theta_1 d\theta_2 + \int_0^1 \int_0^1 \theta_2 d\theta_2 d\theta_1 = \int_0^1 \int_0^1 \frac{1 - \theta_2^2}{2} d\theta_2 + \int_0^1 \frac{1 - \theta_1^2}{2} d\theta_1
\]

\[
= 1 - \int_0^1 \theta_1^2 d\theta_1 = 1 - \frac{1}{3} = \frac{2}{3}.
\]

So no social choice function can yield total expected utility greater than \( \frac{2}{3} \).
23.F.6 in MGW. The monopolist has demand function \( x(p) \) with \( x'(p) < 0 \) and private information \( \theta \) about the marginal cost of production which is drawn from \([\underline{\theta}, \bar{\theta}]\) according to cdf \( \Phi(\cdot) \) and strictly positive density \( \phi(\cdot) > 0 \). Assume \( \frac{\Phi(\theta)}{\phi(\theta)} \) is nondecreasing in \( \theta \).

The regulator proposes the set of outcome: \( X = \{ (p, t) : p > 0 \text{ and } t \in \mathbb{R} \} \).

The monopolist’s profit is \( \pi(p, t, \theta) = (p - \theta) x(p) + t \) which should be non-negative: \( \pi(p, t, \theta) \geq 0 \).

(a) \[
\pi(p, t, \theta) = (p - \theta) x(p) + t = \theta [-x(p)] + [px(p) + t].
\]

Denote \( T(\theta) = p(\theta) x(p(\theta)) + t(\theta), v(\theta) = -x(p(\theta)). \)

Proposition 23.D.2 says that the social choice functions \( p(\theta) \) and \( T(\theta) \) are implementable (correspondingly, \( p(\theta) \) and \( t(\theta) \) are implementable) iff:

1. \( v(\theta) = -x(p(\theta)) \) is nondecreasing. Since \( x'(p) < 0 \Rightarrow \bar{v}(\theta) \) is nondecreasing iff \( p(\theta) \) is nondecreasing in \( \theta \).
2. \( \pi(\theta) = \pi(\theta) + \int_{\underline{\theta}}^{\theta} \bar{v}(s) ds \) for all \( \theta \)

i.e.

\[
\Rightarrow \pi(\theta) = -\theta x(p(\theta)) + T(\theta) = \pi(\theta) - \int_{\underline{\theta}}^{\theta} x(p(s)) ds
\]  \( \quad \) (5)

(b) The regulator chooses \( \{ p(\theta), T(\theta) \} \) to maximize:

\[
\max_{p(\theta), T(\theta)} \int_{\underline{\theta}}^{\theta} \left[ \int_{p(\theta)}^{\infty} x(s) ds - t(\theta) + \alpha \pi(\theta) \right] d\Phi(\theta).
\]

Substituting out \( T(\theta) = \pi(\theta) + \theta x(p(\theta)) = p(\theta) x(p(\theta)) + t(\theta), \) we want to maximize:

\[
\max_{\pi(\theta), p(\theta)} \int_{\underline{\theta}}^{\theta} \left[ \int_{p(\theta)}^{\infty} x(s) ds - \pi(\theta) - \theta x(p(\theta)) + p(\theta) x(p(\theta)) + \alpha \pi(\theta) \right] d\Phi(\theta)
\]

\[
= \max_{\pi(\theta), p(\theta)} \left\{ \int_{\underline{\theta}}^{\theta} \left[ \int_{p(\theta)}^{\infty} x(s) ds - \theta x(p(\theta)) + p(\theta) x(p(\theta)) \right] d\Phi(\theta) - (1 - \alpha) \int_{\underline{\theta}}^{\theta} \pi(\theta) d\Phi(\theta) \right\}.
\]
Note that:

\[
\int_\theta^\tilde{\theta} \pi (\theta) d\Phi(\theta) = \pi (\theta) \Phi (\theta) \Big|_\theta^{\tilde{\theta}} - \int_\theta^\tilde{\theta} \pi' (\theta) \Phi (\theta) d\theta \\
= \pi (\tilde{\theta}) \Phi (\tilde{\theta}) - \pi (\theta) \Phi (\theta) - \int_\theta^\tilde{\theta} \pi' (\theta) \Phi (\theta) d\theta \\
= \pi (\tilde{\theta}) - \int_\theta^\tilde{\theta} \pi' (\theta) \Phi (\theta) d\theta.
\]

Since \( \pi' (\theta) = -x (p (\theta)) \) from (5), we have

\[
\int_\theta^\tilde{\theta} \pi (\theta) \phi (\theta) d\theta = \pi (\tilde{\theta}) + \int_\theta^\tilde{\theta} x (p (\theta)) \frac{\Phi (\theta)}{\phi (\theta)} d\Phi (\theta).
\]

Hence, the objective function becomes:

\[
\max_{\pi(\theta), p(\theta)} \left\{ \int_0^\tilde{\theta} \left[ \int_0^\infty x (s) ds - \theta x (p (\theta)) + p (\theta) x (p (\theta)) - (1 - \alpha) x (p (\theta)) \frac{\Phi (\theta)}{\phi (\theta)} \right] \phi (\theta) d\theta - \alpha \pi (\tilde{\theta}) \right\}
\]

subject to:

1. \( \pi (\tilde{\theta}) \geq 0 \) and since \( \pi (\tilde{\theta}) < \pi (\theta) \) from (5), this implies \( \pi (\theta) \geq 0 \).
2. \( p (\theta) \) is non-decreasing.

Pointwise differentiation w.r.t. \( p (\theta) \) yields:

\[
-x (p (\theta)) - \theta x' (p (x)) + x (p (\theta)) + p (\theta) x' (p (\theta)) - (1 - \alpha) x' (p (\theta)) \frac{\Phi (\theta)}{\phi (\theta)} = 0
\]

\[
\Rightarrow -\theta + p (\theta) - (1 - \alpha) \frac{\Phi (\theta)}{\phi (\theta)} = 0, \text{ since } x' (.) < 0
\]

\[
\Rightarrow p (\theta) = \theta + (1 - \alpha) \frac{\Phi (\theta)}{\phi (\theta)}.
\]

Since \( \frac{\Phi (\theta)}{\phi (\theta)} \) is non-decreasing in \( \theta \), \( p (\theta) \) is non-decreasing in \( \theta \).

If \( \alpha < 1 \), (6) is maximized at \( \pi (\tilde{\theta}) = 0 \).

If \( \alpha = 1 \), (6) is maximized at any value of \( \pi (\tilde{\theta}) \geq 0 \).

If \( \alpha > 1 \), we can always increase (6) by making \( \pi (\tilde{\theta}) \) slightly larger, corresponding to a lack of optimality.