Problem Set 4. Implementation

Suggested Answer

1 (a) Suppose \( a \in F^{PO} (R) \): \( \forall b \in A, \exists \text{ an individual, say } j \), such that \( aR_j b \). If we replace \( R \) with \( R' \) such that for \( \forall i, aR_i b \Rightarrow aR'_i b \), then it must be \( aR'_j b \). Hence, according to definition, \( a \in F^{PO} (R') \). This establishes the monotonicity of \( F^{PO} \).

If \( \left[ \forall \, j \neq i, \forall b \in A, aR_j b \right] \), then \( \left[ \exists \, j \neq i, \forall b \in A, aR_j b \right] \). By definition, \( a \in F^{PO} (R) \). This shows that \( F^{PO} \) satisfies NPV.

(b) Suppose \( a \in F^{CON} (R) \): \( \forall b \in A, \# \{ i|aR_i b \} \geq \# \{ i|bR_i a \} \). Now we replace \( R \) with \( R' \) such that for \( \forall i, aR_i b \Rightarrow aR'_i b \), therefore we have \( \# \{ i|aR'_i b \} \geq \# \{ i|aR_i b \} \). If \( \# \{ i|bR'_i a \} > \# \{ i|bR_i a \} \), then \( \exists \) at least one agent, say \( j \), \( bR'_j a \) and \( aR_j b \), i.e., he switches his preference. But this contradicts the assumption that \( \forall i, aR_i b \Rightarrow aR'_i b \). Therefore, this can never happen. So, it must be \( \# \{ i|bR'_i a \} \leq \# \{ i|bR_i a \} \), then we immediately have \( \# \{ i|aR'_i b \} \geq \# \{ i|bR'_i a \} \). Thus, \( a \in F^{CON} (R') \). This establishes the monotonicity of \( F^{CON} \).

If \( \left[ \forall \, j \neq i, \forall b \in A, aR_j b \right] \), then \( \# \{ k|aR_k b \} \geq n - 1 \). Since \( R \) is strict, \( \# \{ k|bR_k a \} = n - \# \{ k|aR_k b \} \leq 1 \). So long as \( n \geq 2 \), we have \( \# \{ k|aR_k b \} \geq \# \{ k|bR_k a \} \) and thus \( a \in F^{CON} (R) \). This shows that \( F^{CON} \) satisfies NPV.

(c) Suppose \( A = \{ a, b, c, d \} \) and \( n = 2 \). Consider the following profile \( R = (R_1, R_2) \):

\[
\begin{array}{cccc}
R_1 & R_2 \\
a & c \\
d & b \\
b & a \\
c & d
\end{array}
\]

Let \( B (\omega) \) be the Borda count of alternative \( \omega \in A \), then \( F^{BORDA} (R) = \arg \max_{\omega \in A} B (\omega) \).

Now, \( B (a) = 4 + 2 = 6, B (b) = 2 + 3 = 5, B (c) = 1 + 4 = 5 \) and \( B (d) = 3 + 1 = 4 \). Therefore, \( a \in F^{BORDA} (R) \).
Next consider the following profile $R' = (R'_1, R'_2)$:

$$
\begin{array}{cc}
R'_1 & R'_2 \\
\hline
a & b \\
b & c \\
d & a \\
c & d \\
\end{array}
$$

Here, $aR_1\omega \Rightarrow aR'_1\omega$, for $\omega \in \{b, c, d\}$ and $aR_2d \Rightarrow aR'_2d$.

Now, $B(a) = 6, B(b) = 3 + 4 = 7, B(c) = 1 + 3 = 4$ and $B(d) = 2 + 1 = 3$. Therefore, $b \in F^{\text{BORDA}}(R)$. Hence, monotonicity is violated.

To check NPV, still use $R = (R_1, R_2)$ and $a \in F^{\text{BORDA}}(R)$. For individual 1, $aR_1\omega$ for $\omega \in \{b, c, d\}$. However, for individual 2, $cR_2a$, but $c \notin F^{\text{BORDA}}(R)$. Therefore, NPV is violated.

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2. The contrapositive form of monotonicity condition is:

A social choice correspondence $F$ is monotonic if for $\forall R, \overline{R} \in \mathcal{R}$, and $a \in F(R)$ but $a \notin F(\overline{R})$, then there exists an agent $i$ and $b \in A$ such that $aR_i b$ and $bP_i a$.

To show that Nash implementation leads to Maskin monotonicity, fix a preference profile $R$ such that an alternative $a \in F(R)$. Suppose that $g$ Nash implements $F$, so there exists a $m \in M$ such that $m$ is a Nash Equilibrium at $R$ and $g(m) = a$. Now consider another preference profile $\overline{R} \in \mathcal{R}$ such that $a \notin F(\overline{R})$. $m$ cannot be a Nash equilibrium at $\overline{R}$, so someone prefers to deviate, i.e., there must exist agent $i$ and a deviation $\overline{m}_i$ that $g(\overline{m}_i, m_{-i}) \overline{P}_i g(m)$. Recall that $m$ is a Nash equilibrium at $R$, so we have $g(m) R_i g(\overline{m}_i, m_{-i})$. Reindex $g(\overline{m}_i, m_{-i}) = b$ and we have established Maskin monotonicity.

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