Problem Set 2

1. Consider a version of the one-armed bandit model in continuous time with discount rate $r$. The problem is to choose how long to keep an trying with an uncertain arm before giving up and changing to an arm with a known deterministic flow payoff of $c$ (the opportunity cost using the uncertain arm). The uncertain arm is of one of two possible types. If the arm is good, then it delivers in the periods where the arm is chosen a positive payoffs of (stock) size $v$ at a Poisson rate $\lambda$ and a payoff of zero in all other periods. In other words, the probability of receiving a positive payoff in time interval $(t, t + dt]$ is $\lambda dt$. If the arm is bad, then it never delivers any payoffs. Let $\pi(0)$ be the prior probability that the uncertain arm is good.

To make the problem interesting, we assume that $\lambda v > c$. This simply means that a good arm should be chosen over the safe arm. Notice also that when the first positive payoff arrives, then all uncertainty is resolved, and the decision problem is effectively over. Hence the only history after which decisions matter is the history where the uncertain arm has been chosen for a time interval of length $t$ and no payoff has been received. We let $\pi(t)$ denote the posterior probability that the uncertain arm is good along such histories. Notice that as a function of $t$, the decision maker becomes more pessimistic along these histories.

In this problem, I ask you to calculate the optimal stopping time for the decision maker, or alternatively the lowest posterior belief on the arm being good where it is optimal to continue using the uncertain arm.
(a) Compute using Bayes’ rule \( \pi(t) \). Differentiate with respect to \( t \) to get the time derivative \( \dot{\pi}(t) \) along these histories. Alternatively use the fact that \( \pi(t) \) is a martingale and has the posterior jumps to 1 with probability \( \lambda dt \) and moves \( \pi(t + dt) \) with probability \( 1 - \lambda dt \) to compute \( \pi(t + dt) \), take difference divide by \( dt \) and let \( dt \to 0 \).

(b) Write the Bellman’s equation for the problem as follows:

\[
V(\pi(t)) = \max \left\{ \frac{c}{r}, \pi(t)\lambda v + e^{-r dt} \left( \frac{\lambda dt \pi(t)v}{r} + (1 - \pi \lambda dt)V(\pi(t + dt)) \right) \right\}.
\]

Let \( dt \to 0 \), and use Taylor’s first order approximation to write this in form:

\[
V(\pi) = \max \left\{ \frac{c}{r}, \frac{\lambda dt \pi v}{r} + (1 - r - \pi \lambda)\frac{c}{r} + V'(\pi)\dot{\pi}dt \right\}.
\]

At the point \( \pi^* \), where the decision maker is indifferent between abandoning the uncertain arm and continuing,

\[
\frac{c}{r} = \frac{\lambda dt \pi v}{r} + (1 - r - \pi \lambda)\frac{c}{r} + V'(\pi^*)\dot{\pi}(t)dt,
\]

Use the monotonicity of the value function in \( \pi \) that \( V(\pi') = V(\pi^*) \) for all \( \pi' < \pi^* \), and therefore \( V'(\pi^*) = 0 \). This is called the smooth pasting condition. Solve for \( \pi^* \).

(c) How does \( \pi^* \) depend on \( \lambda, r, v, c \)?
2. Consider the following consumer choice problem over two periods, \( t \in \{0, 1\} \) (Assume no discounting for simplicity). There are two goods \( i \in \{1, 2\} \) and product 1 yields a certain utility of 1 per period. Good 2 is of an uncertain nature and yields a utility of either zero or \( v > 1 \) in each period. Let \( \pi \) be the prior probability of the good yielding \( u \) per period.

(a) Characterize the optimal choice behavior in terms of \( \pi \) by the consumer in this setting. I.e. keep other parameters fixed and let \( \pi \) vary.

(b) Assume next that the two goods are offered by strategic sellers setting their prices in a subgame perfect Nash equilibrium. Assume that the result of the buyer’s trial with the uncertain product is observable to both sellers as well. Let \( p^i_1(j, k) \) denote the price of seller \( i \) in period 1 given that the buyer chose seller \( j \) in period 0 and utility \( k \in \{0, 1, v\} \) was realized. Start by backwards induction and characterize the Nash equilibrium price offers of the two sellers for the three possible continuation games. Assume here that firms never make price offers that would leave the firm worse off if accepted. (Notice that this restriction really only matters to the seller that does not sell in equilibrium). Compute also the induced profits and consumer surpluses for these subgames.

(c) Solve for the equilibrium in first period price competition given that the second period profits and consumer surpluses are as in the previous item. Compare to the solution in case a.

(d) Assume next that there is a unit mass of buyers that purchase the product simultaneously. If mass \( x \) of consumers buy the uncertain product, then its quality is revealed for the second period with probability \( x \) and the prior is maintained with probability \( 1 - x \).
In this case, we assume that the buyers just collect their prior value of \( \pi v \) in the first period. If the prices of the two goods are fixed at zero, show that the equilibrium induces the buyers to buy if and only if \( pv > 1 \). Compare to the previous items.

(e) Suppose finally that the goods are offered by strategic sellers as in part b. and c. but that there is a unit mass of consumers as in part d. Show now that the lowest \( \pi \) at which the buyers purchase the uncertain good is lower than the value calculated in parts a. and c. in ll equilibria where the sellers make price offers such that their payoff is not reduced if their offer is accepted. Comment on the reason for this discrepancy.