1 Introduction

The proposition proved here helps to understand the relationship between dynamic games of incomplete information with only private signals and dynamic games of incomplete information with both public and private signals. The proposition states that in games of incomplete information with public and private signals, the equilibrium strategy played by any player upon realization of specific public and private signals must also be an equilibrium strategy played upon the realization of the same private signal in a game with only private signals where the common prior over private signals and the state of the world is equal to the common prior in the game of public and private signals conditioned on the private signal.

This can be viewed as something similar to a sub-game perfection type argument. The theorem can also be useful computationally as it can simplify the procedure of finding equilibria in game of incomplete information with public and private signals by allowing us to focus on games of incomplete information with only private signals.

2 Public Information and Dynamic Games of Incomplete Information

2.1 Perfect Bayesian Equilibrium with Private Information

Consider a game of incomplete information with private signals. There are $n$ players. The underlying state of the world is $\theta \in \Theta$. Individual $i$ takes a single action, $a_i \in A_i$. Denote $a = (a_1, a_2, \ldots a_n)$. Player $i$'s utility function is $U_i(a_i, a_{-i}, \theta)$. We will allow for sequential moves in the game over $j$ periods. Without loss of generality $j \leq n$ (i.e. we disregard periods where no one moves).

There is an exogenously given order of moves:

- In period 1 players $1 \ldots k_1$ move.
- In period 2 players $k_1 + 1 \ldots k_2$ move.
• In period $i$ players $k_{i-1} + 1 \ldots k_i$ move.

• In period $j$ players $k_{j-1} + 1 \ldots n$ move.

For notational consistency define $k_0 = 0$. For example in a Stackelberg duopoly, there are two periods, the leader moves in period 1 and the follower in period 2. Note if $k_1 = n$ we are in a static environment.

We assume that players are able to observe the moves of other players who have moved in prior periods. Denote $\mathbb{I}_i := A_1 \times \ldots \times A_{k_i-1}$. Thus $(a_1, \ldots, a_{k_i-1}) = i \in \mathbb{I}_i$ represent the actions take by previous players who move before period $i$ which are observable in period $i$.

Players receive a private signal over the true state $\theta$, $x_i$ where $x_i \in X_i \ \forall i$. There is a common prior over the joint distribution of $\theta$ and the realization of private signals: $f(\theta, x_1 \ldots x_n)$. Thus player $i$ may form expectations of $\theta$ or $x_j$ conditional on observing $x_i$:

$$E[\theta|x_i] = \int_{t \in \Theta} \int_{x_{-i} \in X_{-i}} t \ f(t, x_i, x_{-i}) \ dx_{-i} \ dt$$

$$E[x_j|x_i] = \int_{x \in X_j} \int_{(\theta, x_{-\{i,j\}}) \in \Theta \times X_{-\{i,j\}}} x \ f(\theta, x, x_{-\{i,j\}}) \ d(\theta, x_{-\{i,j\}}) \ dx$$

Define a mixed strategy of a player, $i$:

$$\sigma_i : A_i \times X_i \times \mathbb{I}_i \rightarrow \mathbb{R}_+$$

Where we have:

$$\int_{a \in A_i} \sigma_i(a, x_i, i_i) \ da = 1, \ \forall x_i \in X_i, \ i_i \in \mathbb{I}_i$$

Denote $\sigma = <\sigma_1 \ldots \sigma_n>$. If we have $i$ such that $i \leq k_1$ we know the players strategy as $\sigma_i(a, x_i, i_i)$ can be written equivalently as $\sigma_i(a, x_i)$ since the player has no additional information, $i_i$.

We define $\mu_i$ to be the beliefs of player $i$ over the $\theta$ and the actions of players who have yet to move. Obviously these beliefs are conditional upon
the available information. Formally, the beliefs of a player, \( i \) who moves in period \( h \geq 1 \) are:

\[
\mu_i : A_i \times X_i \times I_i \times \Theta \times A_{k_h-1+1} \times A_{i-1} \times A_{i+1} \times A_n \rightarrow \mathbb{R}_+
\]

We may write this belief function as \( \mu_i(\theta, a_{k_h-1+1}, \ldots a_{i-1}, a_{i+1}, \ldots a_n|a_i, i_i, x_i) \).

Since this is also a proper conditional probability density function, we have:

\[
\int_{s \in \Theta \times A_{k_h-1+1}, \ldots A_{i-1}, A_{i+1}, \ldots A_n} \mu_i(s|a_i, i_i, x_i) \, ds = 1, \quad \forall x_i \in X_i, \ i_i \in I_i
\]

Denote \( \mu = <\mu_1 \ldots \mu_n> \).

We may now define the notion of a perfect bayesian equilibrium for this game:

**Definition 1.** The strategy profile \( \sigma \) and belief structure \( \mu \) constitute a **perfect bayesian equilibrium with private signals** of this game of incomplete information with common prior \( F(\theta, x_1 \ldots x_n) \) if and only if:

- **Players best respond given their beliefs:**

  If player \( i \) moves in period \( h \geq 1 \) (i.e. \( k_{h-1} + 1 \leq i \leq k_h \)):

  \[
  \forall x_i \in X_i, \forall i_i \in I_i, \ a_i \in \arg\max_{a \in A_i} \int_{s \in \Theta \times A_{k_h-1+1}, \ldots A_{i-1}, A_{i+1}, \ldots A_n} U_i(a, i_i, s) \mu_i(s|a, i_i, x_i) \, ds
  \]

  \[
  \forall a_i \ s.t. \ \sigma_i(a_i, i_i, x_i) > 0
  \]

- **Beliefs are consistent with bayes rule:**

  If player \( i \) moves in period \( h \geq 1 \) (i.e. \( k_{h-1} + 1 \leq i \leq k_h \)):

  \[
  \mu_i(\theta, a_{k_h-1+1}, \ldots a_{i-1}, a_{i+1}, \ldots a_n|a_i, i_i, x_i) =
  \]

  \[
  \int_{x_{-i} \in X_{-i}} \Pi_{c=k_h-1+1 \ldots i-1, i+1 \ldots n} \sigma_c(a_c, i_c, x_c) f(\theta, x_i, x_{-i}) dx_{-i}
  \]
2.2 Perfect Bayesian Equilibrium with Private and Public Information

Alternatively, consider the same game of incomplete information yet with an additional public signal, \( y \), of the true state \( \theta \) where \( y \in Y \). Once again, there is a common prior over the joint distribution of \( \theta \) and the realization of signals, \( G(\theta, x_1 \ldots x_n, y) \). We assume:

\[
f(\theta, x_1 \ldots x_n) = \int_{y \in Y} g(\theta, x_1 \ldots x_n, y) dy, \quad \forall (\theta, x_1 \ldots x_n) \in (\Theta \times X_1 \ldots X_n)
\]

That is the existence of the public signal does not affect the original common prior of \( \theta \) and \( x \).

**Definition 2.** The strategy profile \( \sigma \) and belief structure \( \mu \) constitute a perfect bayesian equilibrium with public and private signals of this game of incomplete information with common prior \( g(\theta, x_1 \ldots x_n, y) \) if and only if:

- **Players best respond given their beliefs:**
  
  If player \( i \) moves in period \( h \geq 1 \) (i.e. \( k_{h-1} + 1 \leq i \leq k_h \)):
  
  \[
  \forall (x_i, y) \in X_i \times Y, \forall i \in \mathbb{I}_i, a_i \in \arg \max_{a \in A_i} \int_{s \in \Theta \times A_{k_{h-1} + 1 \ldots A_i - 1, A_{i+1} \ldots A_n}} U_i(a, i, s) \mu_i(s | a, i, x_i, y) \ ds \\
  \forall a_i \ s.t. \ \sigma_i(a_i, i, x_i, y) > 0
  \]

- **Beliefs are consistent with bayes rule:** If player \( i \) moves in period \( h \geq 1 \) (i.e. \( k_{h-1} + 1 \leq i \leq k_h \)):

  \[
  \mu_i(\theta, a_{k_{h-1} + 1 \ldots a_i - 1, a_{i+1} \ldots a_n | a, i, x_i, y}) = \int_{x_i \in X_i} \Pi_{c = k_{h-1} + 1 \ldots i - 1, i+1 \ldots n} \sigma_c(a_c, i_c, x_c, y) g(\theta, x_i, x_{-i}, y) dx_{-i}
  \]
2.3 The connection of the two

Define the notation
\[ \forall y \in Y, \quad f^y(\theta, x) := g(\theta, x, y) \]
\[ \sigma^y(a_c, i_c, x_c) := \sigma(a_c, i_c, x_c, y) \]
and
\[ \mu^y_i(\theta, a_{-i}|a, x_i) := \mu_i(\theta, a_{-i}|a, x_i, y) \]

**Proposition 1.** The action profile \( \sigma \) and belief profile \( \mu \) are a PBE with public and private information with common prior \( g \) if and only if the action profile \( \sigma^y \) and belief profile \( \mu^y \) are a PBE of the game with private information and common prior \( f^y, \forall y \in Y \).

**Proof.** First we show that if the action profile \( \sigma^y \) and belief profile \( \mu^y \) are a PBE of the game with private information and common prior \( f^y, \forall y \in Y \), then the action profile \( \sigma \) and belief profile \( \mu \) are a PBE with public and private information with common prior \( g \): 

If player \( i \) moves in period \( h \geq 1 \) (i.e. \( k_{h-1} + 1 \leq i \leq k_h \)) we know \( \forall y \in Y \):

\[ \forall x_i \in X_i, \forall i_1 \in I_1, \quad a_i \in \arg \max_{a_i \in A_i} \int_{s \in \Theta \times A_{kh-1} \cdots A_{i-1}, A_{i+1}, \ldots, A_n} U_i(a_i, i_i, s) \mu_i^y(s|a_i, i_i, x_i) \, ds \]

\[ \forall a_i \text{ s.t. } \sigma^y_i(a_i, i_i, x_i) > 0 \]

and

\[ \mu_i^y(\theta, a_{k_{h-1}+1}, \ldots a_{i-1}, a_{i+1}, \ldots a_n|a, i_i, x_i) = \int_{x_{-i} \in X_{-i}} \prod_{c=k_{h-1}+1 \ldots i-1, i+1, \ldots n} \sigma^y_c(a_c, i_c, x_c) f^y(\theta, x_i, x_{-i}) \, dx_{-i} \]

Switching notation gives:

\[ \forall (x_i, y) \in X_i \times Y, \forall i_1 \in I_1, \quad a_i \in \arg \max_{a_i \in A_i} \int_{s \in \Theta \times A_{kh-1} \cdots A_{i-1}, A_{i+1}, \ldots, A_n} U_i(a_i, i_i, s) \mu_i(s|a_i, i_i, x_i, y) \, ds \]

\[ \forall a_i \text{ s.t. } \sigma_i(a_i, i_i, x_i, y) > 0 \]

\[ \mu_i(\theta, a_{k_{h-1}+1}, \ldots a_{i-1}, a_{i+1}, \ldots a_n|a, i_i, x_i, y) = \]
\[ \int_{x_{-i} \in X_{-i}} \prod_{c=k_{h-1}+1 \leq i \leq k_{h}} \sigma_c(a_c, i_c, x_c, y) g(\theta, x_i, x_{-i}, y) dx_{-i} \]

Which is by definition a PBE with Public and Private signals. Now we may show that the action profile if \( \sigma \) and belief profile \( \mu \) are a PBE with public and private information with common prior \( g \) then action profile \( \sigma^y \) and belief profile \( \mu^y \) are a PBE of the game with private information and common prior \( f^y \), \( \forall y \in Y \):

Fix a given \( y \in Y \). We know by definition of a PBE with public and private information:

If player \( i \) moves in period \( h \geq 1 \) (i.e. \( k_{h-1} + 1 \leq i \leq k_{h} \)):

\[ \forall x_i \in X_i, \forall i \in I_i, \ a_i \in \arg \max_{a \in A_i} \int_{s \in \Theta \times A_{k_{h-1}+1 \leq \cdots \leq A_{i-1} \leq \cdots \leq A_n}} U_i(a, i_i, s) \mu_i(s | a, i_i, x_i, y) \]

\[ \forall a_i \ s.t. \ \sigma_i(a_i, i_i, x_i, y) > 0 \]

and

\[ \mu_i(\theta, a_{k_{h-1}+1 \leq \cdots \leq a_{i-1} \leq \cdots \leq a_n} | a, i_i, x_i, y) = \]

\[ \int_{x_{-i} \in X_{-i}} \prod_{c=k_{h-1}+1 \leq i \leq k_{h}} \sigma_c(a_c, i_c, x_c, y) g(\theta, x_i, x_{-i}, y) dx_{-i} \]

Switching notation gives:

\[ \forall x_i \in X_i, \forall i \in I_i, \ a_i \in \arg \max_{a \in A_i} \int_{s \in \Theta \times A_{k_{h-1}+1 \leq \cdots \leq A_{i-1} \leq \cdots \leq A_n}} U_i(a, i_i, s) \mu_i^y(s | a, i_i, x_i) \]

\[ \forall a_i \ s.t. \ \sigma_i^y(a_i, i_i, x_i) > 0 \]

and

\[ \mu_i^y(\theta, a_{k_{h-1}+1 \leq \cdots \leq a_{i-1} \leq \cdots \leq a_n} | a, i_i, x_i) = \]

\[ \int_{x_{-i} \in X_{-i}} \prod_{c=k_{h-1}+1 \leq i \leq k_{h}} \sigma_c^y(a_c, i_c, x_c) f^y(\theta, x_i, x_{-i}) dx_{-i} \]

Thus we have a PBE with private signals with common prior \( f^y \). This holds for all \( y \in Y \). \qed