Information Aggregation and Large Auctions

Motivation

• Where does the price for a Walrasian market come from?

• Maybe the Walrasian outcome can be seen as the equilibrium of a large auction.

• Can we formalize this in a private values setting or is Myerson-Satterhwaite theorem working against us?

• What about the foundations of a revealing rational expectations equilibrium?

• Recall that there price summarizes all information available to the individual players

• The proper setting for studying this latter question is within common values games.

Private values auctions

We start with the simplest possible model. There are \( k_n \) objects for sale and \( n \) potential buyers. The valuations \( v_i \) are drawn independently from a common distribution \( F(v_i) \) for all potential bidders. We assume that \( F \) has a continuous density function \( f \).

Suppose now that the objects are allocated according to the following rule: \( k_n \) buyers with the highest bids receive the object and pay the price of the \((k_n + 1)^{st}\) highest bid. In case of a tie between the \( k_n^{th} \) and the \((k_n + 1)^{st}\) bidder the object is randomly allocated between them.
It is an easy exercise to show that this is a VCG mechanism for the allocation problem. Therefore there exists a dominant strategy equilibrium where the bidders bid according to their valuation. I.e.

\[ b_i(v_i) = v_i \quad \forall i, v_i. \]

Recall the concept of order statistic. The random variable \( Y_n(k_n) \) denotes the \( k_n \)th highest draw amongst \( n \) independent draws from the distribution \( F \).

(We omit the dependence of \( Y \) on \( F \) for notational simplicity). The questions about the convergence of the bids and prices in the auction can then be cast simply in terms of the order statistics. We have

\[ P_n = Y_n(k_n + 1) \]

in the auction with \( n \) bidders and \( k \) objects. Assume that \( F \) is concentrated on \([0, 1]\). Let

\[ \kappa = \lim_{n \to \infty} \frac{k_n}{n}. \]

Then

\[ P_n \to F^{-1}(1 - \kappa). \]

All limits of random variables in this lecture are understood in the sense of convergence in probability.

I.e.

\[ X_n \to Y_n \iff \forall \delta, \epsilon > 0, \exists N \text{ such that } n > N \Rightarrow P(|X_n - Y_n| > \epsilon) < \delta. \]

The result follows as a simple application of the (weak) law of large numbers.

Hence we have for our very first simple auction the result that as the number of bidders grows, the auction converges to a fixed price mechanism with the Walrasian price.
Private value double auctions

The next step is to consider a model where the supply as well as demand is determined endogenously. Therefore we start in a setting where all the \( k_n \) identical objects are initially held by potential sellers. We let the (private) values of the sellers \( c_j \) be i.i.d. according to a continuously differentiable distribution \( G \). Recall that in the case where \( k_1 = n = 1 \), the Myerson-Satterthwaite theorem tells us that efficient allocation is not possible under balanced budgets. In a double auction, sellers receive all payments made by the buyers and all players can secure their no-trade interim utility. The relevant question is then whether this inefficiency persists and whether the market price converges to the Walrasian price. For the remainder in this section, we set \( k_n = n \) and have an equal number of buyers and sellers in the market.

Let’s start by analyzing the ‘limit case’. Assume that there is a continuum of buyers and sellers, and assume that the law of large numbers holds for this case. Then the demand curve in the economy is given by:

\[
D(p) = 1 - F(p).
\]

Similarly, the supply curve is given by

\[
S(p) = G(p).
\]

A competitive equilibrium price \( p^* \) satisfies:

\[
1 - F(p^*) = G(p^*).
\]

Clearly a unique equilibrium exists for our model. The aim in this subsection is to show that this equilibrium is the unique limit of large double auctions as well. What is the issue here? In contrast to the Walrasian market where the price is exogenously given, the price in a double auction market
must be formed based on the bids (or in direct mechanism terms types) of the bidders. There will always be an incentive to manipulate the bid or the report so as to make a gain in the market. The key for the efficiency limit result that we get is an argument that shows that the probability that the individual bid changes the equilibrium price is small. Therefore bidders behave almost as if they were faced with a fixed price and therefore the result follows.

The simplest method for guaranteeing convergence to the walrasian outcome would be through the following approximately efficient fixed price mechanism. Calculate $p^*$ as above and ask the buyers and seller that are willing to trade at that price to report at a market place. Buyers and sellers are matched randomly so that all traders on the short side are matched, but the traders on the long side of the market are matched with equal probability. Within the matches, the trades are executed at price $p^*$. Unmatched players that reported at the market place leave with their initial endowments.

Claim 1 In this fixed price mechanism, sellers have a dominant strategy to report at the market place if and only of their valuation is below $p^*$ and buyers have a dominant strategy to report at the market place if and only if their valuation exceeds $p^*$. Furthermore, this mechanism is efficient in the limit in the sense that for large markets, the expected share of realized gains from trade to total gains from trade converges to 1 as $n$ grows.

Proof. The payoff from not reporting to a seller is $c_j$ and for a buyer, it is 0. Reporting at the market place gives a higher payoff if matched and the same if unmatched. This proves the first claim.

Let $B_n$ denote the random number of buyers that report and $S_n$ the random number of sellers that report when the total number of sellers is $n$. An upper bound for the efficiency loss is given by

$$ | B_n - S_n | .$$
Efficiency loss per trader is then bounded from above by
\[ \frac{|B_n - S_n|}{n}, \]
and weak law of large numbers implies that
\[ \frac{|B_n - S_n|}{n} \to 0 \]
as \( n \to 0 \).

A problem with the above mechanism is that it does not use information well at all. The rationing stage is assumed to be independent of the true valuations of the traders. This is essential to get the mechanism to have the dominant strategy solution. It is clear that efficient rationing would do a much better job, but this particular mechanism does not allow that. Hence we move to discuss briefly a better mechanism for the private values double auction setting.

We discuss next \((n + 1)^{st}\) price auctions. My exposition follows closely Mark Satterthwaite and Steven Williams: "The Rate of Convergence to Efficiency in the Buyers Bid Double Auction as the Market Becomes Large", *Review of Economic Studies*, (1989).

In these auctions, buyers submit bids \(b(v_i)\) and sellers submit offers \(s(c_j)\). Put all bids and offers in an decreasing order: \(s_1 \leq s_2 \leq \ldots \leq s_{2n}\). The price that each buyer receiving the object pays is equal to the \(p_n = s_{n+1}\) highest bid. The allocation is organized as follows: all sellers that have submitted a bid strictly below \(p_n\) trade with a buyer whose bid is at least \(p_n\). If there are no ties at the lowest winning bid, the market clears, if there are ties and the market does not clear, then market is cleared according to efficient rationing.

The payoffs are simply \(v_i - p_n\) for the buyers that trade, 0 for the buyers that do not trade, \(p_n\) for the sellers that trade, and \(c_j\) for the sellers that do not trade.

It is clear that bidding one’s valuation is not a dominant strategy for the buyers. (Remember that the \((n + 1)^{st}\) bid might come from a buyer, and in
this case the seller’s bid would affect the transfer even for a fixed allocation. For the sellers, bidding the value is still a dominant strategy.

We shall look for an equilibrium where the sellers bid according to their dominant strategies, and the buyers use symmetric strategies. In other words, we are looking for a pair of functions $S(c), B(v)$ such that whenever other players in the double auction follow these strategies, it is optimal for an individual buyer to bid according to $B(v)$ and for the seller to offer according to $S(c)$.

**Proposition 2** If $(S, B)$ is an equilibrium in the double auction, then the function $B$ has the following properties: (i) $0 < B(v)$ for all $v \in (0, 1]$, (ii) $B(v) \leq v$ for all $v \in [0, 1)$, (iii) $B(v)$ is strictly increasing on $[0, 1]$ and differentiable almost everywhere.

**Proof.** Note that buyer $i$ trades with a strictly positive probability at price $b$ if she submits any bid $b \in (0, 1)$. It is never optimal to bid $b > v$ (because with positive probability, the price paid exceeds her valuation). Write $P(b; S, B)$ for the probability that an individual buyer trades if she submits the bid $b$ and other players follow their equilibrium strategies. By standard arguments, incentive compatibility implies that:

$$P(B(v); S, B) \geq P(B(v'); S, B)$$

whenever $v > v'$. Since $P(\cdot; S, B)$ is increasing, we conclude that $B(v) \geq B(v')$.

We now show by contradiction that $B$ cannot be constant over any interval with non-empty interior. Suppose that $B(v) = b'$ for all $v$ in such an interval $I$. By i) and ii), $0 < b' < 1$. We claim: $P(b; S, B)$ is discontinuous at $b = b'$. This is true because the following events occur simultaneously with positive probability: (i) each buyer’s reservation value is in $I$ and therefore all buyers bid $b'$, (ii) at least one seller’s offer is less than $b'$, and (iii) at least
one seller’s offer is greater than $b'$. Note that (ii) and (iii) require that $n \geq 2$. Then price is $b'$, the market fails to clear at this price. Therefore there is an $\epsilon > 0$ such that $P(b''; S, B) > P(b'; S, B) + \epsilon$ for all $b'' > b'$.

The rise in the buyer’s expected payment $\Delta C(b'', b')$ is bounded from above by

$$b''(P(b''; S, B) - P(b'; S, B)) + (b'' - b').$$

By ii), there is a $v'$ such that $B(v') = b' < v'$. We claim that this type $v'$ can gain by an upwards deviation. Raising the bid to $b'' \in (b', v')$ induces a change in payoff calculated as:

$$v'(P(b''; S, B) - P(b'; S, B)) - \Delta C(b'', b')$$

$$> (v' - b'')\epsilon + (b' - b'')$$

by the earlier results. Since $b''$ can be chosen arbitrarily close to $b'$, this establishes the desired contradiction. All increasing functions are almost everywhere differentiable.

We consider next what the equilibria look like and how close they are to efficient. We start by defining probabilities of some events that turn out to be important for the incentives. Consider a buyer submitting a bid at $b$ and consider the following three events. i) The price is determined by the offer of one of the sellers and the price is $b$. ii) The price is $b$ and determined by another buyer, and iii) Own bid determines the prices. Letting $B^{-1}(b)$ be the type of the buyer whose equilibrium bid is $b$, we can define the following quantities:

$$K_n(b) = \sum_{i=0}^{n-1} \binom{n-1}{i} \binom{n-1}{n-i-1} G(b)^{n-1-i}(1-G(b))^i F(B^{-1}(b))^i (1-F(B^{-1}(b)))^{n-1-i},$$

$$L_n(b) = \sum_{i=1}^{n-1} \binom{n}{i} \binom{n-2}{i-1} G(b)^{n-i}(1-G(b))^i F(B^{-1}(b))^{i-1} (1-F(B^{-1}(b)))^{n-i-1},$$
\[ M_n(b) = \sum_{i=0}^{n-1} \binom{n-1}{i} G(b)^{n-i} (1-G(b))^i F(B^{-1}(b))^i (1-F(B^{-1}(b)))^{n-1-i}. \]

The interpretation of these is as follows: \( K_n(b) \) is the probability that \( s_{n-1} < b < s_n \) in a sample of \( n - 1 \) buyers and sellers using equilibrium strategies. \( L_n(b) \) is the probability that \( s_{n-1} < b < s_n \) in a sample of \( n - 2 \) buyers and \( n \) sellers using equilibrium strategies. \( M_n(b) \) is the probability that \( s_n < b < s_{n+1} \) in a sample of \( n - 1 \) buyers and \( n \) sellers using equilibrium strategies.

The densities of the three events are then:

i) \( n g(b) K_n(b) \)

ii) \( (n-1) \frac{f(v)}{B'(v)} L_n \)

iii) \( M_n(b) \)

Consider a buyer with valuation \( v \) that raises her bid from \( b \) to \( b + \Delta b \). The change in her expected utility is

\[ [n g(b) K_n(b) \Delta b + (n-1) \frac{f(v)}{B'(v)} L_n(b) \Delta b] (v - b - \Delta b) - M_n(b) \Delta b, \]

where \( \overline{v} \) solves \( B(\overline{v}) = b \).

Letting \( \Delta b \to 0 \) and requiring no profitable deviations from the equilibrium bid gives:

\[ \frac{1}{B'(v)} = \frac{M_n - (v - B(v)) n g(B(v)) K_n}{(v - B(v)) (n-1) f(v) L_n}. \]

Since \( \frac{1}{B'(v)} > 0 \), we get

\[ (v - B(v)) \leq \frac{M_n}{ng(B(v))K_n}. \]

Hence we have a strong convergence result if it can be shown that

\[ \lim_{n \to \infty} \frac{M_n}{K_n} = \gamma \in \mathbb{R}. \]
This intuitive result can be shown, but it is a semi-involved exercise in combinatorics and therefore omitted here.

**Discussion** A good result on the convergence of strategies to truthful. This implies good efficiency properties of the mechanism. What about existence? In the case of a symmetric model, usual existence results for solutions to differential equations guarantee existence.

**Notes on Further Literature**

'Convergence to Efficiency in a Simple Market with Incomplete Information' by Aldo Rustichini, Mark A. Satterthwaite, Steven R. Williams in Econometrica, Vol. 62, No. 5 (Sep., 1994), pp. 1041-1063 obtains a slightly more efficient mechanism than the one discussed above, but the setup is essentially the same, and the gain is marginal in terms of economic understanding.

'Efficiency of Large Double Auctions', Econometrica, vol. 74(1) Jan 2006, pages 47-92 by Martin Cripps and Jeroen Swinkels gives a very thorough investigation of equilibria in large double auctions with private values as the number of participants increases. The paper relaxes the assumptions of symmetry, independence and unit demands and supplies from the earlier literature. Main conclusions are:

- Equilibrium exists. (For non-trivial equilibria, this result is based on earlier work by Jackson and Swinkels).

- Equilibria either involve no trade (trivial equilibria) or are almost efficient (non-trivial equilibria).

- Efficiency losses are asymptotically proportional to $\frac{1}{n^2}$ as $n \to \infty$.

'Existence of equilibrium in large double auctions' by Drew Fudenberg, Markus Mobius, Adam Szeidl in JET, 2007 introduces a nice perturbation
technique for proving equilibrium existence in setting similar to Cripps and Swinkels (2007).
First Steps in Monotone Comparative Statics

Supermodularity

Recall comparative statics of simple optimization problems:

\[
\max_x u(x, y)
\]

First order condition for the problem is\n\[
\frac{\partial u(x, y)}{\partial x} := u_x(x, y) = 0.
\]

Differentiating w.r.t. \(x\) and \(y\) and solving for \(\frac{dx}{dy}\) gives:

\[
\frac{dx}{dy} = -\frac{u_{xy}(x, y)}{u_{xx}(x, y)}.
\]

Second order condition gives that \(u_{xx}(x, y) \geq 0\), and therefore the sign of the comparative statics is given by the cross partial term \(u_{xy}(x, y)\). If \(u_{xy}(x, y) \geq 0\), at all \((x, y)\), then we say that \(u\) is supermodular in \((x, y)\).

A more general definition for supermodularity for functions of real vectors \((x_1, x_2, ..., x_n)\) is given as follows. For any two real numbers \(z, y\), let \(z \wedge y\) denote the minimum and \(z \vee y\) denote the maximum. For vectors \(x, x'\), let

\[
x \wedge x' = (x_1 \wedge x'_1, ..., x_n \wedge x'_n)
\]

and

\[
x \vee x' = (x_1 \vee x'_1, ..., x_n \vee x'_n).
\]

A lattice in \(\mathbb{R}^n\) is a set \(X \subset \mathbb{R}^n\) such that for all \(x, y \in X\), \(x \wedge y \in X\) and \(x \vee y \in X\). A function \(f\) on a lattice \(X\) is called supermodular if for all \(x, x' \in X\), we have

\[
f(x \wedge x') + f(x \vee x') \geq f(x) + f(x').
\]

It is easy to see that if \(X \subset \mathbb{R}^n\) and \(f\) is twice differentiable, then supermodularity in terms of cross partials agrees with this definition.

A positive function \(f\) on a lattice \(X\) is called logsupermodular if \(\log f\) is supermodular. Therefore \(f\) is logsupermodular if
\[ f(x \land x')f(x \lor x') \geq f(x)f(x'). \]

**Monotone Likelihood Ratio Property**

Consider a logsupermodular joint density function \( f(x, \theta) \) on \( \mathbb{R}^2 \). Then for \( x < x' \) and \( \theta < \theta' \), we have:

\[ f(x, \theta)f(x', \theta') \geq f(x', \theta)f(x, \theta'). \]

Let \( f(x) \) and \( g(\theta) \) be the prior marginal distributions on \( x \) and \( \theta \) respectively. Then since

\[ f(x, \theta) = g(\theta)f(x \mid \theta), \]

and since

\[ f(x, \theta) = f(x)g(\theta \mid x), \]

logsupermodularity of \( f(x, \theta) \) immediately implies that for \( x < x' \) and \( \theta < \theta' \), we have

\[ \frac{f(x' \mid \theta')}{f(x \mid \theta')} \geq \frac{f(x' \mid \theta)}{f(x \mid \theta)}, \]

and

\[ \frac{g(\theta' \mid x')}{g(\theta \mid x')} \geq \frac{g(\theta' \mid x)}{g(\theta \mid x)}. \]

These last two inequalities are known as Monotone Likelihood Ratio Property in the literature. They assert a positive association between the two random variables \( x \) and \( \theta \) in the sense that relatively higher values of \( x \) increase the likelihood of relatively higher values of \( \theta \). It is an easy exercise to show that for \( x' > x \), the conditional distribution \( G(\theta \mid x') \) dominates \( G(\theta \mid x) \) in the sense of first order stochastic dominance.

Some observations:
\[ \int g(\theta \mid x) \, d\theta = 1 \text{ for all } x. \]

Therefore
\[ \int \frac{\partial g(\theta \mid x)}{\partial x} \, d\theta := \int g_x(\theta \mid x) \, d\theta = 0. \]

A very useful Lemma for determining comparative statics is the following:

**Lemma 3** Suppose \( H(\theta) \) and \( J(\theta) \) are both increasing and for some measure \( \mu(\theta) \) on the real line, we have
\[
\int H(\theta) \, d\mu(\theta) = 0.
\]

Then we have
\[ \int J(\theta) H(\theta) \, d\mu(\theta) \geq 0. \]

**Proof.** Since \( H \) is increasing and \( \int H(\theta) \, d\mu(\theta) = 0 \), there must be a point \( \theta_0 \) such that \( H(\theta) \leq 0 \) for \( \theta < \theta_0 \), and \( H(\theta) \geq 0 \) for \( \theta > \theta_0 \). Since \( J \) is also increasing, we have that \( J(\theta) - J(\theta_0) \leq 0 \) for \( \theta < \theta_0 \) and \( J(\theta) - J(\theta_0) \geq 0 \) for \( \theta > \theta_0 \). Since \( \int J(\theta_0) H(\theta) \, d\mu(\theta) = 0 \), we have
\[
\int J(\theta) H(\theta) \, d\mu(\theta) = \int (J(\theta) - J(\theta_0)) H(\theta) \, d\mu(\theta)
\[
= \int_{-\infty}^{\theta_0} (J(\theta) - J(\theta_0)) H(\theta) \, d\mu(\theta) + \int_{\theta_0}^{\infty} (J(\theta) - J(\theta_0)) H(\theta) \, d\mu(\theta) \geq 0.
\]

To conclude this discussion, I define a slightly more general form of positive association than supermodularity. Consider for simplicity the case of a function \( f \) on \( \mathbb{R}^2 \).

We say that \( f \) has satisfies the single crossing property (SCP) in \( x \) for \( y \) if for \( x' > x \) and \( y' > y \), we have that \( f(x', y) > f(x, y) \) implies \( f(x', y') > f(x, y') \) and \( f(x', y) \geq f(x, y) \) implies \( f(x', y') \geq f(x, y') \).
This condition is sometimes called the condition of increasing differences since it says that the difference \( f(x', y) - f(x, y) \) is increasing as a function of \( y \). Much of what we have said and will say about supermodular or logsupermodular functions would in fact go through with the more general class of functions satisfying SCP. It is an easy exercise to show that supermodular and logsupermodular functions satisfy SCP.

One last definition must be introduced before we can state the main monotone comparative statics result. Consider subsets \( A \) and \( B \) of a lattice \( X \). We say that \( A \) is higher than \( B \) in the strong set order if \( x \in A \) and \( y \in B \) imply that \( x \lor y \in A \) and \( x \land y \in B \). We write then \( A \leq_S B \).

Denote the set of maximizers in the problem

\[
\max_{x \in S} f(x, t)
\]

by \( M(t, S) \) We state the main theorem for real \( x \) and \( t \), but generalizations for higher dimensions are possible and covered in Milgrom and Shannon (1994).

**Theorem 4** (Milgrom and Shannon, 1994) A function \( f(x, t) \) satisfies the single crossing property if and only if \( M(t, S) \leq M(t', S') \) for all \( t \leq t' \) and \( S \leq_S S' \).

**Proof.** i) \( \Rightarrow \) Let \( t \leq t' \) and \( S \leq_S S' \) and \( x \in M(t, S) \) and \( x' \in M(t', S') \). If \( x' > x \), then there is nothing to prove. Hence assume that \( x > x' \). Then we must show that \( x \in M(t', S') \) and \( x' \in M(t, S) \). But these follow immediately from the definition of single crossing. ii) \( \Leftarrow \) Left as an exercise. ■

**Comparative Statics in Choice Under Uncertainty**

Consider a problem where a decision maker must decide an optimal action \( x \) before knowing what the true state of the world \( \theta \in \Theta \) is. Her utility is state dependent: \( u(x, \theta) \), and prior to deciding, she Observes a signal \( S \) that
is correlated with $\Theta$ according to joint density $f(s, \theta)$. We assume that $f$
is MLRP and thus is logsupermodular. This assumption is also called affiliation between $X$ and $\Theta$.

The maximization problem is then to

$$
\max_x v(x, s) := \int u(x, \theta) f(\theta | s) d\theta.
$$

If we know that $v(x, s)$ satisfies SCP, then by the Milgrom-Shannon theorem, we know that the optimal $x$ is increasing in $s$. A useful Theorem due to Karlin (1968) gives a sufficient condition for this.

**Theorem 5** If $f(x, \theta)$ and $g(\theta, s)$ are logsupermodular, then $h(x, s) = \int f(x, \theta) g(\theta, s) d\theta$
is also logsupermodular.

**Proof.** Write for $x < x'$ and $s < s'$

$$
\Delta = h(x', s')h(x, s) - h(x', s)h(x, , s)
$$

$$
= \int \int f(x, \theta_1)f(x', \theta_2)[g(\theta_1, s)g(\theta_2, s') - g(\theta_1, s')g(\theta_2, s)]d\theta_1d\theta_2.
$$

Consider next $\Delta$ as the double integral over $\{\theta_1 < \theta_2\}$ plus the double integral over $\{\theta_1 > \theta_2\}$ and combine to get

$$
\Delta = \int \int_{\theta_1 < \theta_2} \left[ f(x, \theta_1)f(x', \theta_2) - f(x, \theta_2)f(x', \theta_1) \right] \times
$$

$$
\left[ g(\theta_1, s)g(\theta_2, s') - g(\theta_1, s')g(\theta_2, s) \right] d\theta_1d\theta_2.
$$

Finally use the fact that $f$ and $g$ are logsupermodular. ■

Since $f(\theta | s)$ is logsupermodular by our MLRP assumption, we know that if $u(x, \theta)$ is logsupermodular, then $v(x, s)$ is also logsupermodular and we can use the Milgrom and Shannon Theorem.

If $u$ and $f(\theta | s)$ are differentiable, then we can proceed slightly differently.

The first order condition for our problem is that
\[ v_x(x, s) = \int u_x(x, \theta) f(\theta \mid s) d\theta = 0. \]

Therefore
\[ v_x(s, x) = \int u_x(x, \theta) f_s(\theta \mid s) d\theta = 0. \]

Since
\[ \int f_s(\theta \mid s) d\theta = 0, \]
we can use the lemma from the previous subsection to conclude that
\[ v_x(s, x) = \int u_x(x, \theta) f_s(\theta \mid s) d\theta \geq 0 \]
if \( u_x(x, \theta) \) is increasing in \( \theta \), i.e. if \( u \) is supermodular.

**Existence of Monotone Equilibria in Pure Strategies**

The following exposition follows essentially Susan Athey, Econometrica, 2001.

A game of incomplete information is given by \( (\mathcal{I}, T, A, u, f) \), where \( \mathcal{I} = \{1, \ldots, I\} \) is the set of players, \( T = \times_{i=1}^I T_i \) is the set of types, \( A = \times_{i=1}^I A_i \) is the set of actions, \( u = \times_{i=1}^I u_i \) is the set of Bernoulli utilities where \( u_i : A \times T \to R \), and \( f \) is a prior probability on \( T \). We assume throughout that actions and types are real numbers with the natural order on them.

**Definition 6** The Single Crossing Condition (SCC) is satisfied if for each \( i \in \mathcal{I} \) and \( a_{-i} (t_{-i}) \) where each \( a_j (t_j) \) for \( j \neq i \) is nondecreasing, we have
\[
U_i (a_i, t_i; a_{-i} (t_{-i})) := \int u_i (a_i, a_{-i} (t_{-i}); t_i, t_{-i}) f (t_{-i} \mid t_i) dt_{-i}
\]
satisfies the single crossing property in \( (a_i, t_i) \).
In the previous lecture, we saw examples of sufficient conditions for this. Log-supermodularity is clearly a key for this.

Consider forst the case of games where \( A \) is a finite set with cardinality \( M \) on the real line and \( T_i \) is an interval of the real line \([t_i, \bar{t}_i]\). Then non-decreasing strategies may be represented by a vector \( x \) of cutoff points. We say that \( x \) is consistent with \( a_i(t_i) \) if \( x \) gives the cutoffs in the pure strategy \( a_i(t_i) \). Notice that each vector has at most \( M \) elements. If there are fewer cutoffs, we can take the remaining cutoffs to be at \( \bar{t}_i \). Then each \( x \in T_i^M \).

We denote by \( a_i^{BR}(t_i, x) \) the best responses of player \( i \) to \( a_{-i}(t_{-i}) \) that are consistent with \( x_{-i} \). Let

\[
\Gamma_i(x) = \{ y \in T_i^M | \exists a_i(t_i) \text{ consistent wth } y \text{ such that for all } t_i, a_i(t_i) \in a_i^{BR}(t_i, x) \}.
\]

Let \( \Gamma = (\Gamma_1, ..., \Gamma_I) \). If we can find a fixed point for the correspondence \( \Gamma \), then we’re done.

Key claim is that \( \Gamma_i(x) \) (and therefore also \( \Gamma(x) \)) is convex if \( a_i^{BR}(t_i, x) \) is increasing in the strong set order in \( t_i \). I draw the picture, you provide the formal proof.

We can apply Kakutani’s fixed point theorem if we show that \( \Gamma(x) \) has a closed graph. But this is a standard continuity/compactness argument (actually even easier since \( A \) is finite) so it is also left as an exercise.

**Theorem 7** Under SCC, the game of incomplete information has a monotone (nondecreasing) pure strategy equilibrium.

So for finite games, all the work is in establishing SCC.

Consider next games with a continuous action space.

**Theorem 8** Assume that \( A_i = [a_i, \pi_i] \) and that \( u_i(a, t) \) is continuous in \( a \) and for all finite subsets \( A' \subset A \), the finite game has a pure strategy Nash
equilibrium in nondecreasing strategies. Then the continuous action game also has a monotone PSNE.

The proof is an application of Helly’s selection theorem that shows that the every sequence of nondecreasing bounded functions has a (pointwise) convergent subsequence. For this you start with an approximating sequence of finite games with $A_n$ and let $A_n \to A$ in the sense that for all $\varepsilon > 0$ and for all $a \in A$, there is a $n(\varepsilon, a)$ such that for all $n > n(\varepsilon, a)$, the distance $d(A_n, a) < \varepsilon$, where

\[ d(A_n, a) = \min_{a' \in A_n} d(a', a). \]

Next you consider a sequence of monotone PSNE $a^*_n(t)$ of $\{A_n\}$ which exist by assumption. Use Helly’s selection theorem to conclude the existence of a convergent subsequence, and the limit along this subsequence must be a monotone PSNE of the continuous game. The details are left as an exercise.

In the paper by Athey, you will find all kinds of sufficient conditions for SCC. These cover, in particular, the case of affiliated auctions that we discuss next.
Information Aggregation in Common Values Auctions

Fixed Supply


Motivation

- In private values markets, price reflect the aggregate demand and supply conditions, and measures the equilibrium level of scarcity.

- With common values, price also reflects aggregate information on the true value of goods for sale. Financial markets are a prime example.

- In general equilibrium theory with private information, the most common solution concept is fully revealing rational expectations equilibrium (REE).

- The price reflects all the private information *regardless* of how individual traders believe.

- Hence my information is reflected in the price even if I trade the same quantity at all my information sets. (This is off the equilibrium path in those models, but as always events off the equilibrium path give incentives for the equilibrium path.)

- Grossman & Stiglitz have an early paper demonstrating the problematic nature of REE. They construct an example trade reflects private information if price is uninformative and trade is independent of information if price is revealing.
• Could large auctions and in particular large double auctions be used as models of almost competitive trade?

• These game-theoretic models account well for the informational effects of actions.

• Does the price converge the Walrasian price in common values models?

• Does the price aggregate all the information that the bidders have?

• We shall see that many properties of the equilibrium outcome can be inferred from certain order statistics of the vector of bidders’ signals.

Setup

True value of the object: random variable $V$ with a realization $v \in [0,1]$. Bidder $i$ receives a private signal $S_i$ with a realization $s_i \in [0,1]$ correlated with $V$.

In an auction with $n$ bidders, the vector of signals $S = (S_1,...,S_n)$
The $l^{th}$ order statistic of $S$ is denoted by $Y_n(l)$.
In the sample of all bidders but $i$, it is denoted by $Y^i_n(l)$.
Price (random) in the auction with $n$ bidders is $P_n$.
Bidding strategy in the auction with $n$ bidders is $b_n(s_i)$

Assumption 9 Prior distribution on $V$ is $F(v)$ and we assume that the densities $f(v)$ of the value and the conditionals density on signals $f(s_i \mid v)$ are well defined.

Assumption 10 Monotone likelihood ratio property (MLRP) holds: For all $i$, $s'_i > s_i$, and $v' > v$,

$$\frac{f(s'_i \mid v')}{f(s_i \mid v')} > \frac{f(s'_i \mid v)}{f(s_i \mid v)}.$$
Furthermore we make the assumption that the information content in individual signals is bounded.

**Assumption 11** There is a constant $\kappa > 0$ such that

$$\forall s_i, v, \quad \frac{1}{\kappa} > f(s_i \mid v) > \kappa.$$ 

We concentrate on the $k_n$ unit auction where $k_n$ identical units of a good are sold. The $k_n$ highest bidders receive the object and each pays the $(k_n+1)^{st}$ highest bid.

Milgrom & Weber (1982) show that this auction has a symmetric equilibrium in strictly increasing strategies where an individual bid is calculated as:

$$b_n(s_i) = \mathbb{E}(V \mid Y_{n}(k) = Y_{n}(k + 1) = s_i).$$

The proof that this is actually an equilibrium is left as an exercise. (Recall the reasoning for the wallet game).

**Analysis**

**Definition 12** An auction is competitive if $\mathbb{E}(P_n) \to \bar{V}$.

**Definition 13** A competitive auction aggregates information if $P_n - V_n \to 0$ in probability.

**Definition 14** An auction is informative if $\mathbb{E}(V_n \mid P_n) - V_n \to 0$ in probability.

Call the bidder whose bid is taken as the price a *pivotal bidder*. Denote the information available to the pivotal bidder by $\mathcal{F}_n$. The following result is an immediate consequence of the characterization of the equilibrium bid function.
Lemma 15

\[ P_n \leq \mathbb{E}(V_n \mid \mathcal{F}_n). \]

The following two theorems establish the informational properties of competitive auctions in general.

**Theorem 16** If the auction is competitive, then \( P_n - \mathbb{E}(V_n \mid \mathcal{F}_n) \to 0. \)

**Theorem 17** If the auction is competitive, then \( P_n - \mathbb{E}(V_n \mid P_n) \to 0. \)

Hence a competitive auction aggregates information if and only if it is informative.

In order to get concrete results for the \( k_n \) unit auction, we assume that the \( s_i \) are i.i.d. conditional on the true \( v \) and consider the case where \( k_n = \alpha n \) for some \( \alpha \in (0,1) \). Then weak law of large numbers implies that

\[ Y_n(\alpha n + 1) \to F^{-1}_{s|v}(1 - \alpha) \]

in probability. Under MLRP, \( F^{-1}_{s|v}(1 - \alpha) \) is strictly increasing when viewed as a function of \( v \), we conclude that

\[ \mathbb{E}(V_n \mid Y_n(k_n + 1)) \to V_n \]

in probability.

Therefore we know that if the auction is competitive, then it aggregates information. This final step is in the following theorem.

**Theorem 18** The \( k_n \) unit auction is competitive.

**Proof.** Let

\[ b_n^*(s) \equiv \mathbb{E}(V_n \mid Y_n(k_n + 1) = s), \]

let

\[ h_n(s) = b_n^*(s) - b_n(s), \]

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and

\[ b^*_n(s) \equiv \mathbb{E}(V_n \mid Y_n(k_n) = s). \]

By MLRP,

\[ b^*_n(s) < b_n(s) < b^*_n(s). \]

By weak law of large numbers, \( b^*_n(s) \to v^* \) and \( b^*_{n}(s) \to v^* \), where \( v^* \) solves

\[ F(s \mid v) = 1 - \alpha. \]

Therefore \( h_n(s) \to 0 \) and the proof follows from the fact that

\[ \nabla - \mathbb{E}(P_n) = \int_0^1 h_n(s)f_{Y_n(k_n)}(s)ds \]

and that the limiting distribution of \( Y_n(k_n) \) is given by \( F^{-1}(1 - \alpha \mid v) \) and therefore \( f_{Y_n(k_n)}(s) \) converges to a finite limit. □

**Double Auctions**

We will not attempt this here. "Toward a Strategic Foundation for Rational Expectations Equilibrium." Philip J. Reny and Motty Perry; Econometrica, 2006 needs a lecture series of its own to be covered.
Information Acquisition

Motivation

Until now, we have represented the information available to the agents by their privately known type. Indeed, the whole paradigm of mechanism design starts with the dispersed information held by different agents as a primitive of the economic situation. The task of the mechanism designer is to construct a game form or a mechanism that simultaneously collects the information and uses it efficiently to arrive at the desired allocation.

It should be clear that in many economically interesting settings, information is not exogenous. To the contrary, information is costly to come by and the amount of information acquisition depends on the economic value that the agents expect to get from it. Hence it is not clear why information ought to be seen as a primitive and independent of the final allocation mechanism. The payoffs from private information depend on the mechanism used, and as a consequence, the choice of the mechanism should have a feedback effect to the information acquisition decisions.

Value of Information

Consider the single-agent decision problem, where the agent chooses the optimal action $a \in A$ to maximize her expected utility. The utility depends on the state of nature $\theta \in \Theta$, and for the remainder of this lecture, we assume that $\Theta = \{\theta_1, ..., \theta_N\}$ is a finite set with $N$ elements. Assume also that the (state dependent) utility function $u(a, \theta)$ is continuous in $a$ for all $\theta$ and $A$ is compact. Let $p(i)$ denote the prior probability of the event $\{\theta = \theta_i\}$, and $p = (p(i), ..., p(N))$. We assume w.l.o.g. that $p(i) > 0$ for all $i$. The decision maker’s problem is then to
\[ \max_a \sum_{i=1}^{N} p(i)u(a, \theta_i). \]

Let \( a(p) \) denote the maximizer in the above program, and let

\[ V(p) = \sum_{i=1}^{N} p(i)u(a(p), \theta_i) \]

be the value function of the program.

**Observation 1.**

\( V(p) \) is convex in \( p \).

...Think back to the properties of profit function... This observation follows immediately from the observation that \( V(p) \) is the maximum over a family of linear functions.

**Corollary**

If there are two states, then \( p \in [0, 1] \). Therefore, Rothschild and Stiglitz from first year Micro implies that Mean preserving spreads are good for the decision maker.

The best the decision maker could hope for is to know the true state. Then \( p \in \{0, 1\} \). Clearly the perfectly revealing statistical experiment is the best for the decision maker (since then she can take optimal actions state by state). Loosely speaking, an experiment is a random variable whose outcome is correlated with the true state of the world. After seeing the outcome in the experiment, the decision maker updates her beliefs on the state and then chooses the optimal action. The above reasoning told us that one experiment is better than a second experiment if the posterior belief resulting from the first is second order stochastically dominated by the belief resulting from the second. Or if the posterior from the first is a mean preserving spread of the posterior from the second.

In this lecture, we shall see how to generalize this comparison to the case where there are multiple states (and hence mean preserving spreads are
not well defined. Blackwell’s theorem shows that one experiment is more valuable to all decision makers than a second experiment if and only if the first experiment is statistically sufficient for the second.

**Statistical experiments and sufficiency**

An experiment is a random variable correlated with the state of the world. Here we will consider two arbitrary experiments $X_1$ and $X_2$. Experiment $X^j$ takes values in $1, ..., K^j$. Denote probability of outcome $k^j$ in experiment $j$ conditional on the state $\theta_i$ by $\pi^j(k^j \mid i)$. Assume w.l.o.g. that for all $k^j$, there is an $i$ such that $\pi^j(k^j \mid i) > 0$. Then the marginals on all signals are strictly positive:

$$\pi^j(k^j) \equiv \sum_i \pi^j(k^j \mid i)p(i) > 0.$$  

As a result, we can write Bayes’ rule:

$$p^j(i \mid k^j) = \frac{\pi^j(k^j \mid i)p(i)}{\pi^j(k^j)}.$$

Let $\Pi^j(k^j \mid i)$ denote the matrix where the $i^{th}$ column is $\pi^j(k^j \mid i)$.

**Definition 19** Experiment $X_1$ is sufficient for $X_2$ ($X_1 \geq X_2$) if there exists a positive $K^2 \times K^1$ matrix $B$ with a typical element $b_{k^2k^1}$ such that

$$\Pi^2(k^2 \mid i) = B\Pi^1(k^2 \mid i, )$$

and

$$\sum_{k^2} b_{k^2k^1} = 1 \text{ for all } k^1.$$  

The interpretation of sufficiency is rather immediate. $X_1$ is sufficient for $X_2$ if you can get the conditional probability of the signals in experiment $X_2$ through a two-stage procedure. First observe $X_1$ and the resulting $k^1$. Perform an additional randomization (that is independent of $\theta$ and sends
the original signals onto 1, ..., $K^2$. It is clear that observing only the signals after the second stage cannot improve information on $\theta$ since the second randomization was by assumption independent of $\theta$. Hence there should be at least as much statistical information about $\theta$ in $X^1$ as in $X^2$.

**Valuable information**

We could think that experiment $X^1$ is more valuable than experiment $X^2$ if for all utility functions $u(a, \theta)$, the optimization after observing signals $k^1$ result in a higher expected payoff than optimization after observing $k^2$. Write the value function of the decision maker after observing signal $k^i$ as $V(p \mid k^i)$. More formally, we define:

**Definition 20** Experiment $X^1$ is more informative than experiment $X^2$ ($X^1 \succeq X^2$) if:

$$\sum_{k^1} \pi^1(k^1)V(p^1 \mid k^1) \geq \sum_{k^2} \pi^2(k^2)V(p \mid k^2).$$

for all $A$ and all $u(a, \theta)$.

**Blackwell’s theorem**

Blackwell’s Theorem connects the above two definitions:

**Theorem 21** $X^1$ is more informative than $X^2$ if and only in $X^1$ is sufficient for $X^2$.

To make the proof simple, interpret all matrices as long vectors. For example, $\Pi^j(k^j \mid i)$ should be understood as a $K^j \times n$ vector where the columns $\pi(k^j \mid i)$ are stacked on top of one another. (I.e. first the conditional on signals given state $\theta_1$, then conditional on $\theta_2$, and so on until conditional distribution on signals given $\theta_N$.) We denote this vector by $\pi(\cdot \mid \cdot)$. Similarly $b_{k^2k^1}$ is the $K^2 \times K^1$ positive vector where the columns of $B$ are stacked on top of each other. Let $B$ denote the set of all such vectors, where $\sum_{k^2} b_{k^2k^1} = 1$. 

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Finally, let $D$ denote the set of $d \in \mathbb{R}^{K^2 \times N}$ (again to be read as the columns of the matrix formed by $d$ stacked on to of each other) such that
\[
\sum_{k^1} b_{k^2k^1} \pi^1(k^1 \mid i) = d_{k^2i} \text{ for some } b_{k^2k^1} \in B.
\]

With this notation, we begin the proof.

**Proof.** The following are equivalent:

1. $X^1$ is sufficient for $X^2$.
2. $\pi^2 \in D$.
3. For all $q \in \mathbb{R}^{K^2 \times N}$, there exists $b_{k^2k^1} \in B$ such that
   \[
   \sum_{k^2, i} q_{k^2i} \pi^2(k^2 \mid i) \leq \sum_{k^2, i} q_{k^2i} \sum_{k^1} b_{k^2k^1} \pi^1(k^1 \mid i) = \sum_{k^1} \sum_{k^2} b_{k^2k^1} \sum_{i} q_{k^2i} \pi^1(k^1 \mid i).
   \]
4. For all $q \in \mathbb{R}^{K^2 \times N}$,
   \[
   \sum_{k^2, i} q_{k^2i} \pi^2(k^2 \mid i) \leq \sum_{k^1} \left[ \max_{k^2} \sum_{i} q_{k^2i} \pi^1(k^1 \mid i) \right].
   \]
5. For all $\phi \in \mathbb{R}^{K^2 \times N}$,
   \[
   \sum_{k^2} \pi^2(k^2) \sum_{i} \phi_{k^2i} \pi^2(i \mid k^2) \leq \sum_{k^1} \pi^1(k^1) \max_{k^2} \sum_{i} \phi_{k^2i} \pi^1(i \mid k^1).
   \]
1 ⇔ 2 by definition of sufficiency. 2 ⇔ 3 by separating hyperplane theorem. 3 ⇔ 4 Follows since \( \sum_{k^1} \{ \max_{k^2} \sum_i q_{k^2i} \pi^1(k^1 \mid i) \} \) maximizes 

\[
\sum_{k^1} \sum_{k^2} b_{k^2k^1} \sum_i q_{k^2i} \pi^1(k^1 \mid i)
\]

over \( b_{k^2k^1} \in B \). 4 ⇔ 5 by change of variables: \( q_{k^2i} = p(i)\phi_{k^2i} \).

With these preliminary results, we can complete the proof.

**more informative implies sufficiency**

By contrapositive. If \( X^1 \) is not sufficient for \( X^2 \), then item 5 above does not hold, and there is a \( \phi \in \mathbb{R}^{K^2 \times N} \) such that

\[
\sum_{k^1} \pi^1(k^1) \max_{k^2} \sum_i \phi_{k^2i} p^1(i \mid k^1) < \sum_{k^2} \pi^2(k^2) \sum_i \phi_{k^2i} p^2(i \mid k^2)
\]

\[
\leq \sum_{k^2} \pi^2(k^2) \max_{\lambda^2 \in K^2} \left[ \sum_i \phi_{\lambda^2i} p^2(i \mid k^2) \right]
\].

Then for \( A = K^2 \) and \( u(k^2, \theta_i) = \phi_{k^2i} \), we have

\[
\sum_{k^1} \pi^1(k^1)V(p \mid k^1) < \sum_{k^2} \pi^2(k^2)V(p \mid k^2).
\]

**sufficiency implies more informative**

Let \( \phi_{k^2i} = u(a(2^2(\cdot \mid k^2)), \theta_i) \) and conclude from item 5 above that the value in \( X^1 \) must be at least as high as in \( X^2 \).

\[\blacksquare\]

**Comments**

Blackwell’s theorem is a fantastic result conceptually. At the same time, its message is somewhat disappointing. The resulting order on experiments
is very incomplete. In other words, for most experiments $X,Y$ neither $X$ is more informative than $Y$ nor is $Y$ more informative than $X$. This follows from the fact that statistical sufficiency is not easily satisfied. Another problem with the Theorem is that in practice, it is of little help for decision theory. It is notoriously difficult to check whether one experiment is sufficient for another. In fact, the theorem is often useful in showing that sufficiency fails by constructing an appropriate decision problem. For economics, this is of course not very helpful.

Consider an example to see this. Suppose that $X = \theta + \epsilon_X$ and $Y = \theta + \epsilon_Y$, where $\theta$ is uniformly distributed on $[-M, M]$, $\epsilon_X$ is uniformly distributed on $[-a, a]$, and $\epsilon_Y$ is uniformly distributed on $[-b, b]$, and $a < b$. Intuitively, one would think that $X$ is more informative than $Y$, nonetheless neither of these experiments is more informative than the other. I leave it as a (very good and somewhat challenging) exercise to verify this. On the other hand, it should be noted that adding another copy of an independent experiment leads to a more informative experiment.

In the next section, we will search for tighter orders by imposing additional structure on the decision problems. The approach is thus similar to that taken in expected utility theory when looking for various notions of stochastic orders. (Recall there that FOSD allowed for all increasing utility functions whereas SOSD was defined when utility functions were restricted to increasing and concave functions).
Lehmann’s order

Lehmann’s condition

Consider two experiments $X, Y$ on a state space $\Theta$. Denote the conditional distribution of signals (i.e. outcomes of the experiment) for $X$ by $F(x \mid \theta)$ and the distribution of signals in $Y$ by $G(y \mid \theta)$. Select two states $\theta_1, \theta_2$ with $\theta_1 < \theta_2$. Plot the following parametric curves in the unit square:

$$(F(x, \theta_1), F(x, \theta_2)) \text{ and } (G(y, \theta_1), G(y, \theta_2)).$$

Clearly both of these curves start at the lower left corner of the square and end at the higher right corner. If we assume MLRP, then the curves start with a slope less than 1 and the slopes increase as the parameter $x$ or $y$ grows. This is seen easily by differentiating with respect to the parameter.

Lehmann’s condition for $X$ to be more accurate than $Y$ is that the graph of $(F(x, \theta_2), F(x, \theta_1))$ lie below $(G(y, \theta_2), G(y, \theta_1))$ for all choices of $\theta_1 < \theta_2$. An equivalent statement of this is that for all $\theta_1 < \theta_2$,

$$F(F^{-1}(p; \theta_1); \theta_2) \leq G(G^{-1}(p; \theta_1); \theta_2).$$

Another equivalent way of stating Lehmann’s condition is that for all $x, y$ in the support of $F, G$ respectively,

$$F(x; \theta) - G(y; \theta)$$

has at most one change of sign from negative to positive as $\theta$ increases.

Yet another equivalent statement for Lehmann’s condition is the following: Experiment $\alpha$ is more accurate than experiment $\beta$ if for all $\theta$,

$$T_{\beta, \alpha, \theta}(x) \equiv (F^\alpha)^{-1}(F^\beta(x \mid \theta) \mid \theta)$$
is nondecreasing in $\theta$. Therefore $T_{\beta,\alpha,\theta}(X^\beta \mid \theta)$ has the same distribution as $X^\alpha \mid \theta$.

Finally, it should be noted that Lehmann’s condition is equivalent to Blackwell’s condition when applied to dichotomies (i.e. two-point sets of states).

**Restrictions on preferences and Persico’s theorem**

**Definition 22** A utility function $u(a, \theta)$ satisfies the single crossing property if for each pair of decisions $a, a' \in A$ with $a < a'$, and $\theta < \theta'$,

$$u(a', \theta) - u(a, \theta) \geq 0 \Rightarrow u(a', \theta') - u(a, \theta') \geq 0.$$  

The single crossing property is said to be strong if the above equation holds with strict inequalities.

It should be clear that supermodular preferences satisfy single crossing. This class of preferences is also important in the literature on monotone comparative statics. Milgrom and Shannon, Econometrica, 1994, explore conditions that guarantee monotone comparative statics results for decision problems and for games. Athey, QJE, 2001, extends this analysis to decisions under uncertainty, and papers Athey, Reny, McAdams (all in Econometrica, early 2000’s) and others have used these notions in the study of existence of equilibria in monotone pure strategies in Bayesian games.

For our purposes the following Theorem (due to Nicola Persico) is of most importance.

**Theorem 23** Consider two experiments $X, Y$ that satisfy the MLRP and that have conditional distributions $F(x; \theta), G(x; \theta)$. Then the following two statements are equivalent.

1. $X$ is more accurate than $Y$ in the sense of Lehmann’s accuracy.
2. For all prior probabilities \( \mu \) on \( \Theta \) and for all utility functions \( u(a, \theta) \) satisfying the single crossing property,

\[
\int_{\theta \in \Theta} \int u(a(x), \theta) dF(x; \theta) d\mu(\theta) \geq \int_{\theta \in \Theta} \int u(a(y), \theta) dG(y; \theta) d\mu(\theta).
\]

Hence a Bayesian decision maker with single crossing preferences values information that is accurate in the sense of Lehmann. We end our discussion on this topic with a decision theoretic example (due to Athey and Levin) of information acquisition under this class of preferences.

**Affiliated Values and Revenue Comparisons of Standard Auction Formats**

- Compare the effects of information acquisition in first and second price auctions.
- Two bidders.
- An object to be auctioned has value \( V \) to each bidder. Hence the model is one of pure common values. The paper by Persico presents the analysis for a general affiliated payoff structure.
- \( V \) is unobservable to both bidders, and they choose a statistical experiment \( X^\alpha_i \) on \( V \), where \( \alpha \in [0, \infty) \).
- The outcome from the experiment induces a posterior on \( V \) and we denote the outcome of the experiment by \( x_i \).
- Assume that \( F^{\alpha_i}(x_i) \) is ordered according to Lehmann’s criterion in \( \alpha_i \). Recall that this requires that MLRP holds for \((v, x_i)\) for all \( \alpha_i \).
• Recall the general structure for comparing experiments (I use the convention that distributions (and densities) on \( v \) are denoted with \( G, g \) and corresponding distributions and densities on \( x \) are denoted with \( F, f \)):

\[
\max_b \int_V u(v, b) dG^\alpha(v \mid x).
\]

• The ex ante value of an experiment given by \( \alpha \) is then:

\[
R(\alpha) = \int_V \int_X u(v, b(x)) dF^\alpha(x \mid v) dG(v).
\]

• Assume also that cost of the experiment \( c(\alpha) \) satisfies: \( c(0) = 0 \).

• The ex ante problem of a decision maker is then to choose \( \alpha \) to

\[
\max_\alpha R(\alpha) - c(\alpha).
\]

• We shall assume throughout that the family of distributions is differentiable with respect to accuracy parameter \( \alpha \). We let

\[
MR(\alpha) \equiv \frac{\partial}{\partial \alpha} R(\alpha).
\]

Remark 24 Recall Persico’s theorem from the previous lecture. An experiment \( \alpha \) yields a higher expected value than experiment \( \alpha' \) to all decision makers with single crossing preferences in \( (v, b) \) if and only if \( \alpha \) is more accurate than \( \alpha' \) in the sense of Lehmann. The following proposition extends these ideas to comparisons between decision makers.

Proposition 25 (Persico, 2000) Consider two separate decision makers with payoff functions \( u_1(v, b_1^\alpha(x)) \) and \( u_2(v, b_2^\alpha(x)) \). If

\[
u_1(v, b_1^\alpha(x)) - u_2(v, b_2^\alpha(x))
\]
is single crossing in \((v, x)\), then \(MR_1(\alpha) > MR_2(\alpha)\) for all \(\alpha\).

**Proof.**

\[
MR_1(\alpha) - MR_2(\alpha) = \frac{d}{d\beta} \int_V \int_X u_1(v, b_1^\beta(x)) - u_2(v, b_2^\beta(x)) dF_\beta(x | v) g(v) dv |_{\beta=\alpha}.
\]

Denoting

\[
u^\beta(v, x) := u_1(v, b_1^\beta(x)) - u_2(v, b_2^\beta(x)),
\]

we want to show that

\[
\frac{d}{d\beta} \int_V \int_X u^\beta(v, x) dF_\beta(x | v) g(v) dv |_{\beta=\alpha} \geq 0.
\]

Using the equation

\[
T_{\beta,\alpha,v}(x) \equiv (F_\alpha)^{-1}(F_\beta(x | v) | v),
\]

we can perform a change of variables to get

\[
\frac{d}{d\beta} \int_V \int_X u^\beta(v, x) dF_\beta(x | v) g(v) dv |_{\beta=\alpha} = \frac{d}{d\beta} \int_V \int_X u^\beta(v, T_{\beta,\alpha,v}(x)) dF_\alpha(x | v) g(v) dv |_{\beta=\alpha}.
\]

Changing the order of integration and differentiating inside the integral we have:

\[
\frac{d}{d\beta} \int_V \int_X u^\beta(v, T_{\beta,\alpha,v}(x)) dF_\alpha(x | v) g(v) dv |_{\beta=\alpha} = \int_X \int_V \frac{d}{d\beta} u^\beta(v, T_{\beta,\alpha,v}(x)) dG_\alpha(v | x) f_\beta(x) dx |_{\beta=\alpha}.
\]

We wish to evaluate the inner integral. By envelope theorem,

\[
\int_V \frac{d}{d\beta} u^\beta(v, T_{\beta,\alpha,v}(x)) dG_\alpha(v | x) |_{\beta=\alpha} = \int_V \frac{d}{dx} u^\alpha(v, x) \frac{d}{d\beta} T_{\beta,\alpha,v}(x) dG_\alpha(v | x) |_{\beta=\alpha}.
\]
By the first order conditions,
\[ \int_V \frac{d}{dx} u^\alpha(v, x) dG^\alpha(v \mid x) \bigg|_{\beta=\alpha} = 0. \]

Now \( \frac{d}{dx} u^\alpha(v, x) \) crosses zero once as a function of \( v \) (implied by the single crossing property) and
\[ \frac{d}{d\beta} T_{\beta,\alpha,v}(x) \bigg|_{\beta=\alpha} = \lim_{\beta \downarrow \alpha} \frac{T_{\beta,\alpha,v}(x) - T_{\alpha,\alpha,v}(x)}{\beta - \alpha}. \]

Therefore \( T_{\alpha,\alpha,v}(x) = x \) and therefore independent of \( v \). Therefore \( \frac{d}{d\beta} T_{\beta,\alpha,v}(x) \bigg|_{\beta=\alpha} \) is increasing in \( v \) and we can use the Lemma from second lecture to conclude that
\[ \int_V \frac{d}{dx} u^\alpha(v, x) dG^\alpha(v \mid x) \bigg|_{\beta=\alpha} \geq 0. \]

We are now in a position to proceed with the analysis of auctions.

- Start by analyzing the optimal bidding strategies for a fixed accuracy.
- Consider the payoff to bidder 1 in a second price auction when bidder 2 uses strategy \( b_S(x_2) \)
\[ \max_b \int_V \int_{x_2}^{b_S^{-1}(b)} (v - b_S(x_2)) dF(x_2 \mid v) g(v \mid x_1) dv. \]
- Similarly for first price auctions,
\[ \max_b \int_V \int_{x_2}^{b_F^{-1}(b)} (v - b) dF(x_2 \mid v) g(v \mid x_1) dv, \]

where \( b_F(x_2) \) is the strategy of bidder 2 in the first price auction.

Remark 26 \( \int_{x_2}^{b_S^{-1}(b)} (v - b_S(x_2)) dF(x_2 \mid v) \) and \( \int_{x_2}^{b_F^{-1}(b)} (v - b) dF(x_2 \mid v) \) are single crossing in \( (v, b) \). Therefore Lehmann’s criterion is the appropriate one for comparing experiments.
**Proposition 27** Consider information acquisition in the first and second price auctions as described above. Then the marginal value of information in a first price auction exceeds the marginal value of information in the second price auction for all $\alpha$.

**Proof.** By the above remarks, we are done if we can show that the payoff difference between the two auctions at a fixed accuracy is single crossing in $(x_1, v)$.

\[
\int_{\xi}^{x_1} \left[ b^\alpha_S(y) - b^\alpha_F(x_1) \right] f^\alpha_{X_2}(y \mid v) dy
\]

is single crossing in $(x_1, v)$. To see this, consider the partial derivative w.r.t. $x_1$:

\[
\left[ -b^\alpha_F(x_1) - b^\alpha_F(x_1) \frac{F^\alpha_{X_2}(x_1 \mid v)}{f^\alpha_{X_2}(x_1 \mid v)} + b^\alpha_S(x_1) \right] f^\alpha_{X_2}(x_1 \mid v).
\]

By MLRP, this is monotonic in $v$, and as a result, the required single crossing property holds. ■

To complete the argument for the comparison of equilibria across the two different auctions, denote the marginal cost of accuracy by $MC(\alpha)$ and the marginal revenue from additional accuracy by $MR_m(\alpha)$ for the auction type $m = \{S, F\}$.

**Proposition 28** Suppose a symmetric pure strategy equilibrium exists for a first and second price information acquisition game described in Section 3.1. If $MC(\cdot)$ only intersects $MR_S(\cdot)$ once, then the equilibrium accuracy in the first price auction $\alpha^F$ will be higher than or equal to $\alpha^S$, the accuracy in a second price auction.

For example if $c(\alpha) = \alpha^\gamma$ for a high enough $\gamma$, then the above condition is satisfied and furthermore, equilibrium exists and is unique. Equilibrium accuracy is higher in the first price auction than in the second price auction.
The working paper version of Persico (2000) contains an example that shows how expected revenue ranking with information acquisition may change from the ranking with fixed information. In particular, it is possible that first price action dominates second price auction.

**Discussion**

- Literature on auctions with information acquisition is far from complete.

- Issues: How to get a general characterization of whether information acquisition decisions are substitutes or complements (Borgers et al. (2008) make some progress on this).

- With correlated values, Cremer and McLean type mechanisms leave enough flexibility for the mechanisms so that ex ante and interim incentives can (often) be satisfied simultaneously.

- Hernando-Veciana (2008) is a state of the art paper on comparing different (suboptimal) mechanisms in the correlated common values case.

**Independent Value Auctions and Information Acquisition**

In this lecture, we look at two settings:

1. Independent Values

   - Private Values: Efficiency of the VCG mechanism
   - Interdependent Values: Impossibility of providing simultaneously ex ante incentives to acquire information and interim incentives to disclose information.
2. Correlated Values: Revenue comparisons of the various auction formats revisited.

Relevant papers for this lecture: Bergemann, Shi and Välimäki (2009), Persico (2000) for the results on auctions, Jewitt (2008) for the material on information acquisition.

Standard Model

- Agent $i$ has private information about his payoff type $\theta_i \in \Theta_i$.
- $\theta_i$ should be interpreted as the interim information that agent $i$ has in the mechanism.
- Common Prior $p : \Theta \to [0, 1]$.
- A social outcome $x \in X$ must be chosen.
- Agent $i$’s preference given by Bernoulli utility function
  \[ u_i : \Theta \times X \to \mathbb{R}. \]
- Direct mechanism is given by social choice function $g$
  \[ g : \Theta \to X \]
- Notice that the mechanism asks the players to report their interim beliefs.

Information Acquisition

- Agents acquire costly information:
  \[ \theta_i \sim F^{\alpha_i}(\theta_i). \]
• In other words, $\alpha_i$ parametrizes the distribution of interim beliefs. Hence one can think of $\alpha_i$ as defining a statistical experiment for $i$.

• Informativeness represented by $\alpha_i$, cost of $\alpha_i$ is $c_i(\alpha_i)$.

• Individual maximization problem

$$\max_{\alpha_i} \int_{\Theta_i} \int_{\Theta_{-i}} u_i(g(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) dF^{\alpha_i}(\theta_i) dF^{\alpha_{-i}}(\theta_{-i}) - c(\alpha_i)$$

Interaction between Mechanism and Information

• Observe that optimal $\alpha_i$ depends on $g$:
  
  – Direct effect via allocation given available information
  
  – Indirect effect through the induced distribution of types $F^{\alpha(g)}(\theta)$.

• Incentive compatibility gives incentives for agents to reveal their information.

• In our view, in many allocation problems, it is important to consider the effects of the mechanism on the realized distribution of types as well.
  
  – Which mechanisms perform well in presence these two effects?
  
  – What is the optimal trade-off between these two effects?

Efficient Allocation of a Single Object

• Two bidders

• Interpret $\theta_i$ as posterior probability that object is valuable for bidder $i$.

• Information acquisition: mean preserving spread of $\theta_i$. 
• Quasilinear utility:

\[ u_i(\theta, y) = q_i v_i(\theta) - y_i, \]

\( q_i \): probability of receiving the object, \( y_i \): monetary transfer

• interdependent values:

\[ v_i(\theta) = \theta_i + \gamma \theta_j \quad \text{(private value: } \gamma = 0). \]

Private Values

• Efficiency:

\( q_i = 1 \implies \theta_i \geq \theta_j \) for all \( j \) and \( \theta_i \geq 0 \).

• Second price auction gives efficient allocation in dominant strategies:

\[ U_i(\theta_i) = E_{\theta_0} \max\{\theta_i - \theta_j, 0\}. \]

• Social surplus:

\[ E_{\theta} \max\{\theta_i, \theta_j\} = E_{\theta} \max\{\theta_i - \theta_j, 0\} + \theta_j. \]

– choice by agent \( i \) leads to surplus maximizing choice of \( \alpha_i \).

– general property of Vickrey-Clarke-Groves with private values (Roger-son (1992)).

Interdependent Values

• Interdependent values:

\[ v_i(\theta) = \theta_i + \gamma \theta_j. \]

• Efficiency still requires that

\( q_i = 1 \implies \theta_i \geq \theta_j \) for all \( j \) and \( \theta_i \geq 0 \).
\( x_i(\theta_i) \) is increasing in \( \theta_i \) only if \( \gamma \leq 1 \) : single crossing condition

- Ex post (rather than dominant) incentive compatible transfer rule

\[
y_i(\theta) = \begin{cases} 
(1 + \gamma) \theta_j, & \text{if } \theta_i > \theta_j \\
0, & \text{if otherwise}
\end{cases}
\]

Incentives to Acquire Information

- Net utility of agent \( i \):

\[
U_i(\theta_i) = \mathbb{E}_{\theta_j} \max\{\theta_i - \theta_j, 0\},
\]

- yet social surplus is given by:

\[
\mathbb{E}_\theta \max\{\theta_i + \gamma \theta_j, \theta_j + \gamma \theta_j\} = \mathbb{E}_\theta \max\{\theta_i - \theta_j, \gamma (\theta_i - \theta_j)\} + (1 + \gamma) \theta_j
\]

- social surplus and private surplus of \( i \) (as a function of \( \theta_i \)) have different slopes if \( \theta_i < \theta_j \).

- Bergemann and Välimäki (2002) relate the incentives for individual over- and underacquisition of information to supermodularity of individual payoff functions.

**Theorem 29** *Every efficient mechanism induces insufficient (excessive) incentives for information acquisition by agent \( i \) if \( \sum_{j \neq i} u_j(x, \theta) \) is supermodular (submodular) in \((x, \theta_i)\).*

- Caveat: Does not necessarily translate to equilibria results.

- Bergemann, Shi and Välimäki (2009) show that the same characterization is valid for the equilibria in a class of auction models.
• In that paper, the true value of the object to bidder $i$ is given by

$$u_i(\theta_i, \theta_{-i}) = \theta_i + \gamma \sum_{j \neq i} \theta_j. \quad (1)$$

• The parameter $\gamma$ is a measure of interdependence. With $\gamma > 0$ the interdependence is positive, and with $\gamma < 0$ the model displays negative interdependence. If $\gamma = 0$, then the model is one of private values and if $\gamma = 1$, then the model is of pure common values.

• Initially, each bidder $i$ only knows that the payoff relevant types $\{\theta_j\}_{j=1}^I$ are independently drawn from a common distribution $F$ with support $[\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$. The distribution $F$ has an associated density $f$ and a mean valuation:

$$\mu \triangleq \mathbb{E}[\theta_i].$$

• Each bidder $i$ can acquire information about her payoff relevant type $\theta_i$ at a positive cost $c > 0$. The decision to acquire information is a binary decision. If bidder $i$ acquires information, then she observes the realization of $\theta_i$; otherwise her information is given by the prior distribution $F$.

• In this model, there is excessive acquisition if $\gamma > 0$ and insufficient acquisition relative to social efficiency if $\gamma < 0$. 