Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications

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Introduction
Motivation

Corporate finance studies the design and issuance of securities

Asset pricing investigates the dynamics of their valuations

How do agency costs affect security design and pricing?
The Need for Dynamic Corporate Finance Models

Most corporate finance models are framed in static frameworks

How robust are their predictions in a dynamic setting?

What restrictions do dynamic models put on time-series data?
Literature : Static Models

Townsend 1979

Diamond 1984

Gale and Hellwig 1985

Innes 1990

Bolton and Scharfstein 1990
Literature : Dynamic Models

Gromb 1994

Hart and Moore 1998

Quadrini 2004

Clementi and Hopenhayn 2006

DeMarzo and Sannikov 2006

DeMarzo and Fishman 2007
Outline of the Talk

I. The discrete-time model
   I.1. The optimal contract
   I.2. Implementation

II. The continuous-time limit
   II.1 Characterization
   II.2 Testable implications
I. The Discrete-Time Model
The Investment Project

Time is discrete and indexed by \( nh, n \in \mathbb{N} \), where \( h > 0 \)

A project requires an investment outlay \( I > 0 \) at date 0 and yields a stream of i.i.d. net cash-flows with mean \( \mu_h \) and variance \( \sigma^2_h \)

\[
c_{nh}^h = \begin{cases} 
  c_+^h = \mu_h + \sigma \varepsilon_+ \sqrt{h} > 0 & \text{with probability } p \\
  c_-^h = \mu_h + \sigma \varepsilon_- \sqrt{h} < 0 & \text{with probability } 1 - p
\end{cases}
\]

\(-c_-^h\) is the operating cost, and \( c_{nh}^h - c_-^h \) the gross income

The liquidation value of the project is set equal to 0
Agents

The project is managed by a risk-neutral entrepreneur who has limited initial wealth $A < I$ and is protected by limited liability.

Risk-neutral financiers can help to fund the initial investment cost and the operating costs.

The entrepreneur discounts future payoffs at a higher rate than the financiers, $\rho > r$. 
The Agency Problem

The entrepreneur privately observes the gross income, which she can convert into personal consumption at a rate $\lambda \in (0, 1]$.

For $h$ close to 0, the agency problem is severe as the temptation to divert funds is high relative to the expected cash-flow

$$p\lambda(c^h_+ - c^h_-) > \mu h$$

With multiple periods, additional incentives can be provided by the promise of future payments.
Long-Term Contracts

Given any history of reports \((\hat{c}_0^h, \ldots, \hat{c}_{(n-1)h}^h)\) by the entrepreneur

1. The project is continued with probability \(x_{nh}^h\)

2. The financiers pay the operating cost \(-c_{-}^h\)

3. Given \(c_{nh}^h\), the entrepreneur reports a gross income \(\hat{c}_{nh}^h - c_{-}^h\)

4. The financiers make a transfer \(u_{nh}^h \geq 0\) to the entrepreneur

One can alternatively interpret \(x_{nh}^h\) as a downsizing factor
Literature: Dynamic Contracting

Green 1987

Spear and Srivastava 1987

Thomas and Worrall 1990

Phelan and Townsend 1991
The Dynamic Programming Formulation

The utility of the entrepreneur is a sufficient statistic for the information provided by the history of reports and payments

Given a promised utility $w$ for the entrepreneur, the optimal contract specifies

1. A continuation probability $x \in [0, 1]$

2. Contingent transfers $(u_+, u_-) \in \mathbb{R}_+^2$

3. Contingent continuation utilities $(w_+, w_-) \in \mathbb{R}_+^2$
The Bellman Equation

The financier’s expected payoff $F^h(w)$ is given by

$$\max x \left\{ \mu h - pu_+ - (1 - p)u_- + \frac{pF^h(w_+) + (1 - p)F^h(w_-)}{1 + rh} \right\}$$

subject to consistency and incentive compatibility constraints

$$w = x \left[ pu_+ + (1 - p)u_- + \frac{pw_+ + (1 - p)w_-}{1 + \rho h} \right]$$

$$u_+ + \frac{w_+}{1 + \rho h} \geq u_- + \frac{w_-}{1 + \rho h} + \lambda(c^h_+ - c^h_-)$$
An Alternative Representation

The total surplus $V^h(w) = F^h(w) + w$ satisfies

$$\max_x \left\{ \mu h + \frac{pV^h(w_+) + (1 - p)V^h(w_-)}{1 + rh} - \frac{(\rho - r)h[pw_+ + (1 - p)w_-]}{(1 + rh)(1 + \rho h)} \right\}$$

subject to

$$w \geq x \left[ \frac{w_-}{1 + \rho h} + p\lambda(c^h_+ - c^h_-) \right]$$

$$w \geq x \left[ \frac{pw_+ + (1 - p)w_-}{1 + \rho h} \right]$$
The Optimal Contract

\[ V^h(w) \]

\[ \text{Slope} \ \frac{(\rho - r)h}{1 + \rho h} \]

\[ w^h_{(n+1)h} = \min\{(1 + \rho h)[w^h_{nh} + \lambda(c^h_{nh} - \mu h)], w^h,r\} \]

No transfers

Transfer if success

Transfers

Liquidation

\[ w^h,l \quad w^h,d \quad w^h,r \quad w^h,m \]
Initializing the Contract

\[ w + I - A \]

\[ V^h(w) \]
Implementation with Securities and Cash Reserves

Financiers hold securities, defined as claims with limited liability.

The firm holds cash reserves on a bank account with interest $r$.

Size-adjusted cash reserves are designed so that

$$m^h_{nh} = \frac{w^h_{nh}}{\lambda^h} \quad \text{with} \quad \frac{\lambda^h}{\lambda} = \frac{1 + \rho h}{1 + rh}$$
Stocks and Bonds

The entrepreneur contributes her initial wealth $A$ and is granted a non-tradeable fraction $\lambda$ of the stocks.

The financiers receive the remaining fraction $1 - \lambda$ of the stocks and the totality of the bonds.

The entrepreneur’s contribution and the issuance proceeds are used to pay the investment cost $I$ and to raise

$$m^h_0 = \frac{w^h_0}{\lambda^h}$$
Liquidation and Payout Decisions

If \( m \in [0, w^{h,l}/\lambda^h) \), the firm is scaled down by \( x = m/(w^{h,l}/\lambda^h) \)

If \( m \in [w^{h,l}/\lambda^h, w^{h,r}/\lambda^h] \), stocks distribute a size-adjusted dividend

\[
\max \left\{ \frac{\lambda^h m}{\lambda} - \frac{w^{h,d}}{\lambda}, 0 \right\}
\]

in case of success, and bonds distribute a size-adjusted coupon

\[
\mu_h - \frac{(\rho - r)hm}{1 + rh}
\]
II. The Continuous-Time Limit
Convergence of the Value Functions

For $w \in [w^{h,l}, w^{h,d}]$, a first-order approximation yields

$$V^h(\tilde{w}) \sim V^h(w) + V^h'(w)[\rho w + \lambda(\bar{c} - \mu h)] + \frac{\lambda^2}{2} V^{h''}(w)(\bar{c} - \mu h)^2 h$$

Substituting this into the Bellman equation yields

$$rV^h(w) \sim \mu - (\rho - r)w + \rho wV^h'(w) + \frac{\lambda^2 \sigma^2}{2} V^{h''}(w)$$
Proposition. As $h$ goes to 0, the value function $V^h$ converges uniformly to the unique solution $V$ to the free boundary problem

$$
\begin{cases}
  rV(w) = \mu - (\rho - r)w + \rho w V'(w) + \frac{\lambda^2 \sigma^2}{2} V''(w) & \text{if } w \leq w^m \\
  rV(w) = rV(w^m) & \text{if } w > w^m \\
  V(0) = 0 \\
  V'(w^m) = 0 \\
  V''(w^m) = 0
\end{cases}
$$
The Continuous-Time Value Function

\[ V(w) \]

\[ 0 \leftarrow w^{h,l} \quad w^{h,d}, w^{h,r}, w^{h,m} \rightarrow w^m \]
Convergence of the Optimal Contracts

The total revenue generated by the project up to date $nh$ is

$$r_{nh}^h = \mu nh + \sigma \sum_{i=1}^{n} \varepsilon_i \sqrt{h}$$

As $h$ goes to 0, $r^h$ converges in law to a drifted Brownian motion

$$R_t = \mu t + \sigma Z_t$$

Over $w \in [w^h,l, w^h,r]$, utility evolves according to

$$w_{(n+1)h}^h = \min\{(1 + \rho h)[w_{nh}^h + \lambda \varepsilon_n \sqrt{h}], w^h,r\}$$
Proposition. As $h$ goes to 0, $w^h$ converges in law to the solution $W$ to the reflected diffusion problem

\[
\begin{align*}
    dW_t &= \rho W_t dt + \lambda \sigma dZ_t - dL_t \\
    W_t &\leq w^m \\
    L_t &= \int_0^t 1_{\{W_s = w^m\}} \, dL_s
\end{align*}
\]

for all $t \in [0, \tau]$, where $\tau = \inf\{t \geq 0 \mid W_t = 0\}$.
Proposition. For any \( w \in [0, w^m] \), let \( F(w) = V(w) - w \). Then

\[
w = \mathbb{E}^w \left[ \int_0^T e^{-\rho t} dL_t \right]
\]

\[
F(w) = \mathbb{E}^w \left[ \int_0^T e^{-rt} (\mu dt - dL_t) \right]
\]
The Continuous-Time Implementation

As $h$ goes to 0, the cash reserve process converges to $M = W/\lambda$

$$dM_t = \rho M_t dt + \sigma dZ_t - \frac{1}{\lambda} dL_t = (rM_t + \mu)dt + \sigma dZ_t - dL_t - dP_t$$

for all $t \in [0, \tau]$, where the payment to the financiers satisfies

$$dP_t = [\mu - (\rho - r)M_t]dt + \frac{1 - \lambda}{\lambda} dL_t$$

The stocks distribute a dividend $dL/\lambda$ when $M$ hits $w^m/\lambda$

The bonds distribute a coupon $\mu - (\rho - r)M$ to the financiers
Stock Prices

The market value of stocks is

\[ S_t = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \frac{1}{\lambda} \, dL_s \right] \]

By Itô’s formula, \( S_t = S(M_t) \) where

\[
\begin{cases}
    rS(m) = \rho m S'(m) + \frac{\sigma^2}{2} S''(m) & \text{if } m \leq \frac{w^m}{\lambda} \\
    S(0) = 0 \\
    S'(\frac{w^m}{\lambda}) = 1
\end{cases}
\]
Bond Prices

The market value of bonds is

\[ D_t = \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} [\mu - (\rho - r)M_s] \, ds \right] \]

By Itô's formula, \( D_t = \mathcal{D}(M_t) \) where

\[
\begin{cases}
    r \mathcal{D}(m) = \mu - (\rho - r)m + \rho m \mathcal{D}'(m) + \frac{\sigma^2}{2} \mathcal{D}''(m) & \text{if } m \leq \frac{w^m}{\lambda} \\

    \mathcal{D}(0) = 0 \\

    \mathcal{D}'\left(\frac{w^m}{\lambda}\right) = 0
\end{cases}
\]
The Firm’s Balance Sheet

At any date $t$, $F(W_t) + M_t = (1 - \lambda)S_t + D_t$

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0$</td>
<td>$D_0$</td>
</tr>
<tr>
<td>$I$</td>
<td>$S_0$</td>
</tr>
<tr>
<td>$G_0$</td>
<td></td>
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</tbody>
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$G_0 = \lambda S_0 + F(W_0) - (I - A) - A$ is the “goodwill” of the firm
Testable Implications : Stock Prices

The stock price is increasing and concave in $M_t$

$$dS_t = rS_t dt + S_t \sigma^S(S_t) dZ_t - \frac{1}{\lambda} dL_t \quad \text{with} \quad \sigma^S(s) = \frac{\sigma S'(S^{-1}(s))}{s}$$

Since $s \sigma^S(s) \geq \sigma$, stock prices are more volatile than cash-flows

Since $\lim_{s \downarrow 0} s \sigma^S(s) > 0$, the stock price can attain 0

Since $\sigma^S$ is decreasing, stock return volatility rises in response to bad news (Black 1976, Nelson 1991)
Testable Implications: Leverage

The financial leverage in market value is \( D_t/S_t = D(M_t)/S(M_t) \)

The leverage is decreasing w.r.t. \( M_t \) and \( S_t \)

Performance shocks and stock price movements create persistent changes in the capital structure (Welch 2004)

There is a “leverage effect”: changes in leverage are tied to changes in stock return volatility (Black 1976, Christie 1982)
Testable Implications : Balance Sheet

The book-to-market ratio of equity is \((S_0 + M_t - M_0)/S_t\)

\[
\begin{array}{cc}
\text{Assets} & \text{Liabilities} \\
\hline
M_t & D_0 \\
I & S_0 + M_t - M_0 \\
G_0 & \\
\end{array}
\]

The book-to-market ratio is U-shaped w.r.t. \(M_t\) and \(S_t\)
**Testable Implications: Default Risk**

Let $\Delta_t$ be the credit yield spread on a consol bond

$$
\int_t^\infty e^{-(r+\Delta_t)(s-t)} \, ds = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \, ds \right]
$$

or equivalently

$$
\Delta_t = \frac{r \mathbb{E}_t[e^{-r(\tau-t)}]}{1 - \mathbb{E}_t[e^{-r(\tau-t)}]}
$$

The credit yield spread is decreasing and convex w.r.t. $M_t$.
Comparative Statics

The investment capacity of the firm and the initial level of cash reserves are decreasing w.r.t. $\lambda$ and $\sigma$.

For any given level of cash reserves, the credit yield spread is increasing w.r.t. $\lambda$.

For any given level of cash reserves, the stock price is decreasing w.r.t. $\lambda$. 
Concluding Remarks
Summary

A simple agency model delivers appealing predictions for

1. Payout and liquidation decisions
2. Security prices
3. Financial leverage
4. Balance sheet dynamics
5. Default risk
Extensions

The model can or should be extended to take into account

1. Renegotiation

2. Investment

3. Risk-aversion

4. General equilibrium effects