Optimal Auction Design: Uncertainty, Robustness, and Revenue Maximization

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Abstract

Recent literature on robust mechanism design focuses on detail-free mechanisms to address issues on the restrictive common knowledge assumption. However, most of the existing studies focus on the implementability of an allocation mechanism in a more robust setting, whereas very little is known regarding how these formulations differ from the revenue perspective in the context of optimal mechanism design.

In this paper, we revisit the optimal auction design problem and propose a robust formulation based on an uncertainty set that characterizes the conservativeness of the bidders’ beliefs, with two special cases being the Bayesian and Ex post formulations. Using the network approach, we identify the necessary and sufficient conditions under which the expected revenues achieved by different formulations are identical. Furthermore, we show that there is no discrepancy between any pair of formulations in the single-object auction even if the bidders’ types are discrete. Nevertheless, in a multiple-object auction, the auctioneer’s expected revenue may strictly decrease as the bidders’ beliefs become more uncertain.

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1 Introduction

In the area of mechanism design, an emerging trend of research attempts to address issues of the very restrictive common knowledge assumption. For example, Bergemann and Morris [2005] argue that the notion of robustness should be examined in the richer universal type space.\textsuperscript{1} Thus, they proceed to study the equivalence/difference between Bayesian and Ex post implementations and find that the equivalence holds in some settings that are termed “separable,” but the discrepancy may arise beyond these environments; see also Bergemann and Morris [2008, 2009] along the same vein. Chung and Ely [2007] argue that a mechanism should be robust against the principal’s (auctioneer’s) own belief about the agents’ (bidders’) types. They show that, for every detail-free mechanism, there must exist a belief for which it is outperformed by the optimal dominant strategy mechanism; thus, this provides a normative justification for dominant strategy mechanisms. Bose et al. [2006] study the optimal auction design when the auctioneer exhibits ambiguity aversion. Most recently, Lopomo et al. [2007] introduce Knightian uncertainty to the players’ beliefs, and identify the conditions on the uncertainty set under which the Ex post formulation is equivalent to the robust formulation in the single-dimensional mechanism design problems.

Despite the insightful elaborations in the aforementioned literature on the robust implementability of auction mechanisms, surprisingly little attention has been paid to the revenue loss associated with a (more) robust formulation. In this paper, we adopt the Knightian uncertainty model introduced by Lopomo et al. [2007] and revisit the optimal auction design problem. We first provide a unified framework that allows us to formally define the bidders’ belief uncertainty. This leads to a continuum of formulations/games, each of which corresponds to different levels of belief uncertainties. In this general framework, the bidders’ valuations are allowed to be interdependent, and the auctioneer may sell more than one object, thereby giving rise to the multi-dimensional mechanism design problem.

We apply the network approach developed by Malakhov and Vohra [2005] to reformulate our optimal auction design problems, and use this to characterize the necessary and sufficient conditions (for a fixed allocation) to achieve the same revenue in a robust formulation as in the Bayesian formulation. This allows us to spot the possible distinction between different formulations and investigate the impact of bidders’ belief uncertainty on the auctioneer. We apply our framework to

\textsuperscript{1}This notion was first formalized by Mertens and Zamir [1985]. Specifically, a universal type space is “large enough” to capture players’ beliefs of others’ payoff functions, along with the knowledge of their payoff functions.
study the two classical problems, namely the single-object auction introduced by Myerson [1981] and the multi-object auction introduced by Armstrong [2000]. Our results show that there is no discrepancy between any pair of formulations in the single-object auction even if the bidders’ types are discrete. This is admittedly intuitive, but it serves as a benchmark that validates our result. Furthermore, we show that in a multiple-object auction, the auctioneer’s expected revenue may strictly decrease as the bidders’ beliefs become more uncertain. Following this, we provide a concrete example for which the ex post formulation gives rise to a strictly lower expected revenue in the multi-dimensional setting.

Our paper belongs to a long-standing literature on auction theory, including the seminal paper by Vickrey [1961], the paper on interdependent values by Milgrom and Weber [1982], the mechanism design approaches of Myerson [1981] and Maskin and Riley [1989], the survey by Klemperer [1999], and the recent book by Krishna [2002]. Recent advances are on the design of multi-object auctions with multi-dimensional valuations (Armstrong [2000]), on the competing auctions (Moldovanu et al. [2008]), and on the characterization of bidding behavior when there are interactions among bidders (Figueroa and Skreta [2009] and Jehiel et al. [1999]). Unlike all the aforementioned papers, we introduce the robust formulation in which the bidders may exhibit belief uncertainty. Our paper is also related to a broader class of papers on mechanism design. Stemming from Mirrlees [1971], this framework with information asymmetry has been applied extensively in a variety of areas, including product line design (Moorthy [1984] and Mussa and Rosen [1978]), taxation policies (Salanie [2003]), managerial compensation schemes (Holmstrom and Milgrom [1991]), and government regulation (Laffont and Tirole [1993]). Please see Laffont and Martimort [2002] for detailed discussions. Our approach to model the belief uncertainty may find its applications in these domains as well.

As aforementioned, we attempt to relax the common knowledge assumption and evaluate the robustness of mechanism design. Thus, our paper also belongs to a rising stream on robust or “detail-free” mechanism design, including Bergemann and Morris [2005, 2008, 2009], Bose et al. [2006], Chung and Ely [2007], and Lopomo et al. [2007]. Unlike Bergemann and Morris [2005], we focus on the conventional “first-order” belief space. While Bose et al. [2006] and Chung and Ely [2007] allow the auctioneer to exhibit belief uncertainty, we assume that the belief uncertainty arises solely from the bidders’ side. A closely related recent paper by Heydenreich et al. [2009] applies the graph theory to mechanism design problems with a continuous type space and a discrete outcome space; their main objective is to identify the conditions under which all Ex post incentive compatible payment schemes can only vary by a constant. Our results complement theirs as we focus on the
characterization of conditions for the existence of Ex post incentive compatible payment scheme that is revenue equivalent to the Bayesian formulation; it can be verified that their conditions imply ours, but not vice versa. Moreover, in the terminology of network approach, their condition is equivalent to that all paths being equal between any pair of nodes, whereas we only require there be overlapping between the sets of longest paths; once again, the former condition implies the latter. Finally, as a relatively minor difference, we focus on a discrete type space and unrestricted outcome space rather than the environment studied by Heydenreich et al. [2009].

The rest of the paper is organized as follows. In Section 2, we describe the basic setting. In Section 3, we formulate the robust mechanism design problems, of which the Bayesian and the Ex post formulations can be taken as special cases. In Section 4, we introduce the network approach and the necessary and sufficient conditions for a fixed allocation mechanism to achieve the same revenue in a robust formulation as in the nominal Bayesian formulation. We then consider robust formulations of Bayesian optimal auction mechanisms in Section 5, in both the single-object and multiple-object settings. Finally, we conclude in Section 6.

2 The model

Let us first introduce the general model in which an auctioneer faces a number of bidders that possess privately known valuations. All players, including the auctioneer and the bidders, are risk neutral. Let $F$ be the set of feasible allocations of the resources amongst the bidders and the auctioneer, and $T = \{t_1, t_2, \ldots, t_m\}$ is a finite set of a bidder’s types (possibly multi-dimensional). That is, each bidder privately observes the realization of this signal $t_i \in T$. A collection of types one for each (of $n$ bidders) will be called a “profile” $t$, and a profile involving only $n-1$ bidders will be denoted $t^{n-1}$. Let $T^n$ denote the set of all possible profiles.

Given an allocation $a \in F$, if a bidder has type $t_i$ while other bidders have type profile $t^{n-1}$, she assigns monetary valuation $v(a|t_i, t^{n-1})$ to the allocation $a$. In other words, $v(a|t_i, t^{n-1})$ is the gross “utility” a bidder receives from the allocation. Here, we do not limit our formulation to the private value model, i.e., the valuation of each bidder, $v(a|t_i, t^{n-1})$, can depend on the realized types of the bidder herself and that of the other bidders. For the purpose of this paper, we assume that bidders are ex ante symmetric. In other words, we assume that bidders’ types are independent draws from a common distribution that is commonly known. Specifically, we let $f_i > 0$ denote the probability that a bidder has type $t_i$. The probability of a profile $t^{n-1} \in T^{n-1}$ being realized is
\[ \pi(t^{n-1}) > 0. \]

By the revelation principle, we can restrict our attention to the direct revelation mechanisms (this holds regardless of the common prior assumption). In such a mechanism, each bidder is asked to announce her own type. The auctioneer, as a function of the announcements, decides what element (allocation) of \( F \) to pick and what payments each bidder has to make. As aforementioned, our primary goal is to provide a unified framework that incorporates various types of belief systems depending on how confident the bidders are regarding their estimation of the game. To this end, in the following we first review the classical Bayesian and Ex post formulations of the optimal auction design problem. We then introduce the general framework, of which the two formulations can be regarded as two polar cases.

3 Formulations and uncertainty set

In this section, we provide the formulations for the classical solution concepts and introduce our solution concepts.

3.1 Bayesian formulation

We now formulate the auctioneer’s optimization problem. To this end, we need to introduce the solution concept for the games played by the bidders. In the standard literature on mechanism design, the solution concept is *Bayesian Nash equilibrium*. Despite its popularity, this solution concept requires a relatively strong assumption on the common prior beliefs. Specifically, it requires that each bidder possesses the correct belief about other bidders’ types, each bidder knows that each bidder possesses the correct belief about other bidders’ types, and so on. Putting it in this particular problem, it translates to the following Bayesian incentive compatibility (BIC) constraint:

\[
\sum_{t^{-1}} v(a_i[t_i, t^{n-1}]) \pi(t^{n-1}) - \sum_{t^{-1}} P(t_i, t^{n-1}) \pi(t^{n-1}) \\
\geq \sum_{t^{-1}} v(a_j[t_j, t^{n-1}]) \pi(t^{n-1}) - \sum_{t^{-1}} P(t_j, t^{n-1}) \pi(t^{n-1}), \forall t_i, t_j \in T,
\]

If the bidders’ valuations are correlated, it is possible for the auctioneer to fully extract the surplus by utilizing complicated lotteries that elaborate on all the bidders’ reports/bids, see Cremer and McLean [1985, 1988].
where, on the right-hand side,
\[
\sum_{t_i \in T} v(a_j[t_j,t_i^{n-1}][t_i,t_i^{n-1}]) \pi(t_i^{n-1})
\]
is the expected utility the type-\(t_i\) bidder receives if she pretends to be type-\(t_j\), and \(P(t_j,t_i^{n-1})\) is the payment associated with this misreporting if other bidders’ report profile is \(t_i^{n-1}\). The left-hand side
is a special case in which the bidder reports truthfully. This inequality guarantees that the bidder
is willing to disclose her type. Moreover, we have to include the Bayesian individual rationality
(BIR) condition to ensure that each bidder receives at least a null payoff:
\[
\sum_{t_i \in T} v(a_i[t_i,t_i^{n-1}][t_i,t_i^{n-1}]) \pi(t_i^{n-1}) - \sum_{t_i \in T} P(t_i,t_i^{n-1}) \pi(t_i^{n-1}) \geq 0, \forall t_i \in T.
\]

For convenience of our analysis, we define \(R(t_i,t_i^{n-1})\) to be the “information rent” to a bidder
with type \(t_i\) while other bidders have type \(t_i^{n-1}\):
\[
R(t_i,t_i^{n-1}) \equiv v(a_i[t_i,t_i^{n-1}][t_i,t_i^{n-1}]) - P(t_i,t_i^{n-1}), \forall t_i \in T.
\]

Using this new notation, we can reformulate the (BIC) and (BIR) constraints as follows:
\[
\sum_{t_i \in T} R(t_i,t_i^{n-1}) \pi(t_i^{n-1}) - \sum_{t_i \in T} R(t_j,t_i^{n-1}) \pi(t_i^{n-1}) \geq 0, \forall t_i \in T, \forall t_i \in T
\]
\[
\sum_{t_i \in T} v(a_j[t_j,t_i^{n-1}][t_i,t_i^{n-1}]) \pi(t_i^{n-1}) - \sum_{t_i \in T} v(a_j[t_j,t_i^{n-1}][t_i,t_i^{n-1}]) \pi(t_i^{n-1}), \forall t_i, t_j \in T,
\]
\[
\sum_{t_i \in T} R(t_i,t_i^{n-1}) \pi(t_i^{n-1}) \geq 0, \forall t_i \in T.
\]

The auctioneer’s problem is to maximize his expected revenue subject to the constraints (BIC)
and (BIR). Therefore, assuming that the allocation \(a \in F\) has been fixed, the auctioneer’s optimization problem under the Bayesian formulation is the following:
\[
S^b(a) = \max_{P(t_i,t_i^{n-1})} \sum_{t_i \in T} f_i \sum_{t_i \in T} \{ v(a_i[t_i,t_i^{n-1}][t_i,t_i^{n-1}]) - R(t_i,t_i^{n-1}) \} \pi(t_i^{n-1})
\]
s.t. (BIC) and (BIR),

where the objective function is simply the expected revenue the auctioneer gets from the bidders.
Note that from this formulation, \(f_i \pi(t_i^{n-1})\) is the probability that a specific bidder’s type is \(t_i\), and
other bidders’ type profile is \(\pi(t_i^{n-1})\). Furthermore, we can write down the auctioneer’s expected
payoff from a specific bidder’s viewpoint precisely because all these bidders are ex ante symmetric.\(^3\) It is worth mentioning that if the allocation is fixed, the problem degenerates to a simple linear program.

Note that in this Bayesian formulation, we have assume that the bidder knows perfectly well the distributions of realized type profile \(t^{n-1}\) of other bidders. This is exactly where the common prior (common knowledge) assumption is used in this particular context. We shall call this case the “nominal model.”

As a remark, with this representation, we force the auctioneer to specify the payment scheme (or equivalently information rent) for every realization of the report profile. Thus, there are much more decision variables (compared to that in Malakhov and Vohra [2005]). However, in our formulation, \(\sum_{t^{n-1} \in T^{n-1}} R(t_i, t^{n-1})\pi(t^{n-1})\) can be conveniently redefined as \(R_i\), which is what really matters to the bidders as well as the auctioneer under the Bayesian formulation. This handy change of variables is adopted by Malakhov and Vohra [2005] and most of the papers using the mechanism design approach, and it gives rise to the classical “reduced form” of the auctioneer’s optimization problem.

### 3.2 Ex post formulation

The second extreme case is when the bidders completely have no idea of the realizations of other bidders’ types. In such a scenario, their equilibrium bidding strategies have to be the best responses, regardless of what other bidders’ strategies are. Equivalently, we can write down the following Ex post incentive compatibility (EPIC) constraint to represent this solution concept:\(^4\)

\[
R(t_i, t^{n-1}) - R(t_j, t^{n-1}) \geq v(a_j[t_j, t^{n-1}]|t_i, t^{n-1}) - v(a_j[t_j, t^{n-1}]|t_j, t^{n-1}), \quad \forall t_i, t_j \in T, \ t^{n-1} \in T^{n-1}.
\]

\(^3\)It is also possible to extend our results to the case with asymmetric bidders (regarding both the belief system and the valuation distribution), as is done in Krishna [2002]. However, in this paper we focus on the symmetric case for ease of exposition. It is conceivable that the notation is a lot more cumbersome as we allow asymmetry since we need to keep track of every bidder.

\(^4\)Ex post incentive compatibility is sometimes mentioned under other names, such as the “uniform equilibrium” in d’Aspremont and Gerard-Varet [1979] and the “uniform incentive compatibility” in Holmstrom and Myerson [1983]. As a remark, in the private value model, the ex post incentive compatibility is equivalent to that of the well-known “dominant strategy equilibrium,” but they make a difference if the bidders’ values are interdependent.
Furthermore, to ensure that each bidder receives at least a null payoff, we need to impose the Ex post individual rationality condition (EPIR):

\[ R(t_i, t^{n-1}) \geq 0, \forall t_i \in T, t^{n-1} \in T^{n-1}. \]  

(EPIR)

Note that both (EPIC) and (EPIR) are much stronger constraints than (BIC) and (BIR) in the Bayesian formulation because they require the inequalities to hold for every instance rather than in expectation. Clearly, (EPIC) implies (BIC) and (EPIR) implies (BIR) as we aggregate these Ex post constraints weighted by the probabilities in the nominal model.

Given these incentive constraints, the auctioneer’s optimization problem in this case (Ex post formulation) is:

\[
S^e(a) = \max_{R(t_i, t^{n-1})} \sum_{t_i \in T} \sum_{t^{n-1} \in T^{n-1}} f_i \left[ v(a_i | t_i, t^{n-1}) | (t_i, t^{n-1}) - R(t_i, t^{n-1}) \right] \pi(t^{n-1})
\]

s.t. (EPIC) and (EPIR).

Apparently, this optimization should yield a weakly lower expected profit \( S^e(a) \) for the auctioneer since the constraints are tighter than those in the Bayesian formulation.

The Ex post incentive compatibility is a commonly accepted response by economists to Wilson doctrine (see, e.g., Bergemann and Morris [2005], Bikhchandani et al. [2006], and Chung and Ely [2007]). The primary reason may be that the mechanisms derived from this ex post formulation is “belief-free” as the bidders’ (agents’) beliefs regarding others and the game do not factor into the formulation. However, under this solution concept, an implicit assumption is that each bidder (agent) has \textit{completely no} information of other bidders’ types (or equivalently, other bidders’ strategies based on their realized type profile).

If instead, a bidder more or less has an estimation regarding roughly the possible type realizations of others and how others perceive the game, she would not have completely abandoned her one estimation while determining her best response. In such a scenario, it seems appropriate to explicitly model the confidence and uncertainty of a bidder and formally incorporate this into the formulation of optimal auction design. Incidentally, in Wilson [1987], the statement is that the players may not completely know the correct prior distributions of other players’ types. It is certainly legitimate to push this argument to the extreme and derive the Ex post formulation (as is typically done in the economics literature for the past two decades). Nevertheless, it might be also useful to construct a flexible formulation that allows for the intermediate cases in which the bidders do not
possess the completely correct beliefs but yet still retain some reasonable expectation/estimations. This is precisely our primary objective in this paper, as we describe next.

### 3.3 Uncertainty set and robust formulation

In the sequel, we propose a continuum of solution concepts that lie in between the two extreme cases — Bayesian and Ex post formulations; these solution concepts allow us to model various situations that account for the bidders’ confidence or uncertainty about the beliefs. From the aforementioned two formulations, we find that only the incentive constraints differ in different solution concepts ((BIC), (BIR), (EPIC), and (EPIR)), and the probability distributions $q(t^{n-1})$, $\pi(t^{n-1})$ are actually the coefficients of linear programs. This motivates us to propose the following formulation for bidders’ belief systems. Specifically, we shall take $\pi(t^{n-1})$ as the nominal model and define the following “uncertainty set”:

$$U^\varepsilon = \{ q \in [0, 1]^{n-1} : |q(t^{n-1}) - \pi(t^{n-1})| \leq \varepsilon, \sum_{t^{n-1} \in T^{n-1}} q(t^{n-1}) = 1 \}.$$

Note that we require $\sum_{t^{n-1} \in T^{n-1}} q(t^{n-1}) = 1$ since a bidder’s belief regarding other bidders’ types have to be consistent even if it does not coincide with the correct prior. This uncertainty set nicely provides a ground for us to represent the confidence, conservatism, and uncertainty the bidders are endowed with. The larger the value of $\varepsilon$, the more conservative the bidders are, (or equivalently, the less confident they are regarding their beliefs on other bidders’ types). An interpretation for the existence of uncertainty set is that bidders are endowed with incomplete preferences (see Lopomo et al. [2007]).

Based on the uncertainty set $U^\varepsilon$, we define the robust incentive compatibility (RIC) constraints as:

$$\sum_{t^{n-1} \in T^{n-1}} R(t_i, t^{n-1}) q(t^{n-1}) - \sum_{t^{n-1} \in T^{n-1}} R(t_j, t^{n-1}) q(t^{n-1}) \geq \sum_{t^{n-1} \in T^{n-1}} [v(a_j[t_j, t^{n-1}])|t_i, t^{n-1}| - v(a_j[t_j, t^{n-1}])|t_j, t^{n-1}|] q(t^{n-1}), \forall t_i, t_j \in T, q \in U^\varepsilon \tag{RIC}$$

Similarly, the robust individual rationality (RIR) constraints are defined as:

$$\sum_{t^{n-1} \in T^{n-1}} R(t_i, t^{n-1}) q(t^{n-1}) \geq 0, \forall t_i \in T, q \in U^\varepsilon \tag{RIR}.$$
The auctioneer’s robust mechanism design problem is defined as follows:

\[
S^\varepsilon(a) = \max_{R(t_i, t_{n-1})} \sum_{t_i \in T} f_i \sum_{t_{n-1} \in T_{n-1}} [v(a_i[t_i, t_{n-1}]|t_i) - R(t_i, t_{n-1})]\pi(t_{n-1})
\]

s.t. (RIC) and (RIR).

If \( \varepsilon = 0 \), the uncertainty set is a singleton that contains only the correct prior: \( U^0(\pi) = \{ \pi \} \). This corresponds to our nominal model and the corresponding solution concept is Bayesian Nash equilibrium, i.e.,

\[
S^0(a) = S^b(a), \forall a \in F.
\]

On the other hand, if \( \varepsilon = 1 \), any prior is contained in the uncertainty set: \( U^1(\pi) = \{ q(t_{n-1}) \in [0, 1]^{n-1} : \sum_{t_{n-1} \in T_{n-1}} q(t_{n-1}) = 1 \} \) because \(|q(t_{n-1}) - \pi(t_{n-1})| \leq 1\) is redundant. The auctioneer’s robust optimization problem becomes equivalent to the Ex post formulation:

\[
S^1(a) = S^e(a), \forall a \in F.
\]

Therefore, the existing two solution concepts are actually two extreme cases of this general class of solution concepts. In general, we should be able to identify a continuum of optimization problems, each of which corresponds to a different solution concept that captures the conservativeness of bidders’ beliefs.

From the above formulations, the objective functions are identical, but the incentive constraints ((RIC) and (RIR)) are more restrictive as we increase the value of \( \varepsilon \). This implies that the feasible region of this class of optimization problems becomes larger when we increase the measure of belief uncertainty, \( \varepsilon \). This observation immediately leads to the following result:

**Lemma 1.** In the auction game among bidders with belief uncertainty, 1) The set of equilibria is nested for all \( \varepsilon \), and is enlarging as \( \varepsilon \) becomes larger; 2) The auctioneer’s expected revenue is decreasing in \( \varepsilon \).

Lemma 1 shows that the auctioneer is averse to the uncertainty the bidders possess regarding their beliefs, and it has a clear economic intuition. When bidders are more uncertain regarding which types of bidders they are bidding with (as \( \varepsilon \) becomes larger), their strategies are more conservative. Thus, the set of mechanisms that the auctioneer can select amongst is smaller; consequently, he collects a (weakly) lower expected revenue.

It is worth mentioning that the idea of using robust optimization to formulate a solution concept was first proposed in Aghassi and Bertsimas [2006]. However, in their paper, the uncertainty
is on the players’ payoff profiles, in games with perfect or private information. In our setting, this

corresponds to the case in which the form of \( v(\cdot|t_i, t_n) \) is unknown, but the beliefs are correct.

Since the players in Aghassi and Bertsimas [2006] are uncertain about their own payoffs (even

though they have observed their types), the players’ goal is to find a strategy that provides the

optimal worst case performance guarantee; on the other hand, the uncertainty in our formulation

generates a larger set of constraints that the auctioneer has to encounter while designing the optimal

auction. Another key difference of our paper and Aghassi and Bertsimas [2006] is that Aghassi

and Bertsimas [2006] mainly focus on proving the existence of “robust optimization equilibrium”

in finite games, while we focus on the revenue difference under different solution concepts.

Having described the general robust formulation of the auction design problem, we next pro-

ceed to characterize the difference between the intermediate case and the two extreme cases, namely

the Bayesian and Ex post formulations. Following this, we then provide some examples to illustrate

the discrepancies explicitly.

4 Robust formulations of fixed allocations

In this section, we first introduce the network approach developed by Malakhov and Vohra [2005]

to reformulate our optimal auction design problems. We then apply it to characterize the necessary

and sufficient conditions (for a fixed allocation) to achieve the same revenue in a robust formulation

as in the nominal Bayesian formulation.

4.1 The network approach

First, we introduce the network approach to solve the robust auction design problem. Our analysis

closely follows the elegant framework by Malakhov and Vohra [2005], except that we work with the

information rent \( R \), instead of the payment \( P \). Our goal is to recast the optimal auction design as

a network design problem and utilize the established techniques in the network flow literature, as

we elaborate in the sequel.

For fixed allocation \( a \in F \), let us define a complete directed graph \( G(a, q) = (N, A, w(a, q)) \),

for each \( q \in U^\varepsilon \). Here, \( N = \{n_0, n_1, \cdots, n_{|T|}\} \) is the set of nodes, with \( n_1, \cdots, n_{|T|} \) each

corresponding to a type in \( T \), and \( n_0 \) being the “pseudo” node. The set \( A = \{(i, j) : 0 \leq i, j \leq |T|\} \)

contains the edges of the graph, each corresponding to an incentive compatible constraint. To
transform our optimal auction design problem to a network design problem, we can denote the weights on the edges as

$$w_{ji}(a, q) = \sum_{t^{n-1} \in T^{n-1}} [v(a_j[t_j, t^{n-1}][t_i]) - v(a_j[t_j, t^{n-1}][t_j])]q(t^{n-1}), \forall 1 \leq i, j \leq |T|,$$

and

$$w_{0i}(a, q) = w_{i0}(a, q) = 0, \forall 1 \leq i \leq |T|.$$

Let $d_i(a, q)$ be the length of the longest path from $n_0$ to $n_i$ in $G(a, q)$, if one exists. We say that an allocation $a \in F$ is robust incentive compatible with respect to uncertainty set $U^\varepsilon$, if we can find a pricing/rent mechanism that satisfies (RIC). Lemma 2 reveals the connection between the robust auction design problem and the longest path problem in $G$.

**Lemma 2.** If $a \in F$ is robust incentive compatible with respect to $U^\varepsilon$, the following results hold:

1. There is no positive cost cycle and a longest path exists between each pair of nodes in $G(a, q)$, for all $q \in U^\varepsilon$;

2. For all $t_i \in T$ and $q \in U^\varepsilon$, any feasible pricing/rent scheme $R$ satisfies

$$\sum_{t^{n-1} \in T^{n-1}} R(t_i, t^{n-1})q(t^{n-1}) \geq d_i(a, q).$$

**Proof.** Let us first prove the first claim. For an arbitrary cycle $C = \{(i_1, i_2), \cdots, (i_{k-1}, i_k), (i_k, i_1)\}$ in $G(a, q)$, it follows that

$$\sum_{(i, j) \in C} w_{ij}(a, q) = \sum_{t^{n-1} \in T^{n-1}} [v(a_{i_1}[t_{i_1}, t^{n-1}][t_{i_2}]) - v(a_{i_1}[t_{i_1}, t^{n-1}][t_{i_1}])]q(t^{n-1}) + \cdots$$

$$+ \sum_{t^{n-1} \in T^{n-1}} [v(a_{i_{k-1}}[t_{i_{k-1}}, t^{n-1}][t_{i_k}]) - v(a_{i_{k-1}}[t_{i_{k-1}}, t^{n-1}][t_{i_{k-1}}])]q(t^{n-1})$$

$$+ \sum_{t^{n-1} \in T^{n-1}} [v(a_{i_k}[t_{i_k}, t^{n-1}][t_{i_1}]) - v(a_{i_k}[t_{i_k}, t^{n-1}][t_{i_k}])]q(t^{n-1})$$

$$\leq [R(t_{i_2}, t^{n-1}) - R(t_{i_1}, t^{n-1})] + \cdots + [R(t_{i_k}, t^{n-1}) - R(t_{i_{k-1}}, t^{n-1})]$$

$$+ [R(t_{i_1}, t^{n-1}) - R(t_{i_k}, t^{n-1})]$$

$$= 0,$$

where the first inequality is implied by (RIC). This suggests that there should be no positive cost cycle in $G(a, t^{n-1})$ and asserts the first claim.
Let us now switch to the second claim. The proof is by induction. Specifically, from the first claim we know that there exist a longest path between each pair of nodes in \( G(a, q) \). For each \( t_i \in T \), let \( P_i(a, q) \) be a longest path from \( n_0 \) to \( n_i \) in \( G(a, q) \), and let

\[
A(a, q) \equiv \bigcup_{t_i \in T} P_i(a, q).
\]

From the first claim, \( A(a, q) \) is acyclic, thus there exists a topological ordering of the nodes \( 0, j_1, \cdots, j_{|T|} \), such that for all edges \( (j_u, j_v) \in A(a, q) \), \( u < v \).

Since \( n_0 \rightarrow n_{j_1} \) is the only path to \( n_{j_1} \) in \( A(a, q) \), it follows that

\[
\sum_{t_{n-1} \in T^{n-1}} R(t_{j_1}, t^{n-1})q(t^{n-1}) \geq 0 = w_{0j_1}(a, q) = d_{j_1}(a, q),
\]

where the first inequality is implied by (RIR). For \( u > 1 \), let \( j_v \) \((v < u)\) be the predecessor of \( j_u \) in \( A(a, q) \). It follows that

\[
\sum_{t_{n-1} \in T^{n-1}} R(t_{j_u}, t^{n-1})q(t^{n-1}) \\
\geq \sum_{t_{n-1} \in T^{n-1}} R(t_{j_v}, t^{n-1})q(t^{n-1}) + \sum_{t_{n-1} \in T^{n-1}} [v(a_{j_v}[t_{j_v}, t^{n-1}])t_{j_u}) - v(a_{j_v}[t_{j_v}, t^{n-1}])t_{j_v})]q(t^{n-1}) \\
\geq d_{j_v}(a, q) + w_{j_vj_u}(a, q) \\
= d_{j_u}(a, q),
\]

where the first inequality follows from (RIC), and the second one is implied by the inductive assumption. □

With the help of Lemma 2, we are now ready to state our main results.

4.2 Main results

Let \( a \in F \) be a robust incentive compatible allocation with respect to \( U^\varepsilon \). We say that the robust formulation achieves the same expected revenue as the Bayesian formulation under allocation \( a \), if there exists a robust incentive compatible payment/rent scheme \( R \) that generates the Bayesian optimal revenue, i.e. \( S^\varepsilon(a) = S^b(a) \). Note that this condition is weaker than the characterization of revenue equivalence in Heydenreich et al. [2009], which requires all Ex post incentive compatible payment schemes to generate the Bayesian optimal revenue.
For each \( q \in U^{\varepsilon} \), we define \( \mathcal{L}_i(a, q) \) to be the set of longest paths from the pseudo node \( n_0 \) to \( n_i \) in graph \( G(a, q) \). Theorem 1 characterizes the necessary and sufficient conditions for a robust formulation to achieve the same revenue as a Bayesian formulation (the nominal case).

**Theorem 1.** For a fixed allocation \( a \in F \) that is robust incentive compatible with respect to \( U^{\varepsilon} \), \( S^{\varepsilon}(a) = S^b(a) \) if and only if

\[
\bigcap_{q \in U^{\varepsilon}} \mathcal{L}_i(a, q) \neq \emptyset, \forall t_i \in T. \tag{1}
\]

**Proof.** If (1) holds, there must exist a path \( P_i(a) \in \mathcal{L}_i(a, q) \), for each \( t_i \in T \) and \( q \in U^{\varepsilon} \). Let \( d_i(a) \) represent the length of \( P_i(a) \). We claim that the optimal solution to the robust formulation is

\[ R^e(t_i, t_n) = d_i(a), \forall t_n \in T^{n-1}, \]

and the proof goes as follows.

First, the above solution satisfies (RIC) because for all \( q \in U^{\varepsilon} \),

\[
\sum_{t_{n-1} \in T^{n-1}} R(t_i, t_{n-1})q(t_{n-1}) - \sum_{t_{n-1} \in T^{n-1}} R(t_j, t_{n-1})q(t_{n-1}) = d_i(a) - d_j(a) \geq w_{ji}(a, q)
\]

where the inequality follows from the properties of longest paths in a network.

Let \( R^B_a \) be the optimal solution to the Bayesian formulation. It follows that

\[
S^e(a) = \sum_{t_i \in T} f_i \sum_{t_{n-1} \in T^{n-1}} \pi(t_{n-1})v(a_i[t_i, t_{n-1}]|t_i, t_{n-1}) - \sum_{t_i \in T} f_id_i(a) \\
\geq \sum_{t_i \in T} f_i \sum_{t_{n-1} \in T^{n-1}} \pi(t_{n-1})v(a_i[t_i, t_{n-1}]|t_i, t_{n-1}) - \sum_{t_i \in T} f_i \sum_{t_{n-1} \in T^{n-1}} \pi(t_{n-1})R^B_a(t_i, t_{n-1}) \\
= S^b(a),
\]

where the inequality follows from Lemma 2. However, \( S^e(a) \leq S^b(a) \) from Lemma 1, which indicates that \( S^e(a) = S^b(a) \).

Conversely, if (1) does not hold, there exists \( q^1 \in U^{\varepsilon} \) such that

\[
\mathcal{L}_i(a, q^1) \cap \mathcal{L}_i(a, \pi) = \emptyset.
\]

Let \( q^2 \in U^{\varepsilon} \) be a probability vector such that \( \pi = \alpha q^1 + (1 - \alpha)q^2 \) for some \( 0 < \alpha < 1 \) (since \( \pi \) is an interior point of \( U^{\varepsilon} \), such a vector always exists). Also let \( P_i(a, q) \) be a longest path from \( n_0 \) to \( n_i \) in \( G(a, q) \), i.e. \( P_i(q) \in \mathcal{L}_i(a, q) \), for each \( 1 \leq i \leq |T| \) and \( q \in U^{\varepsilon} \).
Since the social surplus is the same in the robust and the Bayesian formulations for fixed allocation, it suffices just to compare the expected information rent. Let \( R_B^a \) and \( R_\varepsilon^a \) be the optimal rent in the Bayesian and the robust formulations, respectively.

Following an argument similar to the first part of the proof, \( R_B^a \) satisfies
\[
\sum_{t_{n-1} \in T} R_B^a(t_i, t_{n-1}) \pi(t_{n-1}) = d_i(a, \pi), \quad \forall t_i \in T.
\]

Therefore, the expected rent in the Bayesian formulation is:
\[
E[R_B^a] = \sum_{t_i \in T} f_i \sum_{t_{n-1} \in T} R_B^a(t_i, t_{n-1}) \pi(t_{n-1}) = \sum_{t_i \in T} f_i d_i(a, \pi),
\]
and the expect rent in the robust formulation is:
\[
E[R_\varepsilon^a] = \sum_{t_i \in T} f_i \sum_{t_{n-1} \in T} R_\varepsilon^a(t_i, t_{n-1}) \pi(t_{n-1}) \\
\geq \sum_{t_i \in T} f_i [(\alpha)d_i(a, q_1^1) + (1 - \alpha)d_i(a, q_2^2)],
\]
where the inequality follows from Lemma 2.

Since \( d_i(a, q_2^2) \) is the length of the longest path to \( n_i \) in \( G(a, q_2^2) \), it is greater than the length of \( P_i(a, \pi) \), which is a longest path in \( G(a, \pi) \), but not necessarily the longest in \( G(a, q_2^2) \), i.e.,
\[
d_i(a, q_2^2) \geq \sum_{(u,v) \in P_i(\pi)} w_{uv}(a, q_2^2).
\]

Moreover, as \( P_i(a, \pi) \) is not a longest path in \( G(a, q_1^1) \), it follows that
\[
d_i(a, q_1^1) > \sum_{(u,v) \in P_i(\pi)} w_{uv}(a, q_1^1).
\]

Putting everything together, we have
\[
E[R_\varepsilon^a] > \sum_{t_i \in T} f_i \sum_{(u,v) \in P_i(\pi)} [(\alpha)w_{uv}(a, q_1^1) + (1 - \alpha)w_{uv}(a, q_2^2)] \\
= \sum_{t_i \in T} f_i \sum_{(u,v) \in P_i(\pi)} w_{uv}(a, \pi) \\
= \sum_{t_i \in T} f_i d_i(a, \pi) \\
= E[R_B^a].
\]
This implies that $S^e(a) < S^b(a)$ and completes our proof. □

Because the Ex post formulation can be taken as a special case of the robust formulation when $\varepsilon = 1$, we can derive the conditions under which it achieves the same expected revenue as in the nominal case, as a corollary to Theorem 1. To simplify the notation in this case, we define $G(a, t^{n-1}) \equiv G(a, q^{n-1})$ and $\mathcal{L}_i(a, t^{n-1}) \equiv \mathcal{L}_i(a, q^{n-1})$, where $q^{n-1}(t^{n-1}) = 1$ and $q^{n-1}(s^{n-1}) = 0$ for all $s^{n-1} \in T^{n-1} \setminus \{t^{n-1}\}$.

**Corollary 1.** For an Ex post incentive compatible allocation $a \in F$, $S^e(a) = S^b(a)$, if and only if

$$\bigcap_{t^{n-1} \in T^{n-1}} \mathcal{L}_i(a, t^{n-1}) \neq \emptyset, \forall t_i \in T. \quad (2)$$

The next corollary provides sufficient conditions for a robust formulation of $a \in F$ to achieve the same expected revenue for the auctioneer as in the Ex post formulation. Here, we introduce new notation: define $\Theta^e$ as the set of extreme points of $U^e$, and $\tau(q) = \{t^{n-1} \in T^{n-1} : q(t^{n-1}) > 0\}$.

**Corollary 2.** For an Ex post incentive compatible allocation $a \in F$, $S^e(a) = S^b(a)$ if

$$\bigcap_{t^{n-1} \in \tau(q)} \mathcal{L}_i(a, t^{n-1}) \neq \emptyset, \forall t_i \in T, q \in \Theta^e. \quad (3)$$

The proofs of Corollaries 1 and 2 follow directly from that of Theorem 1 and thus are omitted from our discussion. It should be noted that condition (2) in Corollary 1 is weaker than the characterization of revenue equivalence in Theorem 1 of Heydenreich et al. [2009]. Specifically, in Heydenreich et al. [2009], the necessary and sufficient condition (under our setting) for an Ex post incentive compatible allocation $a \in F$ to satisfy revenue equivalence is

$$d_{ij}(a, t^{n-1}) = -d_{ji}(a, t^{n-1}), \forall t_i, t_j \in T, t^{n-1} \in T^{n-1}, \quad (4)$$

where $d_{ij}(a, t^{n-1})$ is the length of the longest path from $n_i$ to $n_j$ in $G(a, t^{n-1})$. It can be verified that (4) is equivalent to the condition that all paths have equal length between any pair of nodes in $G(a, t^{n-1})$ for all $t^{n-1} \in T^{n-1}$. Clearly, (4) indicates (2), but not vice versa. Thus, our results can be regarded as complementary to those in Heydenreich et al. [2009].

In the next section, we analyze Bayesian optimal allocations in single and multiple object auctions; we then focus on the revenue difference between various robust formulations.
5 Robust formulations given Bayesian optimal allocations

In this section, we demonstrate whether and when the different formulations give rise to different expected revenues.

5.1 Single object auction- Myerson’s case

We first investigate the setting of Myerson [1981], in which the auctioneer intends to sell a single object to bidders with private valuations. The valuation follows a continuous distribution that satisfies the monotone hazard rate condition. Malakhov and Vohra [2005] consider a similar problem under the discrete setting that allows for more transparent network representations. In this section, we adopt this discrete setting and their assumptions:

\[ v(\rho_i|t_i) = t_i \rho_i, \quad \text{and} \quad \frac{1 - F_i}{f_i} \geq \frac{1 - F_j}{f_j}, \quad \text{if} \ t_i \geq t_j, \]

where \( \rho_i \) is the expected quantity allocation to a type \( t_i \) bidder, and \( F_i = \sum_{j: t_j \leq t_i} f_j \) is the cumulative distribution of a bidder’s type. Malakhov and Vohra [2005] show that the optimal allocation for the Bayesian formulation is a standard auction with reservation price \( x^* = \min\{t_i \in T : t_i - \frac{1 - F_i}{f_i} \geq 0\} \), which coincides with results in the continuous setting in Myerson [1981]. Here, we show that this optimal allocation has a robust formulation that achieves the same expected revenue, irrespective of the associated uncertainty set.

**Theorem 2.** The Bayesian optimal allocation for Myerson’s single-object auction, \( a^B \), achieves the same optimal expected revenue in any robust formulation, i.e.,

\[ S^e(a^B) = S^b(a^B), \quad \forall 0 < 1 < \varepsilon. \]

**Proof.** Without loss of generality, we assume that the types are ordered such that \( t_1 < t_2 < \cdots < t_m \). We first show that for each \( i \) (\( 1 \leq i \leq |T| \)), a longest path from \( n_0 \) to \( n_i \) in \( G(a^B, q) \) is \( n_0 \rightarrow n_1 \rightarrow \cdots \rightarrow n_i \). That is, the longest path is the same for all \( q \in U^\varepsilon \), or equivalently,

\[ d_i(a^B, q) = \sum_{k=1}^{i} w_{k-1,k}(a^B, q), \quad \forall q \in U^\varepsilon. \]

To show that \( d_i(a^B, q) \) is indeed the length of a longest path, let \( t_i > t_j \) be two arbitrary types.
It follows that
\[ d_i(a^B, q) - d_j(a^B, q) = \sum_{k=1}^{i} w_{k-1,k}(a^B, q) - \sum_{k=1}^{j} w_{k-1,k}(a^B, q) \]
\[ = \sum_{k=j+1}^{i} w_{k-1,k}(a^B, q) \]
\[ = \sum_{k=j+1}^{i} \sum_{t^{n-1} \in T^{n-1}} [v(a^B_{k-1}[t_{k-1}, t^{n-1}][t_k]) - v(a^B_{k-1}[t_{k-1}, t^{n-1}][t_{k-1}])q(t^{n-1})] \]
\[ \geq \sum_{k=j+1}^{i} \sum_{t^{n-1} \in T^{n-1}} [v(a^B_{j}[t_j, t^{n-1}][t_k]) - v(a^B_{j}[t_j, t^{n-1}][t_{k-1}])q(t^{n-1})] \]
\[ = \sum_{t^{n-1} \in T^{n-1}} v(a^B_{j}[t_j, t^{n-1}][t_i]) - v(a^B_{j}[t_j, t^{n-1}][t_j])q(t^{n-1}) \]
\[ = w_{ij}(a^B, q), \]
where the inequality follows from the monotonicity of \( a^B \) in a standard auction.

Following a similar argument, we can show that \( d_i(a^B, q) - d_j(a^B, q) \geq w_{ij}(a^B, q) \) if \( t_i < t_j \). This implies that \( d_i(a^B, q) \) is indeed the length of a longest path in \( G(a^B, q) \). Clearly, \( \bigcap_{q \in U^\epsilon} \mathcal{L}_i(a^B, q) \neq \emptyset \), for all \( t_i \in T \). Following Theorem 1, \( S^b(a^B) = S^e(a^B) \). □

It is well known that in the single-object auction design problem, when the bidders’ valuations are private-valued and follow a continuous distribution, the optimal auction can be implemented by a second-price auction with an appropriately chosen reservation price (Myerson [1981]). Since the second-price auction can sustain truth-telling as a dominant strategy equilibrium, it is not surprising that all the intermediate case under any robust formulation should yield the same expected revenue for the auctioneer in the continuous setting. Note that the proof of Theorem 2 also applies to arbitrary feasible allocation, as we do not explicitly use any property of the optimal allocation. Thus, as anticipated, it can be slightly generalized to establish the equivalence between the Bayesian solution and any other robust formulation. On a related note, Theorem 2 also follows as a corollary of Theorem 1 in Heydenreich et al. [2009], which implies that any Bayesian incentive compatible mechanism can be implemented in dominant strategy, in a single object auction with discrete value. However, as we show in Section 5.2, this result no longer holds in a multiple object auction.
5.2 Multiple object auction - Armstrong’s case

We now consider the multi-object auction design problem introduced by Armstrong [2000]. In this setting, the auctioneer intends to sell two objects, denoted as $A$ and $B$. A bidder’s type $t$ is described by a pair $(\nu^A, \nu^B)$, where $\nu^\ell$ is the bidder’s valuation for object $\ell$. If an allocation $a = (p^A, p^B)$ awards object $A$ ($B$) to this bidder with probability $q^A$ ($q^B$), her gross utility is
\[ v(a|t) = p^A\nu^A + p^B\nu^B. \]

In Armstrong [2000], it is assumed that $\nu^\ell \in \{\nu^\ell_L, \nu^\ell_H\}$, where $\Delta^\ell = \nu^\ell_H - \nu^\ell_L > 0$. Thus there are four types of bidder corresponding to the four realizations $\{(\nu^A_L, \nu^B_L), (\nu^A_L, \nu^B_H), (\nu^A_H, \nu^B_L), (\nu^A_H, \nu^B_H)\}$, and these types are denoted by $LL$, $LH$, $HL$, and $HH$. The probability that a bidder has type $ij$ is $f_{ij}$, where $f_{LL} + f_{LH} + f_{HL} + f_{HH} = 1$. Let $f^\ell_L$ and $f^\ell_H = 1 - f^\ell_L$ denote the marginal probability of having a high or low valuation for object $\ell$, respectively; in other words, $f^\ell_L = f_{LL} + f_{LH}$, $f^\ell_H = f_{HL} + f_{HH}$, and likewise for object $B$. This is arguably the simplest possible setting that fully demonstrates the complicated nature of the multi-object auction design problem. Furthermore, this four-type framework has been adopted as a fixture in the multi-dimensional mechanism design problem, see, e.g., Armstrong and Rochet [1999] and Asker and Cantillon [2009].

5.2.1 Results from Armstrong [2000]

Before we demonstrate how the various solution concepts apply to this multi-object auction design problem, let us first revisit the results in Armstrong [2000]. Following the convention in the auction theory, Armstrong [2000] focuses exclusively on the Bayesian formulation of this problem. He finds that, the optimal auction crucially depends on the “correlation” between the bidders’ valuations for the two objects, defined as follows:
\[ \lambda^A = \frac{f_{HH}f_{LH}^A}{f_{LH}f_{HH}^A}, \quad \lambda^B = \frac{f_{HH}f_{LH}^B}{f_{LH}f_{HH}^B}. \]

Based on the above model characteristics, Armstrong [2000] identifies the structural properties of the optimal auctions. If $1/\lambda^A + 1/\lambda^B \leq 1$, i.e., there is strong positive correlation, the optimal allocation takes the format of an Independent Auction, in which $\rho^\ell_{ij}$, the expected quantity of object $\ell$ that is allocated to a type-$ij$ bidder is prescribed as:
\[ \rho^A_{HH} = \rho^A_{HL} = \frac{1 - (f^A_L)^n}{nf^A_H}, \quad \text{and} \quad \rho^A_{LL} = \rho^A_{LH} = \frac{(f^A_L)^{n-1}}{n}; \]
\[ \rho^B_{HH} = \rho^B_{HL} = \frac{1 - (f^B_L)^n}{nf^B_H}, \quad \text{and} \quad \rho^B_{LL} = \rho^B_{LH} = \frac{(f^B_L)^{n-1}}{n}. \]
When $1/\lambda^A + 1/\lambda^B \geq 2$, there is negative correlation. Armstrong [2000] characterizes some other technical conditions (for example, the symmetric case and the case in which there are sufficiently many bidders) and shows that the optimal allocation takes the format of a Bundling Auction. Under this format, the allocation of an object to a bidder with high valuation for the object is the same as in the Independent Auction, and the allocation to low-valuation bidders is given below:

$$
\rho^A_{LH} = \frac{(f^A_L)^n - f^n_{LL}}{n f^A_{HL}}, \text{ and } \rho^A_{LL} = \frac{f^n_{LL}}{n};
$$

$$
\rho^B_{HL} = \frac{(f^B_L)^n - f^n_{LL}}{n f^B_{HL}}; \text{ and } \rho^B_{LL} = \frac{f^n_{LL}}{n}.
$$

In all other cases, the optimal allocation is a Mixed Auction, which is a convex combination of the Independent Auction and the Bundling Auction.

### 5.2.2 Difference between the formulations

We now introduce other solution concepts for this problem. Our first result is that the auctioneer’s expected revenue may be strictly less if the Ex Post formulation is used instead of the Bayesian formulation. To this end, let us introduce a new notation, in which each profile $t^{n-1} \in T^{n-1}$ is specified with a triplet $(k_1, k_2, k_3)$, where $k_1$, $k_2$, and $k_3$ respectively are the number of bidders with types LH, HL and HH, and $0 \leq k_1 + k_2 + k_3 \leq n - 1$.

**Theorem 3.** Unless the Bayesian optimal allocation is the Independent Auction for both objects, the auctioneer’s expected revenue under the Ex post formulation is strictly lower than that under the Bayesian formulation, i.e., $S^e(a^B) > S^b(a^B)$.

**Proof.** Suppose that $a^B$ is an Bayesian optimal allocation (specified by the expected quantities $\rho$). We can then construct a detailed allocation $p$ that satisfies

$$
\rho_{ij}^\ell = \sum_{t^{n-1} \in T^{n-1}} p_{ij}^\ell(t^{n-1}) \pi(t^{n-1}), \forall (i, j) \in T, \ell \in \{A, B\}.
$$

It is straightforward that an optimal allocation to the high-valuation bidders should satisfy

$$
p^A_{HH}(k_1, k_2, k_3) = p^A_{HL}(k_1, k_2, k_3) = \frac{1}{k_2 + k_3 + 1}, \text{ and } p^B_{HH}(k_1, k_2, k_3) = p^B_{HL}(k_1, k_2, k_3) = \frac{1}{k_1 + k_3 + 1}.
$$

To characterize the allocation to the low-valuation bidders, we restrict our attention to the case where $k_3 = 0$ and $k_1 k_2 = 0$, since otherwise the solution is trivial.(6) Now there are three

(6) If $k_3 > 0$ or $k_1 k_2 > 0$, there exist bidders with high valuation for both objects, therefore the optimal allocation to the low-valuation bidder is zero.
possible scenarios, depending on the structural properties of the Bayesian optimal allocation $a^B$. In the following we discuss them separately.

**Case 1. Independent Auction for both objects.**

It is clear that if the Bayesian optimal allocation is an independent auction, the auctioneer can obtain the same expected revenue under the Ex post formulation by utilizing “the same” allocation. Specifically, we can construct a detailed allocation that is equivalent to the independent auction as follows

$$p^A_{LL}(k, 0, 0) = p^A_{LH}(k, 0, 0) = \frac{1}{n}, \text{ and } p^B_{LL}(0, k, 0) = p^B_{HL}(0, k, 0) = \frac{1}{n}.$$  

It is easy to verify that the detailed allocation $p$ satisfy condition (2) in Corollary 1; therefore, it achieves the same expected revenue under the Ex post formulation.

**Case 2. Bundling Auction for both objects.**

In this case, the unique equivalent detailed allocation is given by

$$p^A_{LL}(0, 0, 0) = \frac{1}{n}, p^A_{LL}(k, 0, 0) = 0, \forall 1 \leq k \leq n - 1,$$

$$p^A_{LH}(k, 0, 0) = \frac{1}{k + 1}, \forall 1 \leq k \leq n - 1,$$

$$p^B_{LL}(0, 0, 0) = \frac{1}{n}, p^B_{LL}(0, k, 0) = 0, \forall 1 \leq k \leq n - 1,$$

$$p^B_{HL}(0, k, 0) = \frac{1}{k + 1}, \forall 1 \leq k \leq n - 1.$$ 

It is straightforward that for $k \geq 1$ and $t^{n-1} = (k, 0, 0)$, $p^A_{LH}(t^{n-1}) > p^A_{LL}(t^{n-1})$. Thus, $LL \to LH \to HH$ is the unique longest path from the $LL$ type to the $HH$ type in graph $G(a^b, t^{n-1})$. Similarly, for $k \geq 1$ and $t^{n-1} = (0, k, 0)$, $p^B_{HL}(t^{n-1}) > p^B_{LL}(t^{n-1})$. Therefore, $LL \to HL \to HH$ is the unique longest path. Clearly, condition (2) in Corollary 1 is violated, which indicate that $S^a(a^B) < S^b(a^B)$.

**Case 3. Bundling Auction for one object and Mixed Auction for the other.**

According to Armstrong [2000], the expected allocation in this case satisfies

$$\Delta^A(\rho_{LH}^A - \rho_{LL}^A) = \Delta^B(\rho_{HL}^B - \rho_{LL}^B).$$  

(5)

Suppose that there exists an equivalent detailed allocation $p$ which achieves the same expected revenue in the Ex post formulation. We first consider the case in which the optimal allocation for object $A$ is Bundling Auction.
For $t^{n-1} = (k, 0, 0)$, $1 \leq k \leq n - 1$, the detailed allocation of the two objects

\[ p^A_{LL}(k, 0, 0) = 0, \quad p^A_{LH}(k, 0, 0) = \frac{1}{k+1}, \quad p^B_{LL}(k, 0, 0) = 0, \quad p^B_{HL}(k, 0, 0) = 0, \]

which implies that

\[ \Delta^B[p^B_{HL}(0, k, 0) - p^B_{LL}(0, k, 0)] = 0 < \Delta^A[p^A_{LH}(0, k, 0) - p^A_{LL}(0, k, 0)]. \] (6)

This detailed allocation also indicates that $LL - LH - HH$ is the unique longest path in $G(a^b, t^{n-1})$ if $t^{n-1} = (k, 0, 0)$. From Corollary 1, $LL - LH - HH$ must also be a longest path in $G(a^b, t^{n-1})$ for all $t^{n-1} \in T^{n-1}$.

For $t^{n-1} = (0, k, 0)$, $1 \leq k \leq n - 1$, this implies

\[ \Delta^A p^A_{LL}(0, k, 0) + \Delta^B p^B_{HL}(0, k, 0) \geq \Delta^B p^B_{LL}(0, k, 0) + \Delta^A p^A_{LH}(0, k, 0), \]

where the left-hand side is the length of path $LL - LH - HH$ and the right-hand side is the length of $LL - HL - HH$.

Since $p^A_{LL}(0, k, 0) = p^A_{LH}(0, k, 0) = 0$, it follows that

\[ \Delta^B[p^B_{HL}(0, k, 0) - p^B_{LL}(0, k, 0)] \leq 0 = \Delta^A[p^A_{LH}(0, k, 0) - p^A_{LL}(0, k, 0)]. \] (7)

Moreover, for $t^{n-1} = (0, 0, 0)$, since $LL - LH - HH$ is no shorter than $LL - HL - HH$,

\[ \Delta^A p^A_{LL}(0, 0, 0) + \Delta^B p^B_{HL}(0, 0, 0) \geq \Delta^B p^B_{LL}(0, 0, 0) + \Delta^A p^A_{LH}(0, 0, 0), \]

or equivalently,

\[ \Delta^B[p^B_{HL}(0, 0, 0) - p^B_{LL}(0, 0, 0)] \leq \Delta^A[p^A_{LH}(0, 0, 0) - p^A_{LL}(0, 0, 0)]. \] (8)

Finally, for $t^{n-1} = (k_1, k_2, k_3)$ where $k_1 k_2 > 0$ or $k_3 > 0$, the detailed allocation is

\[ p^A_{LL}(k_1, k_2, k_3) = p^A_{LH}(k_1, k_2, k_3) = 0, \quad p^B_{LL}(k_1, k_2, k_3) = p^B_{HL}(k_1, k_2, k_3) = 0. \]

It follows that

\[ \Delta^B[p^B_{HL}(k_1, k_2, k_3) - p^B_{LL}(k_1, k_2, k_3)] = 0 = \Delta^A[p^A_{LH}(k_1, k_2, k_3) - p^A_{LL}(k_1, k_2, k_3)]. \] (9)

Adding (6)-(9) leads to

\[ \Delta^B(\rho^B_{HL} - \rho^B_{LL}) < \Delta^A(\rho^A_{LH} - \rho^A_{LL}), \]

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which leads to a contradiction to (5).

Following a similar argument, one can show that there does not exist a detailed allocation that achieves the same revenue in the Ex post formulation, in the case where the optimal allocation is Bundling Auction for object $B$. □

Theorem 3 indicates the possibility of discrepancies between different formulations in terms of the auctioneer’s expected revenue. A natural question, that follows, is whether there is indeed a gap between any two formulations. To this end, we present an example for which different formulations give rise to distinct maximum expected revenues, thereby asserting this possibility.

5.2.3 An example

In this example, there are two bidders, i.e., $n = 2$. The probabilities of bidder types are

$$f_{LL} = \frac{1}{6}, f_{LH} = \frac{1}{4}, f_{HL} = \frac{5}{12}, \text{ and } f_{HH} = \frac{1}{6},$$

and the common prior $\pi$ is given by

$$\pi(LL) = \frac{1}{6}, \pi(LH) = \frac{1}{6}, \pi(HL) = \frac{1}{2}, \text{ and } \pi(HH) = \frac{1}{6}.$$  

Also, let $\Delta^A = \Delta^B = \Delta$. Since $1/\lambda^A + 1/\lambda^B > 2$, the Bayesian optimal allocation is Bundling Auction for both objects according to Armstrong [2000]. Following Theorem 3, the Ex post formulation achieves a strictly lower expected revenue than the Bayesian formulation. In the following steps, we demonstrate how the auctioneer’s revenue changes according to the level of uncertainty $\varepsilon$.

Since both objects are always sold, the social surpluses under the two formulations are the same. It suffices to only consider the difference in information rents.

According to Armstrong [2000], the Bayesian optimal (expected) allocation in this case is:

$$\rho^A_{LL} = \frac{f_{LL}^{n-1}}{n} = \frac{1}{12}, \rho^A_{LH} = \frac{(f^A_L)^n - f^A_{LL}}{nf_{LH}} = \frac{7}{24},$$

$$\rho^B_{LL} = \frac{f_{LL}^{n-1}}{n} = \frac{1}{12}, \rho^B_{HL} = \frac{(f^B_L)^n - f^B_{LL}}{nf_{HL}} = \frac{3}{8},$$ 

Thus, $R_{ij}^b$, the minimum expected information rent received by a type-$ij$ bidder, is:

$$R_{LL}^b = 0, \quad R_{LH}^b = \Delta B \rho_{LL}^B = \frac{1}{12} \Delta;$$

$$R_{HL}^b = \Delta A \rho_{LL}^A = \frac{1}{12} \Delta, \quad R_{HH}^b = \Delta B \rho_{LL}^B + \Delta A \rho_{LH}^A = \frac{3}{8} \Delta.$$ 

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Ex ante, the expected information rent for any bidder in the Bayesian formulation is therefore

\[ E[R^b] = \sum_{(i,j) \in T} f_{ij} R_{ij} = \frac{17}{144} \Delta. \]

For the Ex post formulation, consider the following equivalent detailed allocation:

- \( p^A_{LL}(0, 0, 0) = \frac{1}{2}, \ p^A_{LL}(0, 0, 0) = 1, \ p^B_{LL}(0, 0, 0) = \frac{1}{2}, \ p^B_{HL}(0, 0, 0) = 1; \)
- \( p^A_{LL}(1, 0, 0) = 0, \ p^A_{LL}(0, 0, 0) = \frac{1}{2}, \ p^B_{LL}(1, 0, 0) = 0, \ p^B_{HL}(1, 0, 0) = 0; \)
- \( p^A_{LL}(0, 1, 0) = 0, \ p^A_{LL}(0, 1, 0) = 0, \ p^B_{LL}(0, 1, 0) = 0, \ p^B_{HL}(0, 1, 0) = \frac{1}{2}; \)
- \( p^A_{LL}(0, 0, 1) = 0, \ p^A_{LL}(0, 0, 1) = 0, \ p^B_{LL}(0, 0, 1) = 0, \ p^B_{HL}(0, 0, 1) = 0. \)

In this case, \( R^e_{ij}(t^{n-1}) \), the minimum information rent in the Ex post formulation, is given by:

- \( R^e_{LL}(0, 0, 0) = 0, \ R^e_{HL}(0, 0, 0) = \frac{1}{2} \Delta, \ R^e_{HL}(0, 0, 0) = \frac{1}{2} \Delta, \ R^e_{HH}(0, 0, 0) = \frac{3}{2} \Delta; \)
- \( R^e_{LL}(1, 0, 0) = 0, \ R^e_{HL}(1, 0, 0) = 0, \ R^e_{HL}(1, 0, 0) = 0, \ R^e_{HH}(1, 0, 0) = \frac{1}{2} \Delta; \)
- \( R^e_{LL}(0, 1, 0) = 0, \ R^e_{HL}(0, 1, 0) = 0, \ R^e_{HL}(0, 1, 0) = 0, \ R^e_{HH}(0, 1, 0) = \frac{1}{2} \Delta; \)
- \( R^e_{LL}(0, 0, 1) = 0, \ R^e_{HL}(0, 0, 1) = 0, \ R^e_{HL}(0, 0, 1) = 0, \ R^e_{HH}(0, 0, 1) = 0. \)

The expected information rent received by a bidder in the Ex post formulation is:

\[ E[R^e] = \sum_{(i,j) \in T} f_{ij} \sum_{t^{n-1} \in T^{n-1}} R^e_{ij}(t^{n-1}) \pi(t^{n-1}) = \frac{11}{72} \Delta, \]

which is clearly higher than that of the Bayesian formulation \( (E[R^b]) \). This indicates that the auctioneer’s expected revenue is strictly lower in the Ex post formulation, i.e. \( S^e(a) > S^b(a) \).

Next, let us consider a more general case where \( 0 < \varepsilon < 1 \). It can be verified that condition (1) in Theorem 1 holds if and only if \( \varepsilon \leq \frac{1}{12} \), which indicates \( S^e(a) < S^b(a) = S^b(a) \). Moreover, condition (3) in Corollary 2 holds if \( \varepsilon \geq \frac{3}{4} \), which implies that \( S^e(a) = S^e(a) < S^b(a) \). It should be noted that for \( \frac{1}{12} < \varepsilon < \frac{3}{4} \), the robust formulation might be different from both the Bayesian and the Ex post formulations. For example, if \( \varepsilon = \frac{1}{6} \), the extreme points of \( U^e \) are:

- \( q^1 = (1/3, 1/12, 1/4, 1/3) \), \( q^2 = (0, 5/12, 7/12, 0) \), \( q^3 = (0, 1/12, 7/12, 1/3) \),
- \( q^4 = (1/3, 1/12, 7/12, 0) \), \( q^5 = (0, 5/12, 1/4, 1/3) \), \( q^6 = (1/3, 5/12, 1/4, 0) \).
Furthermore, it can be verified that a payment/rent scheme is robust incentive compatible if and only if \((RIC)\) holds for \(q^k, k = 1, \ldots, 6\), since \((RIC)\) is linear and all probability vectors in \(U^\varepsilon\) can be written as a convex combination of the extreme points. Solving the corresponding linear program yields the following robust optimal rent \(R^\varepsilon\):

\[
R^\varepsilon_{LL}(0, 0, 0) = 0, R^\varepsilon_{LH}(0, 0, 0) = \frac{1}{2}\Delta, R^\varepsilon_{HL}(0, 0, 0) = \frac{1}{2}\Delta, R^\varepsilon_{HH}(0, 0, 0) = \frac{3}{2}\Delta;
\]

\[
R^\varepsilon_{LL}(1, 0, 0) = 0, R^\varepsilon_{LH}(1, 0, 0) = 0, R^\varepsilon_{HL}(1, 0, 0) = 0, R^\varepsilon_{HH}(1, 0, 0) = \frac{7}{32}\Delta;
\]

\[
R^\varepsilon_{LL}(0, 1, 0) = 0, R^\varepsilon_{LH}(0, 1, 0) = 0, R^\varepsilon_{HL}(0, 1, 0) = 0, R^\varepsilon_{HH}(0, 1, 0) = \frac{15}{32}\Delta;
\]

\[
R^\varepsilon_{LL}(0, 0, 1) = 0, R^\varepsilon_{LH}(0, 0, 1) = 0, R^\varepsilon_{HL}(0, 0, 1) = 0, R^\varepsilon_{HH}(0, 0, 1) = 0.
\]

Accordingly, the expected information rent received by a bidder in this robust formulation is:

\[
E[R^\varepsilon] = \sum_{(i,j) \in T} f_{ij} \sum_{t^{n-1} \in T^{n-1}} R^\varepsilon_{ij}(t^{n-1})\pi(t^{n-1}) = \frac{5}{36}\Delta,
\]

which indicates that \(S^e(a) < S^\varepsilon(a) < S^h(a)\). Thus, we have constructed an example for which all the three formulations (Bayesian, Ex post, and general robust formulations) lead to different expected revenues for the auctioneer. It is worth mentioning that in Lopomo et al. [2007], they identify the conditions on the uncertainty set under which the Ex post formulation is equivalent to the robust formulation in single-dimensional mechanism design problems. Specifically, they demonstrate that as long as the uncertainty set has full dimensionality, then the incentive compatibility constraints in fact expand to cover every possible scenario of belief systems; thus, the set of feasible solutions is the same under the two formulations. Our result complements theirs by explicitly finding an example for which the results differ in the multiple-dimensional setting.

### 6 Conclusions

In this paper, we revisit the optimal auction design problem and propose a robust formulation that incorporates the bidders’ belief uncertainty. For a fixed allocation mechanism, we identify the necessary and sufficient conditions under which the robust formulation achieves the same expected revenue as the Bayesian formulation. We apply this result to show that the optimal allocation of a single-object auction achieves the same expected revenue in any robust formulation, even though the bidders’ valuations are discrete rather than continuous. We also find that, there may be a
discrepancy between different formulations in a multiple-object auction, and we provide a concrete example for which the ex post formulation gives rise to a strictly lower expected revenue in the multi-dimensional setting. Our results imply that the auctioneer may have to sacrifice the expected revenue for a more robust formulation.

References


