Reputation in Continuous-Time Games, Supplementary Material: Public Randomization

Eduardo Faingold
Department of Economics
Yale University
eduardo.faingold@yale.edu

Yuliy Sannikov
Department of Economics
Princeton University
sannikov@gmail.com

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Abstract

We show that, in the reputation games of Faingold and Sannikov (2010, Section 7), if a continuous-time public randomization device is avaliable, then a Markov perfect equilibrium in publicly randomized strategies exists under Conditions 1 and 4.

As explained in Remark 7 of Faingold and Sannikov (2010, Section 7), when Condition 1 holds but Condition 2 fails, some reputation games may fail to have a sequential equilibrium in continuous time (even when the action sets are finite and mixed strategies are allowed). In this appendix, we enlarge the model of Section 7 to allow the players to condition their behavior on the outcomes of a *continuous-time public randomization device*. In this augmented game an equilibrium is guaranteed to exist under Conditions 1 and 4.

Before turning to the formalism of continuous-time public randomization, we draw an analogy with discrete time in order to attach a meaning to our continuous-time mixtures. In a discrete-time game, there is a well-defined order of events within each period: first, the outcome of the randomization device is realized and publicly observed, then players take their actions simultaneously, then the signals are realized and publicly observed, and finally the players update their beliefs. Heuristically, in our continuous-time game it is also possible to capture a similar order of events as follows: at any time t, suppose the current value of the small players' posterior on the behavioral type is ϕ; then,

1. the outcome of the randomization device is realized and publicly observed;
2. the normal type chooses an action a and, simultaneously, each small player i chooses an action b^i, which gives rise to a distribution $\bar{b}$;
3. the public signal $dX = \mu(a, \bar{b})\, dt + \sigma(\bar{b})\, dZ$ is realized;
4. players receive their flow payoffs $rg(a, \bar{b})\, dt$ and $rh(a, b^i, \bar{b})\, dt$;

5. the posterior is updated to $\phi + d\phi$.

Thus, calculated at the stage prior to the realization of the randomization device, the expected flow payoff of the normal type is

$$rg(\pi)\,dt,$$

where $\pi \in \Delta(A \times \Delta(B))$ designates the distribution over action profiles induced by the randomization device.[1] Also, from the viewpoint of the normal type, the change in posterior beliefs, $d\phi$, has mean and variance given by

$$-\frac{\Gamma^2(\pi,\phi)}{1-\phi}\,dt \quad \text{and} \quad \Gamma^2(\pi,\phi)\,dt$$

respectively, where

$$\Gamma^2(\pi,\phi) \stackrel{\text{def}}{=} \int_{A\times\Delta(B)} |\gamma(a,\bar{b},\phi)|^2\,\pi(da \times d\bar{b}).$$

Then, following the logic of Sections 6 and 7, in a sequential equilibrium in which the players use a publicly randomized strategy $\pi_t = \pi(\phi_t) \in \Delta(A \times \Delta(B))$ in which the actions are determined by the small players' posterior belief after every history, we expect the value function $U : (0,1) \to \mathbb{R}$ of the normal type to solve the differential equation

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\pi(\phi)))}{\Gamma^2(\pi(\phi),\phi)}, \tag{1}$$

with

$$(a,\bar{b}) \in \mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r) \quad \forall (a,\bar{b}) \in \operatorname{supp} \pi(\phi), \tag{2}$$

since in equilibrium the incentive constraints must hold following each realization of the randomization device.

This motivates the following definition.

Definition. A *Markov perfect equilibrium with public randomization* is a measurable function $\pi : (0,1) \to \Delta(A \times \Delta(B))$ for which there exists a bounded differentiable function $U : (0,1) \to \mathbb{R}$ that has an absolutely continuous derivative and satisfies (1) and (2) almost everywhere. Function U is the *value function of the normal type* and, for each prior $p \in (0,1)$ on the behavioral type, $U(p)$ is the *expected discounted payoff of the normal type* under the equilibrium π and the prior p.

Although in this definition we have ultimately assumed (1) and (2), we do believe it is possible to derive these conditions, as we did in the setting without public randomization of Section 7. However, this would require a rich mathematical framework in which it were possible to formulate an exact law of large numbers for a continuum of i.i.d. random variables. Various such frameworks have been provided in the Large Games literature and we believe it is possible to extend such frameworks to

[1] As usual, the function g is linearly extended to $\Delta(A \times \Delta(B))$.

deal with our dynamic setting in continuous time as well.[2] Providing this extension, however, is beyond the scope of our paper.

Finally, we note that neither the existence problem nor the issue of randomization in continuous time is new to our paper. Public randomization is used by Harris, Reny, and Robson (1995) to establish existence of subgame perfect equilibria in extensive-form games with continuum of actions. In the experimentation literature, Bolton and Harris (1999) make use of continuous-time randomization to obtain existence of Markov perfect equilibria. Like us, they define Markov perfect equilibria directly using a differential equation for the value. The heuristic derivation presented above follows that paper closely.

The main result of this appendix is:

Proposition. *Under Conditions 1 and 4, a Markov perfect equilibrium with public randomization exists.*

Proof. It is enough to show that the differential inclusion

$$U''(\phi) \in \Lambda(\phi, U(\phi), U'(\phi)) \quad a.e. \tag{3}$$

has a bounded solution, where $\Lambda : (0,1) \rightrightarrows \mathbb{R}$ is the correspondence defined by

$$\Lambda(\phi, u, u') \overset{\text{def}}{=} \left\{ \frac{2u'}{1-\phi} + \frac{2r(u - g(\pi))}{\Gamma^2(\pi, \phi)} \;\middle|\; \begin{array}{l} \pi \in \Delta(A \times \Delta(B)) \text{ such that} \\ \text{supp } \pi \subseteq \mathcal{N}(\phi, \phi(1-\phi)u'/r) \end{array} \right\} \quad \forall (\phi, u, u') \in (0,1) \times \mathbb{R}^2.$$

Indeed, if such a solution exists then for each ϕ we can find $\pi(\phi)$ satisfying (1) and (2), by the definition of $\mathcal{N}$. Moreover, such $\pi(\phi)$ can be chosen measurably as a function of ϕ, by a standard measurable selection argument.

We proceed with the proof that (3) has a bounded solution.

STEP 0 (Quadratic growth): *Given any closed interval* $[p, q] \subset (0, 1)$ *there exists* $K > 0$ *such that*

$$|\Lambda(\phi, u, u')| \leq K(1 + |u'|^2) \quad \forall (\phi, u, u') \times [p, q] \times [\underline{g}, \bar{g}] \times \mathbb{R}.$$

Given Conditions 1 and 4, this is immediate from Corollary B.1 of Faingold and Sannikov (2010).

STEP 1 (Existence away from 0 and 1): *Given any closed interval* $[p, q] \subset (0, 1)$*, the differential inclusion has a* C^2*-solution on* $[p, q]$ *that takes values in the set of feasible payoffs* $[\underline{g}, \bar{g}]$.

This follows directly from an existence theorem due to Bebernes and Kelley (1973) for boundary value problems for second-order differential inclusions. Bebernes and Kelley's (1973) sufficient condition for existence is that the non-linearity Λ satisfy the Nagumo condition. This condition is implied by the quadratic growth condition from STEP 0.

[2] See Sun (1998) and Khan and Sun (1999) for an approach based on hyperfinite Loeb spaces, and Al-Najjar (2009) for an approach based on finitely-additive probability measures.

STEP 2 (A priori bound on the derivative): *Given any closed interval* $[p,q] \subset (0,1)$*, there is a constant* $R > 0$ *such that every* C^2*-function* $U : [p,q] \to \mathbb{R}$ *that takes values in* $[\underline{g}, \bar{g}]$ *and solves the differential inclusion on* $[p,q]$ *satisfies* $|U'| \leq R$.

Let $L = (\bar{g} - \underline{g})/(q-p)$ and choose $R > 0$ large enough so that

$$\frac{1}{2K} \log \frac{1+R^2}{1+L^2} > \bar{g} - \underline{g}. \tag{4}$$

Let $U : [p,q] \to [\underline{g}, \bar{g}]$ be an arbitrary C^2-function that solves the differential inclusion on the interval $[p,q]$. Suppose, towards a contradiction, that $U'(\phi_1) > R$ for some $\phi_1 \in [p,q]$. (The proof for the case $U'(\phi_1) < -R$ is similar.) By the mean value theorem, there is some $\bar{\phi} \in [p,q]$ such that $U'(\bar{\phi}) \leq L$. Hence, we can find an interval $[\phi_0, \phi_1]$ or $[\phi_1, \phi_0]$ such that $U'(\phi_0) = L$ and $U' \geq 0$ on this interval. Thus,

$$\frac{1}{2K} \log \frac{1+R^2}{1+L^2} = \int_L^R \frac{v\, dv}{K(1+v^2)} \leq \int_{U'(\phi_0)}^{U'(\phi_1)} \frac{v\, dv}{K(1+v^2)} =$$
$$= \int_{\phi_0}^{\phi_1} \frac{U'(\phi)U''(\phi)\, d\phi}{K(1+U'(\phi)^2)} \leq \Big| \int_{\phi_0}^{\phi_1} U'(\phi)\, d\phi \Big| \leq |U(\phi_1) - U(\phi_0)| \leq \bar{g} - \underline{g},$$

where the second inequality follows from the differential inclusion and the quadratic growth condition of STEP 0. By (4) the inequalities above yield a contradiction.

STEP 3 (Extension to $(0,1)$): For each n, let $U_n : [1/n, 1-1/n] \to \mathbb{R}$ be a C^2 solution of the differential inclusion on $[1/n, 1-1/n]$ and let $R_n > 0$ denote the corresponding a priori bound on derivative from STEP 2. Fix n and consider $m \geq n$. The restriction of U_m to $[1/n, 1-1/n]$ is a solution of the differential inclusion that takes values in the set of feasible payoffs on $[1/n, 1-1/n]$. Hence STEP 2 applies and we have $|U_m'(\phi)| < R_n$ for all $\phi \in [1/n, 1-1/n]$, which implies a uniform bound on the second derivative U_m'' because of the quadratic growth condition. By the Arzelá-Ascoli Theorem, for every n the sequence $(U_m)_{m>n}$ has a sub-sequence that converges in the C^1 topology on $[1/n, 1-1/n]$. Hence, by a standard diagonalization argument, $(U_n)_{n\geq 1}$ has a sub-sequence $(U_{n_k})_{k\geq 1}$ that converges pointwise to some function $U : (0,1) \to [\underline{g}, \bar{g}]$. Moreover, on any closed sub-interval J of $(0,1)$, the convergence takes place in $C^1(J)$, i.e. U is C^1 and $(U_{n_k}, U_{n_k}') \to (U, U')$ uniformly on I.

STEP 4 (Absolute continuity of U'): First, note that

(a) $U_{n_k}' \to U'$ pointwise, and

(b) for every $[p,q] \subset (0,1)$, U_{n_k}' is Lipschitz continuous on $[p,q]$ uniformly over all k with $n_k > \max\{1/p, 1/(1-q)\}$. This is because by STEP 2 there exists $R > 0$ such that for all k with $1/n_k < \min\{p, 1-q\}$ we have $|U_{n_k}'| \leq R$ on $[p,q]$, which implies a uniform bound on U_{n_k}'' by the quadratic growth condition of STEP 0.

It follows that U' is locally Lipschitz, and hence absolutely continuous.

STEP 5 (U is a solution): Fix an arbitrary $\varepsilon > 0$ and a ϕ_0 at which U'' exists. Since Λ is u.h.c. and $(U_{n_k}, U'_{n_k}) \to (U, U')$ uniformly on a nbd of ϕ_0 (by STEP 3), there exist $\delta > 0$ and $K \geq 1$ such that for all $k \geq K$ and all $\phi \in (\phi_0 - \delta, \phi_0 + \delta)$ we have

$$\Lambda(\phi, U_{n_k}(\phi), U'_{n_k}(\phi)) \subset \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon].$$

Thus, for almost all $\phi \in (\phi_0 - \delta, \phi_0 + \delta)$ and all $k \geq K$,

$$U''_{n_k}(\phi) \in \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon]. \tag{5}$$

On the other hand, for all $h > 0$ and $k \geq 1$ we have

$$\frac{U'_{n_k}(\phi_0 + h) - U'_{n_k}(\phi_0)}{h} = \frac{1}{h} \int_{\phi_0}^{\phi_0 + h} U''_{n_k}(x)\, dx.$$

This implies that for each $k \geq K$ and $|h| < \delta$,

$$\frac{U'_{n_k}(\phi_0 + h) - U'_{n_k}(\phi_0)}{h} \in \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon],$$

by the differential inclusion (5) and the fact that Λ has closed convex values. Thus, letting $k \to \infty$ first and then $h \to 0$ yields

$$U''(\phi_0) \in \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon].$$

Finally, taking ε to zero yields the desired result.

■

References

Al-Najjar, Nabil I. (2009), "Large games and the law of large numbers." *Games and Economic Behavior*, 64, 1–34.

Bebernes, J. and W. Kelley (1973), "Some boundary value problems for generalized differential equations." *SIAM J. Appl Math*, 25, 16–23.

Bolton, Patrick and Christopher Harris (1999), "Strategic experimentation." *Econometrica*, 67, 349–374.

Faingold, Eduardo and Yuliy Sannikov (2010), "Reputation in continuous-time games." mimeo., Yale University.

Harris, Christopher, Philip Reny, and Arthur Robson (1995), "The existence of subgame-perfect equilibrium in continuous games with almost perfect information: a case for public randomization." *Econometrica*, 63, 507–44.

Khan, M. and Y. Sun (1999), "Non-cooperative games on hyperfinite Loeb spaces." *Journal of Mathematical Economics*, 31, 455–492.

Sun, Y. (1998), "A theory of hyperfinite processes: The complete removal of individual uncertainty via exact LLN." *Journal of Mathematical Economics*, 29, 419–503.