Reputation in Continuous-Time Games*

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Abstract

We study reputation dynamics in continuous-time games in which a large player (e.g. government) faces a population of small players (e.g. households) and the large player's actions are imperfectly observable. The major part of our analysis examines the case in which public signals about the large player's actions are distorted by a Brownian motion and the large player is either a *normal type*, who plays strategically, or a *behavioral type*, who is committed to playing a stationary strategy. We obtain a clean characterization of sequential equilibria using ordinary differential equations and identify general conditions for the sequential equilibrium to be unique and Markovian in the small players' posterior belief. We find that a rich equilibrium dynamics arises when the small players assign positive prior probability to the behavioral type. By contrast, when it is common knowledge that the large player is the normal type, every public equilibrium of the continuous-time game is payoff-equivalent to one in which a *static* Nash equilibrium is played after every history. Finally, we examine variations of the model with Poisson signals and multiple behavioral types.

Keywords: Reputation, repeated games, incomplete information, continuous time.

JEL Classification: C70, C72.

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1 Introduction

Reputation plays an important role in long-run relationships. Firms can benefit from reputation to fight potential entrants (Kreps and Wilson (1982), Milgrom and Roberts (1982)), to provide high quality to consumers (Klein and Leffler (1981)), or to generate good returns to investors (Diamond (1989)). Reputation can help time-inconsistent governments commit to non-inflationary monetary policies (Barro (1986), Cukierman and Meltzer (1986)), low capital taxation (Chari and Kehoe (1993), Celentani and Pesendorfer (1996)), and repayment of sovereign debt (Cole, Dow, and English (1995)). In credence goods markets, strategic concerns to avoid a bad reputation can create perverse incentives that lead to market breakdown (Ely and Valimaki (2003)).

We study reputation dynamics in repeated games between a *large player* (e.g. firm, government) and a population of *small players* (e.g. consumers, households) in which the actions of the large player are *imperfectly observable*. For example, the observed quality of a firm's product may be a noisy outcome of the firm's effort to maintain quality standards; the realized inflation rate may be a noisy signal of the central bank's target monetary growth. Our setting is a continuous-time analogue of the Fudenberg and Levine (1992) model. Specifically, we assume that a noisy signal about the large player's actions is publicly observable and that the evolution of this signal is driven by a Brownian motion. The small players are anonymous and hence behave myopically in every equilibrium, acting to maximize their instantaneous expected payoffs. We follow the incomplete information approach to reputation pioneered by Kreps and Wilson (1982) and Milgrom and Roberts (1982), which assumes that the small players are uncertain as to which *type* of large player they face. Specifically, the large player can be either a *normal type*, who is forward looking and behaves strategically, or a *behavioral type*, who is committed to playing a certain strategy. As usual, we interpret the small players' posterior probability on the behavioral type as a measure of the large player's reputation.

Recall that in discrete time two main limit results about reputation effects are known. First, in a general model with multiple behavioral and non-behavioral types, Fudenberg and Levine (1992) provide upper and lower bounds on the set of equilibrium payoffs of the normal type which hold in the limit as he gets arbitrarily patient; when the public signals satisfy an identifiability condition and the stage-game payoffs satisfy a non-degeneracy condition, these asymptotic bounds are tight and equal the stage-game Stackelberg payoff. Second, Cripps, Mailath, and Samuelson (2004) show that in a wide range of repeated games the power of reputation effects is only temporary: in any equilibrium the large player's type must be asymptotically revealed in the long run.[1] However, apart from these two *limit* results (and their extensions to various important settings), not much is known about equilibrium behavior in reputation games. In particular, in the important case in which the signal distribution has full support, the explicit construction of even one sequential equilibrium appears to be an elusive, if not intractable, problem.

By contrast, we obtain a clean characterization of sequential equilibria for fixed discount rates using ordinary differential equations, by setting the model in continuous time and restricting attention to a single behavioral type. Using the characterization, we find that a rich equilibrium dynamics arises when the

[1] For this result, Cripps, Mailath, and Samuelson (2004) assume that: (i) the public monitoring technology has full support and satisfies an identifiability condition; (ii) there is a single non-behavioral type and finitely many behavioral types; (iii) the action of each behavioral type is not part of a static Nash equilibrium of the complete information game; (iv) the small players have a unique best-reply to the action of each behavioral type.

small players assign positive prior probability to the behavioral type, but not otherwise. Indeed, when the small players are certain that they are facing the normal type, we show that the only sequential equilibria of the continuous-time game are those which yield payoffs in the convex hull of the set of *static* Nash equilibria. By contrast, the possibility of behavioral types gives rise to non-trivial intertemporal incentives. In this case, we identify conditions for the sequential equilibrium to be unique and Markovian in the small players' posterior belief and examine when a reputation yields a positive value for the large player. Then, under the equilibrium uniqueness conditions, we examine the impact of the large player's patience on the equilibrium strategies and obtain a reputation result for equilibrium *behavior*, which strengthens the conclusion of the more standard reputation results concerning equilibrium *payoffs*. Finally, we extend some of our results to settings with multiple equilibria, Poisson signals and multiple behavioral types.

Our characterization relies on the recursive structure of repeated games with public signals and on stochastic calculus. Both in discrete time and in continuous time, sequential equilibria in public strategies—hereafter, *public sequential equilbria*—can be described by the stochastic law of motion of two state variables: the large player's reputation and his continuation value. The law of motion of the large player's reputation is determined by Bayesian updating, while the evolution of his continuation value is characterized by promise keeping and incentive constraints. If and only if the evolution of the state variables satisfies these restrictions (as well as the condition that the continuation values are bounded), there is a public sequential equilibrium corresponding to these state variables. It follows that the set of equilibrium belief-continuation value pairs can be characterized as the greatest bounded set inside which the state variables can be kept while respecting Bayes' rule and the promise keeping and incentive constraints (Theorem 2).[2]

In continuous time, it is possible to take a significant step further and use stochastic calculus to connect the equilibrium law of motion of the state variables with the *geometry* of the set of equilibrium beliefs and payoffs. This insight—introduced in Sannikov (2007) in the context of continuous-time games without uncertainty over types—allows us to characterize public sequential equilibria using differential equations.[3] We first use this insight to identify an interesting class of reputation games that have a unique sequential equilibrium. Our main sufficient condition for uniqueness is stated in terms of a family of auxiliary one-shot games in which the stage-game payoffs of the large player are adjusted by certain "reputational weights." When these auxiliary one-shot games have a unique Bayesian Nash equilibrium, we show that the reputation game must also have a unique public sequential equilibrium, which is Markovian in the large player's reputation and characterized by a second-order ordinary differential equation (Theorem 4).[4] We also provide sufficient conditions in terms of the primitives of the game, i.e. stage-game payoffs, drift and volatility (Proposition 4).

We use our characterization of unique Markovian equilibria to derive a number of interesting results about behavior. First, we show that when the large player's static Bayesian Nash equilibrium payoff increases in reputation, his sequential equilibrium payoff in the continuous-time game also increases in reputation. Second, while the normal type of large player benefits from imitating the behavioral type, in

[2]In discrete time, this is a routine extension of the recursive methods of Abreu, Pearce, and Stacchetti (1990) to repeated games with uncertainty over types.

[3]See also Sannikov (2008) for an application of this methodology to agency problems.

[4]Recall that in a Markov perfect equilibrium (cf. Maskin and Tirole (2001)) the equilibrium behavior is fully determined by the payoff-relevant state variable, which, in our reputation game, is the small players' posterior belief.

equilibrium this imitation is necessarily imperfect; otherwise, the public signals would be uninformative about the large player's type and imitation would have no value. Third, we find a square root law of substitution between the discount rate and the volatility of the noise: doubling the discount rate has the same effect on the equilibrium as rescaling the volatility matrix of the public signal by a factor of $\sqrt{2}$. Finally, we derive the following reputation result for equilibrium *behavior*: If the small players assign positive probability to a behavioral type committed to the Stackelberg action and the signals satisfy an identifiability condition, then, as the large player gets patient, the equilibrium strategy of the normal type approaches the Stackelberg action at every reputation level (Theorem 5).

By contrast, when the small players are certain that they are facing the normal type, we find that every equilibrium of the continuous-time game must be degenerate, i.e. a *static* Nash equilibrium must be played after every history (Theorem 3). This phenomenon has no counterpart in the discrete-time framework of Fudenberg and Levine (1992), where non-trivial equilibria of the complete information game are known to exist (albeit with payoffs bounded away from efficiency, as shown in Fudenberg and Levine (1994)). In discrete time, the large player's incentives to play a non-myopic best reply can be enforced by the threat of a punishment phase, which is triggered when the public signal about his hidden action is sufficiently "bad." However, such intertemporal incentives may unravel as actions become more frequent, as first demonstrated in a classic paper of Abreu, Milgrom, and Pearce (1991) using a game with Poisson signals. Such incentives also break down under Brownian signals, as in the repeated Cournot duopoly with flexible production of Sannikov and Skrzypacz (2007) and also in the repeated commitment game with long-run and short-run players of Fudenberg and Levine (2007). The basic intuition underlying these results is that, under some signal structures, when players take actions frequently the information they observe within each time period becomes excessively noisy, and so the statistical tests that trigger the punishment regimes produce false positives too often. However, this phenomenon arises only under some signal structures (Brownian signals and "good news" Poisson signals), as shown in contemporaneous work of Fudenberg and Levine (2007) and Sannikov and Skrzypacz (2010). We further discuss the discrete-time foundations of our equilibrium degeneracy result in Section 5 and in our concluding remarks in Section 10.

While a rich structure of intertemporal incentives can arise in equilibrium only when the prior probability on the behavioral type is positive, there is no discontinuity when the prior converges to zero: Under our sufficient conditions for uniqueness, as the reputation of the large player tends to zero, for any fixed discount rate the equilibrium behavior in the reputation game converges to the equilibrium behavior under complete information. In effect, the Markovian structure of reputational equilibria is closely related to the equilibrium degeneracy without uncertainty over types. When the stage game has a unique Nash equilibrium, the only equilibrium of the continuous-time game without uncertainty over types is the repetition of the static Nash equilibrium, which is trivially Markovian. In our setting with Brownian signals, continuous time prevents non-Markovian incentives created by rewards and punishments from enhancing the incentives naturally created by reputation dynamics.

We go beyond games with a unique Markov perfect equilibrium and extend our characterization to more general environments with multiple sequential equilibria. Here, the object of interest is the correspondence of sequential equilibrium payoffs of the large player as a function of his reputation. In Theorem 6, we show that this correspondence is convex-valued and that its upper boundary is the greatest bounded solution of a *differential inclusion* (see, e.g., Aubin and Cellina (1984)), with an analogous characterization for the lower boundary. We provide a computed example illustrating the solutions to these differential inclusions.

While a major part of our analysis concerns the case of a single behavioral type, most of the work on reputation effects in discrete time examines the general case of multiple behavioral types, as in Fudenberg and Levine (1992) and Cripps, Mailath, and Samuelson (2004). To be consistent with this tradition, we feel compelled to shed some light on continuous-time games with multiple types, and so we extend our recursive characterization of public sequential equilibrium to this case, characterizing the properties that the reputation *vector* and the large player's continuation value must satisfy in equilibrium. With multiple types, however, we do not go to the next logical level to characterize equilibrium payoffs via *partial* differential equations, thus leaving this important extension for future research. Nevertheless, we use our recursive characterization of sequential equilibria under multiple behavioral types to prove an analogue of the Cripps, Mailath, and Samuelson (2004) result that the reputation effect is a short-lived phenomenon, i.e. eventually the small players learn when they are facing a normal type. The counterpart of the Fudenberg-Levine bounds on the equilibrium payoffs of the large type as he becomes patient, for continuous-time games with multiple types, is shown in Faingold (2008).

Finally, to provide a more complete analysis, we also address continuous-time games with Poisson signals and extend many of our results to those games. However, as we know from Abreu, Milgrom, and Pearce (1991), Fudenberg and Levine (2007), Fudenberg and Levine (2009) and Sannikov and Skrzypacz (2010), Brownian and Poisson signals have different informational properties in games with frequent actions or in continuous time. As a result, our equilibrium uniqueness result extends only to games in which Poisson signals are "good news." Instead, when the Poisson signals are "bad news," multiple equilibria are possible even under the most restrictive conditions on payoffs that yield uniqueness in the Brownian and in the Poisson good news case.

The rest of the paper is organized as follows. Section 2 presents an example in which a firm cares about her reputation concerning the quality of her product. Section 3 introduces the continuous-time model with Brownian signals and a single behavioral type. Section 4 provides the recursive characterization of public sequential equilibria. Section 5 examines the underlying complete information game. Section 6 presents the ODE characterization when the sequential equilibrium is unique, along with the reputation result for equilibrium strategies and the sufficient conditions for uniqueness in terms of the primitives of the game. Section 7 extends the characterization to games with multiple sequential equilibria. Section 8 deals with multiple behavioral types. Section 9 considers games with Poisson signals and proves the equilibrium uniqueness and characterization result for the case in which the signals are good news. Section 10 concludes by drawing analogies between continuous and discrete time.

2 Example: product choice

Consider a firm who provides a service to a continuum of identical consumers. At each time $t \in [0, \infty)$, the firm exerts a costly effort, $a_t \in [0, 1]$, which affects the quality of the service provided, and each consumer $i \in [0, 1]$ chooses a level of service to consume, $b_t^i \in [0, 3]$. The firm does not observe each consumer individually but only the aggregate level of service in the population of consumers, denoted $\bar{b}_t$. Likewise, the consumers do not observe the firm's effort level; instead, they publicly observe the quality of the service, dX_t, which is a noisy signal of the firm's effort and follows

$$dX_t = a_t dt + dZ_t,$$

where $(Z_t)_{t\geq 0}$ is a standard Brownian motion. The unit price for the service is exogenously fixed and normalized to one. The discounted profit of the firm and the overall surplus of consumer i are, respectively,

$$\int_0^\infty re^{-rt}(\bar{b}_t - a_t)\,dt \qquad \text{and} \qquad \int_0^\infty re^{-rt}(b_t^i(4-\bar{b}_t)\,dX_t - b_t^i\,dt),$$

where $r > 0$ is the discount rate. Thus, the payoff function of the consumers features a *negative externality*: greater usage $\bar{b}_t$ of the service by other consumers leads each consumer to enjoy the service less. This feature is meant to capture a situation in which the quality of the service is adversely affected by congestion, as in the case of Internet service providers.

Note that in every equilibrium of the continuous-time game the consumers must optimize *myopically*, i.e. they must act to maximize their expected instantaneous payoff. This is because the firm can only observe the *aggregate* consumption in the population, so no individual consumer can have an impact on future play.

In the unique static Nash equilibrium, the firm exerts zero effort and the consumers choose the lowest level of service to consume. In Section 5, we show that the unique equilibrium of the continuous-time repeated game is the repeated play of this static equilibrium, irrespective of the discount rate r. Thus, it is impossible for the consumers to create intertemporal incentives for the firm to exert effort, even when the firm is patient and despite her behavior being *statistically identified*—i.e. different effort levels induce different drifts for the quality signal X_t. This stands in sharp contrast to the standard setting of repeated games in discrete time, which is known to yield a great multiplicity of non-static equilibria when the non-myopic players are sufficiently patient (Fudenberg and Levine, 1994). However, if the firm were able to commit to any effort level $a^* \in [0,1]$, this commitment would influence the consumers' choices, and hence the firm could earn a higher profit. Indeed, each consumer's choice, b^i, would maximize their expected flow payoff, $b^i(a^*(4-\bar{b})-1)$, and in equilibrium all consumers would choose the same level $b^* = \max\{0, 4-1/a^*\}$. The service provider would then earn a profit of $\max\{0, 4-1/a^*\} - a^*$, and at $a^* = 1$ this function achieves its maximal value of 2, the firm's *Stackelberg payoff*.

Thus, following Fudenberg and Levine (1992), it is interesting to explore the repeated game with *reputation effects*. Assume that at time zero the consumers believe that with probability $p \in (0,1)$ the firm is a *behavioral type*, who always chooses effort level $a^* = 1$, and with probability $1-p$ the firm is a *normal type*, who chooses a_t to maximize her expected discounted profit. What happens in equilibrium?

The top panel of Figure 1 displays the unique sequential equilibrium payoff of the normal type as a function of the population's belief p, for different discount rates r. In equilibrium the consumers continually update their posterior belief ϕ_t— the probability assigned to the behavioral type—using the observations of the public signal X_t. The equilibrium is Markovian in ϕ_t, which uniquely determines the equilibrium actions of the normal type (bottom left panel) and the consumers (bottom right panel). Consistent with the bounds on equilibrium payoffs obtained in Faingold (2008), which extends the reputation bounds of Fudenberg and Levine (1992) to continuous time, the computation shows that as $r \to 0$ the large player's payoff converges to the Stackelberg payoff of 2. We can also see from Figure 1 that the aggregate consumption in the population, $\bar{b}$, increases towards the commitment level of 3 as the discount rate r decreases towards 0. While the normal type chooses action 0 for all levels of ϕ_t when $r = 2$, as r is closer to 0 his action increases towards the Stackelberg action $a^* = 1$. However, the "imitation" of the behavioral type by the normal type is never perfect, even for very low discount rates.

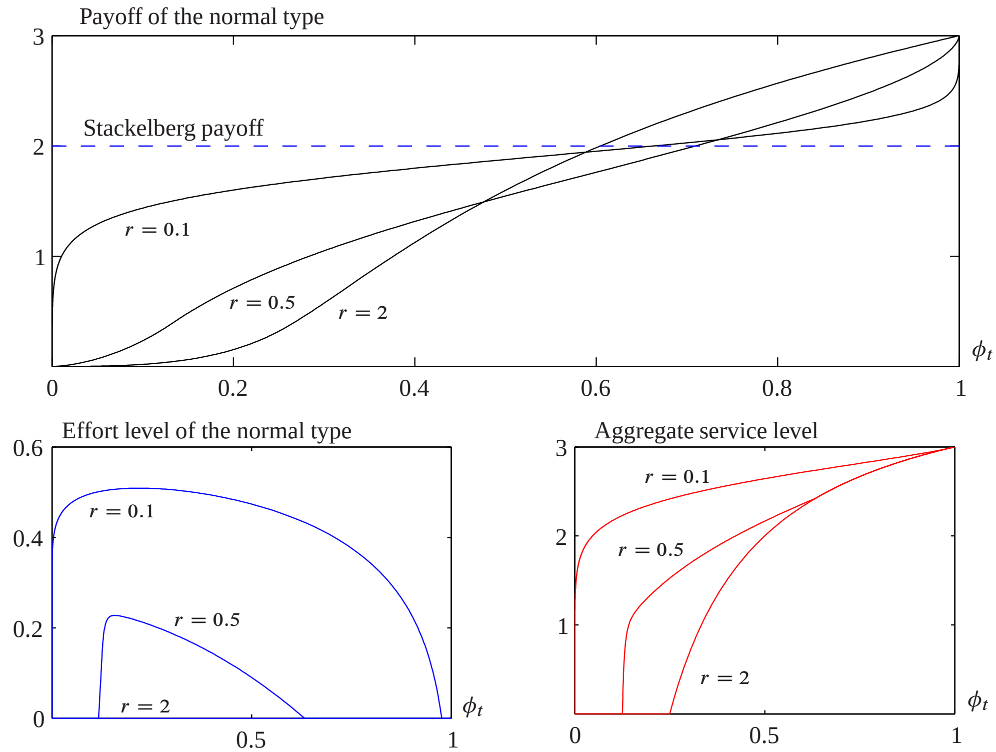

Figure 1: Equilibrium payoffs and actions in the product choice game.

3 The reputation game

A *large player* faces a continuum of *small players* in a continuous-time repeated game. At each time $t \in [0, \infty)$, the large player chooses an action $a_t \in A$ and each small player $i \in I \stackrel{\text{def}}{=} [0, 1]$ chooses an action $b_t^i \in B$, where the action spaces A and B are compact subsets of an Euclidean space. The small players are *anonymous*: at each time t the public information includes the aggregate distribution of the small players' actions, $\bar{b}_t \in \Delta(B)$, but not the action of any individual small player.[5] We assume that the actions of the large player are not directly observable by the small players. Instead, there is a noisy public signal $(X_t)_{t \geq 0}$, whose evolution depends on the actions of the large player, the aggregate distribution of the small players' actions and noise. Specifically,

$$dX_t = \mu(a_t, \bar{b}_t)\, dt + \sigma(\bar{b}_t)\, dZ_t,$$

where $(Z_t)_{t \geq 0}$ is a d-dimensional Brownian motion and the drift and volatility coefficients are determined by Lipschitz continuous functions $\mu : A \times B \to \mathbb{R}^d$ and $\sigma : B \to \mathbb{R}^{d \times d}$, which are linearly extended to

[5] The aggregate distribution over the small players' actions is the probability distribution $\bar{b}_t \in \Delta(B)$ such that $\bar{b}_t(B') = \int_{\{i : b_t^i \in B'\}} di$ for each Borel measurable subset $B' \subseteq B$.

$A \times \Delta(B)$ and $\Delta(B)$, respectively.[6,7] For technical reasons, we assume that there is a constant $c > 0$ such that $|\sigma(b)y| \geq c|y|$ for all $y \in \mathbb{R}^d$ and $b \in B$. We write $(\mathcal{F}_t)_{t\geq 0}$ to designate the filtration generated by $(X_t)_{t\geq 0}$.

The small players have identical preferences.[8] The payoff of player $i \in I$ depends on his own action, the aggregate distribution of the small players' actions and the action of the large player:

$$\int_0^\infty re^{-rt} h(a_t, b_t^i, \bar{b}_t)\, dt,$$

where $r > 0$ is the discount rate and $h : A \times B \times B \to \mathbb{R}$ is a continuous function, which is linearly extended to $A \times B \times \Delta(B)$. As is standard in the literature on imperfect public monitoring, we assume that the small players do not gather any information from their payoff flow beyond the information conveyed in the public signal. An important case is when $h(a_t, b_t^i, \bar{b}_t)$ is the *expected* payoff flow of player i, whereas his *ex post* payoff flow is privately observable and depends only on his own action, the aggregate distribution of the small players' actions, and the flow, dX_t, of the public signal. In this case, the *ex post* payoff flow of player i takes the form

$$u(b_t^i, \bar{b}_t)\, dt \;+\; v(b_t^i, \bar{b}_t) \cdot dX_t,$$

so that

$$h(a, b^i, \bar{b}) = u(b^i, \bar{b}) + v(b^i, \bar{b}) \cdot \mu(a, \bar{b}), \tag{1}$$

as in the product choice game of Section 2. While this functional form is natural in many applications, none of our results hinges on it, so for the sake of generality we take the payoff function $h : A \times B \times B \to \mathbb{R}$ to be a primitive and do not impose (1).

While the small players' payoff function is common knowledge, there is uncertainty about the type θ of the large player. At time $t = 0$ they believe that with probability $p \in [0, 1]$ the large player is a *behavioral type* ($\theta = \mathsf{b}$) and that with probability $1 - p$ he is a *normal type* ($\theta = \mathsf{n}$). The behavioral type plays a fixed action $a^* \in A$ at all times, irrespective of history. The normal type plays strategically to maximize the expected value of his discounted payoff,

$$\int_0^\infty re^{-rt}\, g(a_t, \bar{b}_t)\, dt\,,$$

where $g : A \times B \to \mathbb{R}$ is a Lipschitz continuous function, which is linearly extended to $A \times \Delta(B)$.

A *public strategy* of the normal type of the large player is a random process $(a_t)_{t\geq 0}$ with values in A and progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$. Similarly, a public strategy of small player $i \in I$ is a progressively measurable process $(b_t^i)_{t\geq 0}$ taking values in B. In the repeated game the small players

[6]Functions μ and σ are extended to distributions over B via $\mu(a, \bar{b}) = \int_B \mu(a, b)\, d\bar{b}(b)$ and $\sigma(\bar{b})\sigma(\bar{b})^\top = \int_B \sigma(b)\sigma(b)^\top d\bar{b}(b)$.

[7]The assumption that the volatility of $(X)_{t\geq 0}$ is independent of the large player's actions corresponds to the *full support* assumption that is standard in discrete-time repeated games. By Girsanov's Theorem (Karatzas and Shreve, 1991, p. 191) the probability measures over the sample paths of two diffusion processes with the same volatility coefficient but different bounded drifts are *mutually absolutely continuous*, i.e. they have the same zero-probabilty events. Since the volatility of a continuous-time diffusion is effectively observable, we do not allow σ to depend on a_t.

[8]All our results can be extended to a setting where the small players observe the same public signal, but have heterogeneous preferences.

formulate a belief about the large player's type following their observations of $(X_t)_{t\geq 0}$. A *belief process* is a progressively measurable process $(\phi_t)_{t\geq 0}$ taking values in $[0,1]$, where ϕ_t designates the probability that the small players assign at time t to the large player being the behavioral type.

Definition 1. A *public sequential equilibrium* consists of a public strategy $(a_t)_{t\geq 0}$ of the normal type of the large player, a public strategy $(b^i_t)_{t\geq 0}$ for each small player $i \in I$ and a belief process $(\phi_t)_{t\geq 0}$ such that at all times $t \geq 0$ and after all public histories,

(a) the strategy of the normal type of the large player maximizes his expected payoff

$$\mathsf{E}_t\left[\int_0^\infty re^{-rs}\, g(a_s,\bar{b}_s)\,ds \;\Big|\; \theta=\mathsf{n}\right],$$

(b) the strategy of each small player i maximizes his expected payoff

$$(1-\phi_t)\mathsf{E}_t\left[\int_0^\infty re^{-rs}\, h(a_s,b^i_s,\bar{b}_s)\,ds \;\Big|\; \theta=\mathsf{n}\right] + \phi_t\,\mathsf{E}_t\left[\int_0^\infty re^{-rs}\, h(a^*,b^i_s,\bar{b}_s)\,ds \;\Big|\; \theta=\mathsf{b}\right]$$

(c) beliefs $(\phi_t)_{t\geq 0}$ are determined by Bayes rule given the common prior $\phi_0 = p$.

A strategy profile satisfying conditions (a) and (b) is called *sequentially rational.* A belief process $(\phi_t)_{t\geq 0}$ satisfying condition (c) is called *consistent*.

This definition can be simplified in two ways. First, because the small players have identical preferences, any strategy profile obtained from a public sequential equilibrium by a permutation of the small players' labels remains a public sequential equilibrium. Given this immaterial indeterminacy, we shall work directly with the aggregate behavior strategy $(\bar{b}_t)_{t\geq 0}$ rather than with the individual strategies $(b^i_t)_{t\geq 0}$. Second, in any public sequential equilibrium the small players' strategies must be *myopically* optimal, for their individual behavior is not observed by any other player in the game and it cannot influence the evolution of the public signal. Thus, with slight abuse of notation, we will say that a tuple $(a_t,\bar{b}_t,\phi_t)_{t\geq 0}$ is a public sequential equilibrium when, for all $t \geq 0$ and after all public histories, conditions (a) and (c) are satisfied as well as the myopic incentive constraint

$$b \in \operatorname*{arg\,max}_{b'\in B}\; (1-\phi_t)\,h(a_t,b',\bar{b}_t) \;+\; \phi_t\,h(a^*,b',\bar{b}_t) \quad \forall b \in \operatorname{supp}\bar{b}_t.$$

Finally, we make the following remarks concerning the definition of strategies and the solution concept:

Remark 1. Since the aggregate distribution over the small players' actions is publicly observable, the above definition of public strategies is somewhat non-standard, in that it requires the behavior to depend only on the sample path of $(X_t)_{t\geq 0}$. Notice, however, that in our game this restricted definition of public strategies incurs no loss of generality. For a given strategy profile, the public histories along which there are observations of $\bar{b}_t$ that differ from those on the path of play correspond to deviations by a positive measure of small players. Since the play that follows such joint deviations is irrelevant for equilibrium incentives our restricted definition of public strategies does not alter the set of public sequential equilibrium outcomes.

Remark 2. The framework can be extended to accommodate public sequential equilibria in mixed strategies. A mixed public strategy of the large player is a random process $(\bar{a}_t)_{t\geq 0}$ progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ with values in $\Delta(A)$. To accommodate mixed strategies, the payoff functions $g(\cdot, \bar{b})$ and $h(\cdot, b^i, \bar{b})$ and the drift $\mu(\cdot, \bar{b})$ are linearly extended to $\Delta(A)$. Because there is a continuum of anonymous small players, the assumption that each of them plays a pure strategy is without loss of generality.

Remark 3. For both pure- and mixed-strategy equilibria, the restriction to public strategies is without loss of generality. For pure strategies, it is redundant to condition a player's current action on his private history, as every private strategy is outcome-equivalent to a public strategy. For mixed strategies, the restriction to public strategies is without loss of generality in repeated games with signals that have a *product structure*, as in the repeated games that we consider.[9] To form a belief about his opponent's private histories, in a game with product structure a player can ignore his own past actions because they do not influence the signal about his opponent's actions. Formally, a mixed private strategy of the large player in our game would be a random process $(a_t)_{t\geq 0}$ with values in A and progressively measurable with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$, which is generated by the public signals and the large player's private randomization. For any private strategy of the large player, an equivalent mixed public strategy can then be defined by letting $\bar{a}_t$ be the conditional distribution of a_t given $\mathcal{F}_t$. Strategies $(a_t)_{t\geq 0}$ and $(\bar{a}_t)_{t\geq 0}$ induce the same probability distributions over public signals and hence give the large player the same expected payoff conditional on $\mathcal{F}_t$.

3.1 Relation to the literature

At this point the reader may be wondering how exactly our continuous-time formulation relates to the canonical model of Fudenberg and Levine (1992). Besides continuous vs. discrete time, a fundamental difference is that we assume the existence of a single behavioral type, while in the Fudenberg and Levine (1992) model the type space of the large player is significantly more general, including multiple behavioral types and, possibly, arbitrary non-behavioral types. Another distinction is that in our model the uninformed players are infinitely-lived small anonymous players, whereas in each period of the canonical model there is a single uninformed player who lives only in that period. But, in this dimension our formulation nests the canonical model, since when the payoff $h(a, b^i, \bar{b})$ is independent of $\bar{b}$ our model is formally equivalent to one in which there is a continuous flow of uninformed players, each of whom lives only for an instant of time. In this case, $b^i = \bar{b}$ and therefore each individual uninformed player can influence the evolution of the public signal—both drift and volatility—, just as in the canonical model.

Moreover, the goal of our analysis is conceptually different from that of Fudenberg and Levine (1992). While their main result determines upper and lower *bounds* on the equilibrium *payoffs* of the normal type which hold in the *limit* as he gets arbitrarily patient, we provide a *characterization* of sequential equilibrium—both *payoffs* and *behavior*—under a *fixed* rate of discounting. For this reason, it is not surprising that in some dimensions our assumptions are more restrictive than the assumptions in Fudenberg and Levine (1992).

[9] A public monitoring technology has a *product structure* if each public signal is controlled by exactly one large player and the public signals corresponding to different large players are conditionally independent given the action profile—cf. Fudenberg and Levine (1994, Section 5). Since our reputation game has only one large player, this condition holds trivially.

Although our focus is on equilibrium characterization for fixed discount rates, it is worth noting that the Fudenberg-Levine limit bounds on equilibrium payoffs have a counterpart in our continuous-time setting, as shown in Faingold (2008). For each $a \in A$, let $\mathsf{B}(a)$ designate the set of Nash equilibria of the partial game between the small players given some action that is *observationally equivalent* to a, i.e.

$$\mathsf{B}(a) \stackrel{\text{def}}{=} \left\{ \bar{b} \in \Delta(B) : \begin{array}{l} \exists \tilde{a} \in A \text{ such that } \mu(\tilde{a}, \bar{b}) = \mu(a, \bar{b}) \text{ and} \\ b \in \arg\max_{b' \in B} h(\tilde{a}, b', \bar{b}) \ \forall b \in \operatorname{supp} \bar{b} \end{array} \right\}.$$

The following theorem is similar to Faingold (2008, Theorem 3.1):[10]

Theorem 1 (Reputation Effect)**.** *For every $\varepsilon \in (0,1)$ and $\delta > 0$ there exists $\bar{r} > 0$ such that in every public sequential equilibrium of the reputation game with prior $p \in [\varepsilon, 1-\varepsilon]$ and discount rate $r \in (0, \bar{r}]$, the expected payoff of the normal type is bounded below by*

$$\min_{\bar{b} \in \mathsf{B}(a^*)} g(a^*, \bar{b}) \ - \ \delta,$$

and bounded above by

$$\max_{a \in A} \max_{\bar{b} \in \mathsf{B}(a)} g(a, \bar{b}) \ + \ \delta.$$

The *upper bound*, $\bar{g}^{\mathsf{s}} \stackrel{\text{def}}{=} \max_{a \in A} \max_{\bar{b} \in \mathsf{B}(a)} g(a, \bar{b})$, is the *generalized Stackelberg payoff*. It is the greatest payoff that a large player with commitment power can get, taking into account the limited observability of the large players' actions and the incentives that arise in the partial game among the small players. Perhaps more interesting is the *lower bound*, $\min_{\bar{b} \in \mathsf{B}(a^*)} g(a^*, \bar{b})$, which can be quite high in some games. Of particular interest are games in which it approximates

$$\underline{g}^{\mathsf{s}} \stackrel{\text{def}}{=} \sup_{a \in A} \min_{\bar{b} \in \mathsf{B}(a)} g(a, \bar{b}),$$

the greatest possible lower bound. While the supremum above may not be attained, given $\varepsilon > 0$ we can find some $a^* \in A$ such that $\min_{\bar{b} \in \mathsf{B}(a^*)} g(a^*, \bar{b})$ lies within ε of $\underline{g}^{\mathsf{s}}$. Thus, if the behavioral type plays such a^*, there is a lower bound on equilibrium payoffs converging to $\underline{g}^{\mathsf{s}} - \varepsilon$ as $r \to 0$.[11]

It is therefore natural to examine when the reputation bounds, $\underline{g}^{\mathsf{s}}$ and $\bar{g}^{\mathsf{s}}$, coincide. A wedge may arise for two reasons: (i) there may be multiple actions that are observationally equivalent to some action a, under some $\bar{b}$; and (ii) the partial game among the small players may have multiple static Nash equilibria. Recall that also in Fudenberg and Levine (1992) the upper and lower bounds may be different. However, in their setting the indeterminacy is less severe, for in their model there is a single myopic player in each period, and hence (ii) is just the issue of multiplicity of optima in a single-agent programming problem,

[10] Faingold (2008) examines reputation effects in continuous-time games with a general type space for the large player, as in Fudenberg and Levine (1992), and public signals that follow a controlled Lévy process. Although Faingold (2008) does not consider cross-section populations of small players, a straightforward extension of the proof of Faingold (2008, Theorem 3.1) can be used to prove Theorem 1 above.

[11] When the support of the prior contains all possible behavioral types, as in Fudenberg and Levine (1992) and Faingold (2008), there is a lower bound on the equilibrium payoffs of the normal type which converges exactly to $\underline{g}^{\mathsf{s}}$. In contrast, when there is only a finite set of behavioral types—e.g., as in the current paper and in Cripps, Mailath, and Samuelson (2004)—, the lower bound maybe strictly less than $\underline{g}^{\mathsf{s}}$.

a less severe problem than the issue of multiplicity of Nash equilibria in a game. Indeed, Fudenberg and Levine (1992) show that when the action sets are finite, the upper and lower bounds must be equal under the following assumptions:

- the actions of the large player are *identified*, i.e. there do not exist $\bar{a}, \bar{a}' \in \Delta(A)$, with $\bar{a} \neq \bar{a}'$, and $\bar{b} \in \Delta(B)$ such that $(\bar{a}, \bar{b})$ and $(\bar{a}', \bar{b})$ generate the same distribution over public signals;
- the payoff matrix of the small players is *non-degenerate*, i.e. there do not exist $b \in B$ and $\bar{b} \in \Delta(B)$ such that $h(\cdot, \bar{b}) = h(\cdot, b)$ and $b \neq \bar{b}$.[12]

However, in our setting these assumptions do not generally imply $\bar{g}^{\mathsf{s}} = \underline{g}^{\mathsf{s}}$, due to the externality across the small players that we allow.

4 The structure of public sequential equilibrium

This section develops a recursive characterization of public sequential equilibria which is used through the rest of the paper. Recall that in a public sequential equilibrium, beliefs must be consistent with the public strategies and the strategies must be sequentially rational given beliefs. For the consistency of beliefs, Proposition 1 presents equation (2), which describes how the small players' posterior evolves with the public signal $(X_t)_{t \geq 0}$. The sequential rationality of the normal type can be verified by examining the evolution of his *continuation value*, i.e. his expected future discounted payoff given the history of public signals up to time t. First, Proposition 2 presents a necessary and sufficient condition for a random process $(W_t)_{t \geq 0}$ to be the process of continuation values of the normal type. Then, Proposition 3 characterizes sequential rationality using a condition that is connected to the law of motion of $(W_t)_{t \geq 0}$. Propositions 2 and 3 are analogous to Propositions 1 and 2 of Sannikov (2007).

We begin with Proposition 1, which characterizes the stochastic evolution of the small players' posterior beliefs.[13]

Proposition 1 (Belief Consistency)**.** *Fix the prior $p \in [0,1]$ on the behavioral type. A belief process $(\phi_t)_{t \geq 0}$ is consistent with a public strategy profile $(a_t, \bar{b}_t)_{t \geq 0}$ if and only if $\phi_0 = p$ and*

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t) \cdot \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\, dt), \tag{2}$$

where for each $(a, \bar{b}, \phi) \in A \times \Delta(B) \times [0,1]$,

$$\gamma(a, \bar{b}, \phi) \stackrel{\text{def}}{=} \phi(1-\phi)\sigma(\bar{b})^{-1}\left(\mu(a^*, \bar{b}) - \mu(a, \bar{b})\right),$$

$$\mu^{\phi}(a, \bar{b}) \stackrel{\text{def}}{=} \phi\mu(a^*, \bar{b}) + (1-\phi)\mu(a, \bar{b}).$$

[12] To be precise, in Fudenberg and Levine (1992) the upper and lower bounds are slightly different from $\bar{g}^{\mathsf{s}}$ and $\underline{g}^{\mathsf{s}}$. Since they assume that the support of the prior contains all possible behavioral types, the myopic players never play a weakly dominated action. Accordingly, they define $\mathsf{B}_0(a)$ to be the set of all *undominated* actions $\bar{b}$ which are a best-reply to some action that is observationally equivalent to a under $\bar{b}$. Then, the definitions of the upper and lower bounds are similar to those of $\bar{g}^{\mathsf{s}}$ and $\underline{g}^{\mathsf{s}}$, but with $\mathsf{B}_0(a)$ replacing $\mathsf{B}(a)$. This turns out to be crucial for their proof that the upper and lower bounds coincide under the identifiability and non-degeneracy conditions above.

[13] Similar versions of the filtering equation (2) have been used in the literature on strategic experimentation in continuous time (cf. Bolton and Harris (1999), Keller and Rady (1999) and Moscarini and Smith (2001)). For a general treatment of filtering in continuous time, see Liptser and Shiryaev (1977).

Proof. The strategy of each type of the large player induces a probability measure over the paths of the public signal $(X_t)_{t\geq 0}$. From Girsanov's Theorem we can find the ratio ξ_t between the likelihood that a path $(X_s; s \in [0,t])$ arises for type b and the likelihood that it arises for type n. This ratio is characterized by

$$d\xi_t = \xi_t\, \rho_t \cdot dZ_s^{\mathsf{n}}, \quad \xi_0 = 1\,, \tag{3}$$

where $\rho_t = \sigma(\bar{b}_t)^{-1}\left(\mu(a^*,\bar{b}_t) - \mu(a_t,\bar{b}_t)\right)$ and $Z_t^{\mathsf{n}} = \int_0^t \sigma(\bar{b}_s)^{-1}(dX_s - \mu(a_s,\bar{b}_s)\,ds)$ is a Brownian motion under the probability measure generated by the strategy of type n.

Suppose that $(\phi_t)_{t\geq 0}$ is consistent with $(a_t,\bar{b}_t)_{t\geq 0}$. Then, by Bayes' rule, the posterior after observing a path $(X_s; s\in[0,t])$ is

$$\phi_t = \frac{p\xi_t}{p\xi_t + (1-p)}. \tag{4}$$

By Itô's formula,

$$\begin{aligned} d\phi_t &= \frac{p(1-p)}{(p\xi_t + (1-p))^2}\, d\xi_t - \frac{2p^2(1-p)}{(p\xi_t+(1-p))^3}\, \frac{\xi_t^2 \rho_t \cdot \rho_t}{2}\, dt \\ &= \phi_t(1-\phi_t)\rho_t \cdot dZ_t^{\mathsf{n}} - \phi_t^2(1-\phi_t)\rho_t\cdot\rho_t\, dt \\ &= \phi_t(1-\phi_t)\rho_t \cdot \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t,\bar{b}_t)\,dt), \end{aligned} \tag{5}$$

which is equation (2).

Conversely, suppose that $(\phi_t)_{t\geq 0}$ solves equation (2) with initial condition $\phi_0 = p$. Define ξ_t using expression (4), i.e.

$$\xi_t = \frac{1-p}{p}\frac{\phi_t}{1-\phi_t}.$$

Then, applying Itô's formula to the expression above gives equation (3), hence ξ_t must equal the ratio between the likelihood that a path $(X_s; s\in[0,t])$ arises for type b and the likelihood that it arises for type n. Thus, ϕ_t is determined by Bayes' rule and the belief process is consistent with $(a_t,\bar{b}_t)_{t\geq 0}$. ■

Note that in the statement of Proposition 1, $(a_t)_{t\geq 0}$ is the strategy that the small players believe that the normal type is following. Thus, when the normal type deviates from his equilibrium strategy, the deviation affects only the drift of $(X_t)_{t\geq 0}$, but not the other terms in equation (2).

Coefficient γ of equation (2) is the volatility of beliefs: it reflects the speed with which the small players learn about the type of the large player. The definition of γ plays an important role in the characterization of public sequential equilibrium presented in Sections 6 and 7. The intuition behind equation (2) is as follows. If the small players are convinced about the type of the large player, then $\phi_t(1-\phi_t) = 0$, so they never change their beliefs. When $\phi_t \in (0,1)$ then $\gamma(a_t,\bar{b}_t,\phi_t)$ is larger, and learning is faster, when the noise $\sigma(\bar{b}_t)$ is smaller or the drifts produced by the two types differ more. From the small players' perspective, the noise driving equation (2), $\sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t,\bar{b}_t)\,dt)$, is a standard Brownian motion and their belief $(\phi_t)_{t\geq 0}$ is a martingale. From equation (5) we see that, conditional on the large player being the normal type, the drift of ϕ_t is non-positive: in the long run, either the small players learn that they are facing the normal type, or the normal type plays like the behavioral type.

We turn to the analysis of the second important state variable in the strategic interaction between the large player and the small players: the *continuation value* of the normal type, i.e. his future expected discounted payoff following a public history, under a given strategy profile. More precisely, given a strategy profile $S = (a_s, \bar{b}_s)_{s \geq 0}$, the continuation value of the normal type at time t is

$$W_t(S) \stackrel{\text{def}}{=} \mathsf{E}_t \left[\int_t^\infty re^{-r(s-t)} g(a_s, \bar{b}_s) \, ds \;\middle|\; \theta = \mathsf{n} \right]. \tag{6}$$

The following proposition characterizes the law of motion of $W_t(S)$. Throughout the paper, we write $\mathcal{L}$ to designate the space of $\mathbb{R}^d$-valued progressively measurable processes $(\beta_t)_{t \geq 0}$ with $\mathsf{E}\big[\int_0^T |\beta_t|^2 \, dt\big] < \infty$ for all $0 < T < \infty$.

Proposition 2 (Continuation Values)**.** *A bounded process $(W_t)_{t \geq 0}$ is the process of continuation values of the normal type under a public strategy profile $S = (a_t, \bar{b}_t)_{t \geq 0}$ if and only if for some $(\beta_t)_{t \geq 0}$ in $\mathcal{L}$,*

$$dW_t = r(W_t - g(a_t, \bar{b}_t)) \, dt \; + \; r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t) \, dt). \tag{7}$$

Proof. First, note that $W_t(S)$ is a bounded process by (6). Let us show that $W_t = W_t(S)$ satisfies (7) for some $(\beta_t)_{t \geq 0}$ in $\mathcal{L}$. Denote by $V_t(S)$ the expected discounted payoff of the normal type conditional on the public information at time t, i.e.

$$V_t(S) \stackrel{\text{def}}{=} \mathsf{E}_t \left[\int_0^\infty re^{-rs} g(a_s, \bar{b}_s) \, ds \;\middle|\; \theta = \mathsf{n} \right] = \int_0^t re^{-rs} g(a_s, \bar{b}_s) \, ds + e^{-rt} W_t(S) \tag{8}$$

Thus, $(V_t)_{t \geq 0}$ is a martingale when the large player is the normal type. By the Martingale Representation Theorem, there exists $(\beta_t)_{t \geq 0}$ in $\mathcal{L}$ such that

$$dV_t(S) = re^{-rt} \beta_t^\top \sigma(\bar{b}_t) \, dZ_t^{\mathsf{n}}, \tag{9}$$

where $dZ_t^{\mathsf{n}} = \sigma(\bar{b}_t)^{-1}(dX_t - \mu(a_t, \bar{b}_t) \, dt)$ is a Brownian motion when the large player is the normal type. Differentiating (8) with respect to t yields

$$dV_t(S) = re^{-rt} g(a_t, \bar{b}_t) \, dt \; - \; re^{-rt} W_t(S) \, dt \; + \; e^{-rt} dW_t(S). \tag{10}$$

Combining equations (9) and (10) yields (7), which is the desired result.

Conversely, let us show that if $(W_t)_{t \geq 0}$ is a bounded process satisfying (7) then $W_t = W_t(S)$. Indeed, when the large player is the normal type, the process

$$V_t = \int_0^t re^{-rs} g(a_s, \bar{b}_s) \, ds + e^{-rt} W_t$$

is a martingale under the strategy profile $S = (a_t, \bar{b}_t)$, because $dV_t = re^{-rt} \beta_t^\top \sigma(\bar{b}_t) \, dZ_t^{\mathsf{n}}$ by (7). Moreover, as $t \to \infty$ the martingales V_t and $V_t(S)$ converge because both $e^{-rt} W_t$ and $e^{-rt} W_t(S)$ tend to 0. Therefore,

$$V_t = \mathsf{E}_t[V_\infty] = \mathsf{E}_t[V_\infty(S)] = V_t(S) \quad \Rightarrow \quad W_t = W_t(S)$$

for all t, as required. ■

Thus, equation (7) describes how $W_t(S)$ evolves with the public history. Note that this equation must hold regardless of the large player's actions before time t. This fact is used in the proof of Proposition 3 below, which deals with incentives.

We finally turn to conditions for sequential rationality. The condition for the small players is straightforward: they maximize their static payoff because a deviation of an individual small player cannot influence the future course of equilibrium play. The characterization of the large player's incentive constraints is more complicated: the normal type acts optimally if he maximizes the sum of his current flow payoff and the expected change in his continuation value.

Proposition 3 (Sequential Rationality)**.** *A public strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ is sequentially rational with respect to a belief process $(\phi_t)_{t\geq 0}$ if and only if there exists $(\beta_t)_{t\geq 0}$ in $\mathcal{L}$ and a bounded process $(W_t)_{t\geq 0}$ satisfying* (7)*, such that for all $t \geq 0$ and after all public histories,*

$$a_t \in \operatorname*{arg\,max}_{a' \in A} \; g(a', \bar{b}_t) + \beta_t \cdot \mu(a', \bar{b}_t), \tag{11}$$

$$b \in \operatorname*{arg\,max}_{b' \in B} \; \phi_t h(a^*, b', \bar{b}_t) + (1-\phi_t) h(a_t, b', \bar{b}_t), \quad \forall b \in \operatorname{supp} \bar{b}_t. \tag{12}$$

Proof. Consider a strategy profile $(a_s, \bar{b}_s)_{s\geq 0}$ and an alternative strategy $(\tilde{a}_s)_{s\geq 0}$ for the normal type. Denote by W_t the continuation payoff of the normal type at time t when he follows strategy $(a_s)_{s\geq 0}$ after time t, while the population follows $(\bar{b}_s)_{s\geq 0}$. If the normal type plays $(\tilde{a}_s)_{s\geq 0}$ up to time t and then switches to $(a_s)_{s\geq 0}$, his expected payoff conditional on the public information at time t is given by

$$\tilde{V}_t = \int_0^t r e^{-rs} g(\tilde{a}_s, \bar{b}_s)\, ds \; + \; e^{-rt} W_t \, .$$

By Proposition 2, the continuation values $(W_t)_{t\geq 0}$ follow equation (7) for some $(\beta_t)_{t\geq 0}$ in $\mathcal{L}$. Thus, the above expression for $\tilde{V}_t$ implies

$$\begin{aligned} d\tilde{V}_t &= r e^{-rt} \left(g(\tilde{a}_t, \bar{b}_t) - W_t\right) dt \; + \; e^{-rt} dW_t \\ &= r e^{-rt} \left(\left(g(\tilde{a}_t, \bar{b}_t) - g(a_t, \bar{b}_t)\right) dt + \beta_t \cdot \left(dX_t - \mu(a_t, \bar{b}_t)\, dt\right)\right), \end{aligned}$$

and hence the profile $(\tilde{a}_t, \bar{b}_t)_{t\geq 0}$ yields the normal type an expected payoff of

$$\begin{aligned} \tilde{W}_0 &= \mathsf{E}[\tilde{V}_\infty] \; = \; \mathsf{E}\left[\tilde{V}_0 \; + \; \int_0^\infty d\tilde{V}_t\right] \; = \; W_0 \; + \; \mathsf{E}\left[\int_0^\infty d\tilde{V}_t\right] \\ &= W_0 \; + \; \mathsf{E}\left[\int_0^\infty r e^{-rt} \left(g(\tilde{a}_t, \bar{b}_t) - g(a_t, \bar{b}_t) \; + \; \beta_t \cdot \left(\mu(\tilde{a}_t, \bar{b}_t) - \mu(a_t, \bar{b}_t)\right) dt\right)\right], \end{aligned}$$

where the expectation is taken under the probability measure induced by $(\tilde{a}_t, \bar{b}_t)_{t\geq 0}$, so that $(X_t)_{\geq 0}$ has drift $\mu(\tilde{a}_t, \bar{b}_t)$.

Suppose that $(a_t, \bar{b}_t, \phi_t)_{t\geq 0}$ satisfies the incentive constraints (11) and (12). Then, for every $(\tilde{a}_t)_{t\geq 0}$ we have $W_0 \geq \tilde{W}_0$, and therefore the normal type must be sequentially rational at time $t = 0$. A similar argument can be used to show that the normal type is sequentially rational at all times $t > 0$, after all public histories. Finally, the small players must also be sequentially rational, since they are anonymous and hence myopic. Conversely, suppose that the incentive constraint (11) fails. Choose a strategy $(\tilde{a}_t)_{t\geq 0}$ satisfying (11) for all $t \geq 0$ and all public histories. Then $\tilde{W}_0 > W_0$, and hence the large player is not sequentially rational at $t = 0$. Likewise, if (12) fails, then a positive measure of small players would not be maximizing their instantaneous expected payoffs. Since the small players are anonymous, and hence myopic, their strategies would not be sequentially rational. ∎

We can summarize our characterization in the following theorem.

Theorem 2 (Sequential Equilibrium)**.** *Fix a prior $p \in [0,1]$ on the behavioral type. A strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ and a belief process $(\phi_t)_{t\geq 0}$ form a public sequential equilibrium with continuation values $(W_t)_{t\geq 0}$ for the normal type if and only if for some process $(\beta_t)_{t\geq 0}$ in $\mathcal{L}$,*

(a) $(\phi_t)_{t\geq 0}$ satisfies (2) with initial condition $\phi_0 = p$,

(b) $(W_t)_{t\geq 0}$ is a bounded process satisfying (7), given $(\beta_t)_{t\geq 0}$,

(c) $(a_t, \bar{b}_t)_{t\geq 0}$ satisfies the incentive constraints (11) and (12), given $(\beta_t)_{t\geq 0}$ and $(\phi_t)_{t\geq 0}$.

Thus, Theorem 2 provides a recursive characterization of public sequential equilibrium. Let $\mathcal{E} : [0,1] \rightrightarrows \mathbb{R}$ denote the correspondence that maps a prior probability on the behavioral type into the corresponding set of public sequential equilibrium payoffs of the normal type. An equivalent statement of Theorem 2 is that $\mathcal{E}$ is the greatest bounded correspondence such that a controlled process $(\phi_t, W_t)_{t\geq 0}$, defined by (2) and (7), can be kept in $\mathsf{Graph}(\mathcal{E})$ by controls $(a_t, \bar{b}_t, \beta_t)_{t\geq 0}$ which are require to satisfy (11) and (12).[14] In Section 5 we apply Theorem 2 to the repeated game with prior $p = 0$, the complete information game. In Sections 6 and 7 we characterize $\mathcal{E}(p)$ for $p \in (0,1)$.

5 Equilibrium degeneracy under complete information

In this section we examine the structure of public equilibrium in the underlying complete information game, i.e. the continuous-time repeated game in which it is common knowledge that the large player is the normal type. We show that non-trivial intertemporal incentives cannot be sustained in equilibrium, regardless of the level of patience of the large player.

Theorem 3 (Equilibrium Degeneracy)**.** *Suppose the small players are certain that they are facing the normal type, i.e. $p = 0$. Then, irrespective of the discount rate $r > 0$, in every public equilibrium of the continuous-time game the large player cannot achieve a payoff outside the convex hull of his static Nash equilibrium payoffs, i.e.*

$$\mathcal{E}(0) \;=\; \mathrm{co}\left\{ g(a,\bar{b}) \;:\; \begin{array}{l} a \in \underset{a' \in A}{\arg\max}\; g(a',\bar{b}) \\ b \in \underset{b' \in B}{\arg\max}\, h(a,b',\bar{b}),\;\; \forall b \in \mathrm{supp}\,\bar{b} \end{array} \right\}.$$

Here is the idea behind this result. In order to give incentives to the large player to play an action that results in a greater payoff than in static Nash equilibrium, his continuation value must respond to the public signal $(X_t)_{t\geq 0}$. But, when the continuation value reaches the greatest payoff among all public equilibria of the repeated game, such incentives cannot be provided. In effect, if the large player's continuation value were sensitive to the public signal when his continuation value equals the greatest public equilibrium payoff, then with positive probability the continuation value would escape above this upper bound, and this is not possible. Therefore, at the upper bound the continuation value cannot be sensitive to the public signal, hence the large player must be playing a myopic best-reply there. This implies that at the upper bound the

[14] This means that the graph of any bounded correspondence with this property must be contained in the graph of $\mathcal{E}$.

flow payoff of the large is no greater than the greatest static Nash equilibrium payoff. Moreover, another necessary condition to prevent the continuation value from escaping is that the drift of the continuation value process must be less than or equal to zero at the upper bound. But since this drift is proportional to the difference between the continuation value and the flow payoff, it follows that the greatest public equilibrium payoff must be no greater than the flow payoff at the upper bound, which, as we have argued above, must be no greater than the greatest static equilibrium payoff.

Proof of Theorem 3. Let $\bar{v}$ be the greatest Nash equilibrium payoff of the large player in the static complete information game. We will show that it is impossible to achieve a payoff greater than $\bar{v}$ in any public equilibrium of the continuous-time game. (The proof for the least Nash equilibrium payoff is similar and therefore omitted.) Suppose there was a public equilibrium $(a_t, \bar{b}_t)_{t\geq 0}$ with continuation values $(W_t)_{t\geq 0}$ for the normal type in which the large player's expected discounted payoff, W_0, was greater than $\bar{v}$. By Propositions 2 and 3, for some $(\beta_t)_{t\geq 0}$ in $\mathcal{L}$ the large player's continuation value must satisfy

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\,dt + r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\,dt),$$

where a_t maximizes $g(a', \bar{b}_t) + \beta_t \cdot \mu(a', \bar{b}_t)$ over all $a' \in A$ and $\bar{b}_t$ maximizes $h(a_t, b', \bar{b}_t)$ over all $b' \in \Delta(B)$. Let $\bar{D} \overset{\text{def}}{=} W_0 - \bar{v} > 0$.

Claim. *There exists $c > 0$ such that, so long as $W_t \geq \bar{v} + \bar{D}/2$, either the drift of W_t is greater than $r\bar{D}/4$ or the norm of the volatility of W_t is greater than c.*

This claim is an implication of the following lemma, whose proof is relegated to Appendix A.

Lemma 1. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \geq 0$ and after all public histories, $|\beta_t| \geq \delta$ whenever $g(a_t, \bar{b}_t) \geq \bar{v} + \varepsilon$.*

Indeed, letting $\varepsilon = \bar{D}/4$ in this lemma yields a $\delta > 0$ such that the norm of the volatility of W_t, which equals $r|\beta_t|$, must be greater than or equal to $c = r\delta$ whenever $g(a_t, \bar{b}_t) \geq \bar{v} + \bar{D}/4$. Moreover, when $g(a_t, \bar{b}_t) < \bar{v} + \bar{D}/4$ and $W_t \geq \bar{v} + \bar{D}/2$, then the drift of W_t, which equals $r(W_t - g(a_t, \bar{b}_t))$, must be greater than $r\bar{D}/4$, and this concludes the proof of the claim.

Since we have assumed that $\bar{D} = W_0 - \bar{v} > 0$, the claim above readily implies that W_t must grow arbitrarily large with positive probability, and this is a contradiction since $(W_t)_{t\geq 0}$ is bounded. ■

While Theorem 3 has no counterpart in discrete time, it is not a result of continuous-time technicalities.[15] The large player's incentives to depart from a static best response become fragile when he is flexible to react to new information quickly. The foundations of this result are similar to the collapse of intertemporal incentives in discrete-time games with frequent actions, as in Abreu, Milgrom, and Pearce (1991) in a prisoners' dilemma with Poisson signals, and in Sannikov and Skrzypacz (2007) and Fudenberg and Levine (2007) in games with Brownian signals. Borrowing intuition from these papers, suppose that the large player must hold his action fixed for an interval of time of length $\Delta > 0$. Suppose that the large player's equilibrium incentives to take the Stackelberg action are created through a statistical test that triggers an equilibrium punishment when the signal is sufficiently "bad." A profitable deviation has a gain

[15] Fudenberg and Levine (1994) show that in discrete-time repeated games with large and small players, often there are public perfect equilibria with payoffs above static Nash, albeit bounded away from efficiency.

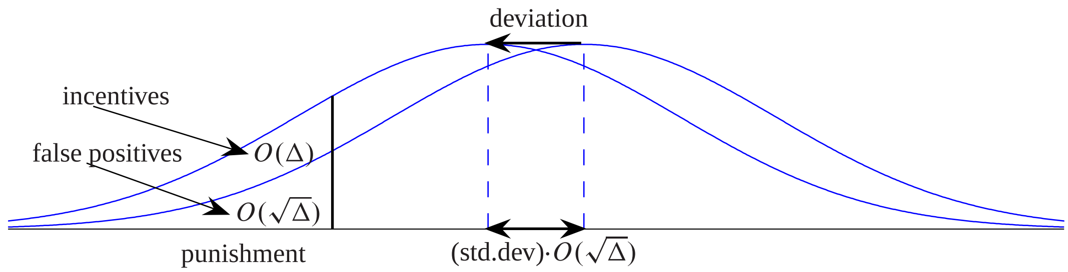

Figure 2: A statistical test to prevent a given deviation.

on the order of Δ, therefore such deviation can be prevented only if it increases the probability of triggering punishment by at least $O(\Delta)$. Sannikov and Skrzypacz (2007) and Fudenberg and Levine (2007) show that, under Brownian signals, the log-likelihood ratio for a test against any particular deviation is normally distributed, and that a deviation shifts the mean of this distribution by $O(\sqrt{\Delta})$. Thus, a successful test against a deviation would generate a false positive with probability of $O(\sqrt{\Delta})$. This probability, which reflects the value destroyed in each period by the punishment, is disproportionately large for small Δ compared to the value created during a period of length Δ. This intuition implies that in equilibrium the large player cannot sustain a payoff above static Nash as $\Delta \to 0$. Figure 2 illustrates the densities of the log-likelihood ratio under the "recommended" action of the large player and a deviation, and the areas responsible for the large player's incentives and for the false positives.

The above result relies crucially on the assumption that the volatility of the public signal is independent of the large player's actions. If the large player could influence the volatility, his actions would become publicly observable, since in continuous time the volatility of a diffusion process is effectively observable. Motivated by this observation, Fudenberg and Levine (2007) consider discrete-time approximations of continuous-time games with Brownian signals, allowing the large player to control the volatility of the public signal. In a variation of the product choice game, when the volatility is decreasing in the large player's effort level, they show the existence of nontrivial equilibrium payoffs in the limit as the period length tends to zero.[16]

Finally, Fudenberg and Levine (2009) examine discrete-time repeated games with public signals drawn from multinomial distributions that depend on the length of the period. They assume that for each profile of stationary strategies, the distribution over signals—properly normalized and embedded in continuous time—satisfies standard conditions for weak convergence to a Brownian motion with drift as the period length tends to zero.[17] When the Brownian motion is approximated by a binomial process, they show that the set of equilibrium payoffs of the discrete-time game approaches the degenerate equilibrium of the continuous-time game. By contrast, when the Brownian motion is approximated by a trinomial process and the discount rate is low enough, they show that the greatest equilibrium payoff of the discrete-time

[16] They also find the surprising result that when the volatility of the Brownian signal is increasing in the large player's effort, the set of equilibrium payoffs of the large player collapses to the set of static Nash equilibrium payoffs, despite the fact that in the continuous-time limit the actions of the large player are effectively observable.

[17] Namely, they satisfy the conditions of Donker's Invariance Principle (Billingsley, 1999, Theorem 8.2).

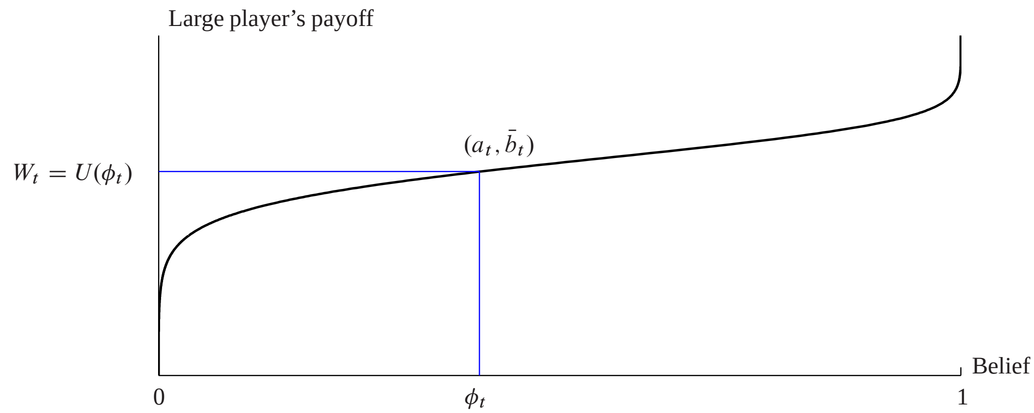

Figure 3: The large player's payoff in a Markov perfect equilibrium.

model converges to a payoff strictly above the static Nash equilibrium payoff.

6 Reputation games with a unique sequential equilibrium

In many interesting applications, including the product choice game of Section 2, the public sequential equilibrium is unique and Markovian in the small players' posterior belief. That is, at each time t, the small players' posterior belief, ϕ_t, uniquely determines the equilibrium actions $a_t = \mathsf{a}(\phi_t)$ and $\bar{b}_t = \mathsf{b}(\phi_t)$, as well as the continuation value of the normal type, $W_t = U(\phi_t)$, as depicted in Figure 3. This section presents general conditions for the sequential equilibrium to be unique and Markovian, and characterizes the equilibrium using an ordinary differential equation.

First, we derive our characterization heuristically. By Theorem 2, in a sequential equilibrium $(a_t, \bar{b}_t, \phi_t)_{t\geq 0}$ the small players' posterior beliefs, $(\phi_t)_{t\geq 0}$, and the continuation values of the normal type, $(W_t)_{t\geq 0}$, evolve according to

$$d\phi_t = -|\gamma(a_t, \bar{b}_t, \phi_t)|^2/(1-\phi_t)\,dt + \gamma(a_t, \bar{b}_t, \phi_t)\cdot dZ_t^{\mathsf{n}} \tag{13}$$

and

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\,dt \;+\; r\beta_t^{\top}\sigma(\bar{b}_t)\,dZ_t^{\mathsf{n}}, \tag{14}$$

for some random process $(\beta_t)_{t\geq 0}$ in $\mathcal{L}$, where $dZ_t^{\mathsf{n}} = \sigma(\bar{b}_t)^{-1}(dX_t - \mu(a_t, \bar{b}_t)\,dt)$ is a Brownian motion under the normal type.[18] The negative drift in (13) reflects the fact that, conditional on the large player being the normal type, the posterior on the behavioral type must be a supermartingale.

If we assume that the equilibrium is Markovian, then by Itô's formula the continuation value $W_t = U(\phi_t)$ of the normal type must follow

$$dU(\phi_t) = \frac{1}{2}|\gamma(a_t, \bar{b}_t, \phi_t)|^2\big(U''(\phi_t) - 2U'(\phi_t)/(1-\phi_t)\big)\,dt + U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\cdot dZ_t^{\mathsf{n}}, \tag{15}$$

[18]Equation (13) is just equation (2) re-written from the point of view of the normal type, rather than the point of view of the small players. This explain the change of drift.

assuming that the value function $U : (0,1) \to \mathbb{R}$ is twice continuously differentiable. Thus, matching the drifts in (14) and (15) yields the *optimality equation*:

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\mathsf{a}(\phi), \mathsf{b}(\phi)))}{|\gamma(\mathsf{a}(\phi), \mathsf{b}(\phi), \phi)|^2}, \quad \phi \in (0,1). \tag{16}$$

Then, to determine the Markovian strategy profile $(a_t, \bar{b}_t) = (\mathsf{a}(\phi_t), \mathsf{b}(\phi_t))$, we can match the volatility coefficients in (14) and (15):

$$r\beta_t^\top = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)^\top \sigma(\bar{b}_t)^{-1}.$$

Plugging this expression in the incentive constraint (11) and applying Theorem 2 yields

$$(a_t, \bar{b}_t) \in \mathcal{N}(\phi_t, \phi_t(1-\phi_t)U'(\phi_t)/r), \tag{17}$$

where $\mathcal{N} : [0,1] \times \mathbb{R} \rightrightarrows A \times \Delta(B)$ is the correspondence defined by[19]

$$\mathcal{N}(\phi, z) \overset{\text{def}}{=} \left\{ (a, \bar{b}) : \begin{array}{l} a \in \operatorname*{arg\,max}_{a' \in A} \; g(a', \bar{b}) + z(\mu(a^*, \bar{b}) - \mu(a, \bar{b}))^\top (\sigma(\bar{b})\sigma(\bar{b})^\top)^{-1} \mu(a', \bar{b}) \\ b \in \operatorname*{arg\,max}_{b' \in B} \; \phi h(a^*, b', \bar{b}) + (1-\phi) h(a, b', \bar{b}) \quad \forall b \in \operatorname{supp} \bar{b} \end{array} \right\}. \tag{18}$$

Effectively, for each $(\phi, z) \in [0,1] \times \mathbb{R}$, correspondence $\mathcal{N}$ returns the set of Bayesian Nash equilibria of an auxiliary one-shot game in which the large player is a behavioral type with probability ϕ and the payoff of the normal type is perturbed by a "reputational term" weighted by z. In particular, $\mathcal{N}(\phi, 0)$ is the set of static Bayesian Nash equilibria when the prior on the behavioral type is ϕ.

Our characterization of unique sequential equilibria, stated below as Theorem 4, is valid under Conditions 1 and 2 below. Condition 1 guarantees that the volatility γ of beliefs, which appears in the optimality equation in the denominator of a fraction, is bounded away from zero across all reputation levels. Condition 2 ensures that correspondence $\mathcal{N}$ is nonempty and single-valued, so that equilibrium behavior is determined by condition (17).

Condition 1. *For each $\phi \in [0,1]$ and each Bayesian Nash equilbrium $(a, \bar{b})$ of the static game with prior ϕ, we have $\mu(a, \bar{b}) \neq \mu(a^*, \bar{b})$.*

Thus, under Condition 1, in every static Bayesian Nash equilibrium the behavior of the normal type is statistically distinguishable from the behavioral type. Therefore, in order for the normal type to play either a^* or some observationally equivalent action, he ought to be given intertemporal incentives. This rules out sequential equilibria in which the posterior beliefs settle in finite time.

Note that when the flow payoff of the small players depends on the actions of the large player only through the public signal (cf. equation (1) and the discussion therein), Condition 1 becomes equivalent to the following simpler condition: *for every static Nash equilibrium $(a, \bar{b})$ of the complete information game, $\mu(a, \bar{b}) \neq \mu(a^*, \bar{b})$.* This condition is similar to the non-credible commitment assumption of Cripps, Mailath, and Samuelson (2004), which we discuss in greater detail in Section 8.

Condition 2. *Either of the following conditions holds:*

[19] Note that $\mathcal{N}(\phi, z)$ also depends on the strategy a^* of the behavioral type, although our notation does not make this dependence explicit.

a. *$\mathcal{N}$ is a non-empty single-valued correspondence that returns a mass-point distribution of small players' actions for each $(\phi, z) \in [0,1] \times \mathbb{R}$. Moreover, $\mathcal{N}$ is Lipschitz continuous on every bounded subset of* $[0,1] \times \mathbb{R}$.

b. *the restriction of $\mathcal{N}$ to $[0,1] \times [0,\infty)$ is non-empty, single-valued and returns a mass-point distribution of small players' actions for each $(\phi, z) \in [0,1] \times [0,\infty)$. Moreover, $\mathcal{N}$ is Lipschitz continuous on every bounded subset of $[0,1] \times [0,\infty)$, and the static Bayesian Nash equilibrium payoff of the normal type, $g(\mathcal{N}(\phi, 0))$, is increasing in the prior ϕ.*

Condition 2.b has practical importance, for it holds in many games in which Condition 2.a fails. Indeed, the payoff of the normal type in (18), when adjusted by a negative reputational weight z, may fail to be concave even when $g(\cdot, \bar{b})$ and $\mu(\cdot, \bar{b})$ are strictly concave for all $\bar{b}$. In Section 6.2 we provide conditions on the *primitives* of the game—stage-game payoffs, drift and volatility—which are sufficient for Conditions 2.b.

Note that Condition 2 depends on the action a^* of the behavioral type: it may hold for some behavioral types but not others. However, to keep the notation concise we have chosen not to index correspondence $\mathcal{N}$ by a^*.

Finally, Condition 1 is not necessary for equilibrium uniqueness. When Condition 2 holds but Condition 1 fails, the reputation game still has a unique public sequential equilibrium, which is Markovian and characterized by (13), (16) and (17) up until the stopping time when the posterior first hits a value ϕ for which there is a static Bayesian equilibrium $(a^\phi, \bar{b}^\phi)$ with $\mu(a^\phi, \bar{b}^\phi) = \mu(a^*, \bar{b}^\phi)$; from this time on, the posterior no longer updates and the behavior is $(a^\phi, \bar{b}^\phi)$ statically.[20] In particular, when the payoff flow of the small players depends on the large player's action only through the public signal (cf. equation (1) and the discussion therein), then, when Condition 1 fails and Condition 2 holds, for every prior p the unique public sequential equilibrium of the reputation game is the repeated play of the static Nash equilibrium of the complete information game, and the posterior is $\phi_t = p$ identically.

Theorem 4. *Assume Conditions 1 and 2. Under Condition 2.a (resp. 2.b), the correspondence of public sequential equilibrium payoffs of the normal type, $\mathcal{E} : [0,1] \rightrightarrows \mathbb{R}$, is single-valued and coincides, on the interval $(0,1)$, with the unique bounded solution (resp. unique bounded increasing solution) of the optimality equation:*

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r))}{|\gamma(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r), \phi)|^2}, \quad \phi \in (0,1). \tag{19}$$

Moreover, at $p = 0$ and 1, $\mathcal{E}(p)$ satisfies the boundary conditions

$$\lim_{\phi \to p} U(\phi) = \mathcal{E}(p) = g(\mathcal{N}(p, 0)) \quad \textit{and} \quad \lim_{\phi \to p} \phi(1-\phi)U'(\phi) = 0. \tag{20}$$

Finally, for each prior $p \in [0,1]$ there exists a unique public sequential equilibrium, which is Markovian in the small players' posterior belief: at each time t and after each public history, the equilibrium actions are determined by (17), the posterior evolves according to (13) with initial condition $\phi_0 = p$ and the continuation value of the normal type is $W_t = U(\phi_t)$.

[20] Indeed, an argument similar to the proof of Theorem 3 can be used to show that when the prior is ϕ the large player cannot achieve any payoff other than $g(a^\phi, \bar{b}^\phi)$.

The intuition behind Theorem 4 is similar to the idea behind the equilibrium degeneracy result under complete information (Theorem 3). With Brownian signals it is impossible to create incentives to sustain a greater payoff than in a Markov perfect equilibrium, for otherwise, in a public sequential equilibrium that achieves the greatest difference $W_0 - U(\phi_0) > 0$ across all priors $\phi_0 \in [0,1]$, at time $t = 0$ the joint volatility of $(\phi_t, W_t)_{t \geq 0}$ must be parallel to the slope of $U(\phi_t)$, since $W_t - U(\phi_t)$ cannot increase for any realization of dX_t. It follows that $r\beta_0^\top \sigma(\bar{b}_0) = U'(\phi_0)\gamma(a_0, \bar{b}_0, \phi_0)^\top$ and hence, when $\mathcal{N}$ is single-valued, the equilibrium action profile played at time zero must equal that played in a Markov perfect equilibrium at reputation ϕ_0. The optimality equation (19) then implies that $W_t - U(\phi_t)$ has a positive drift at time zero, which implies that with positive probability $W_t - U(\phi_t) > W_0 - U(\phi_0)$ for some $t > 0$, and this is a contradiction.

Proof of Theorem 4. The proofs of existence and uniqueness of a bounded solution of the optimality equation (19) are presented in Appendix C, along with a number of intermediate lemmas. First, Proposition C.3 shows that under Conditions 1 and 2.a the optimality equation has a unique bounded solution $U : (0,1) \to \mathbb{R}$. Then, Proposition C.2 shows that U must satisfy the boundary conditions (39), which include (20). Finally, Proposition C.4 implies that under Conditions 1 and 2.b the optimality equation has a unique increasing bounded solution U, which also satisfies the boundary conditions.

We now show that for each prior $p \in (0,1)$ there is no public sequential equilibrium in which the normal type receives a payoff different from $U(p)$. Towards a contradiction, suppose that for some $p \in (0,1)$ there is a public sequential equilibrium $(a_t, \bar{b}_t, \phi_t)_{t \geq 0}$ in which the normal type receives payoff $W_0 > U(p)$. By Theorem 2, the small players' belief process $(\phi_t)_{t \geq 0}$ follows (13), the continuation value of the normal type $(W_t)_{t \geq 0}$ follows (14) for some $(\beta_t)_{t \geq 0}$ in $\mathcal{L}$, and the equilibrium actions $(a_t, \bar{b}_t)_{t \geq 0}$ satisfy the incentive constraints (11) and (12). Moreover, by Itô's formula, the process $(U(\phi_t))_{t \geq 0}$ follows (15). Then, using (14) and (15), the process $D_t \stackrel{\text{def}}{=} W_t - U(\phi_t)$, which starts at $D_0 > 0$, has drift

$$\underbrace{rD_t + rU(\phi_t)}_{rW_t} - rg(a_t, \bar{b}_t) + |\gamma(a_t, \bar{b}_t, \phi_t)|^2 \left(\frac{U'(\phi_t)}{1 - \phi_t} - \frac{U''(\phi_t)}{2} \right),$$

and volatility

$$r\beta_t^\top \sigma(\bar{b}_t) - \gamma(a_t, \bar{b}_t, \phi_t)^\top U'(\phi_t).$$

Claim. *There exists $\delta > 0$ such that, so long as $D_t \geq D_0/2$, either the drift of D_t is greater than $rD_0/4$ or the norm of the volatility of D_t is greater than δ.*

This claim is an implication of Lemma C.8 from Appendix C, which shows that for each $\varepsilon > 0$ we can find $\delta > 0$ such that for all $t \geq 0$ and after all public histories, either the drift of D_t is greater than $rD_t - \varepsilon$, or the norm of the volatility of D_t is greater than δ. Thus, letting $\varepsilon = rD_0/4$ in Lemma C.8 proves the claim.

The proof of Lemma C.8 is relegated to Appendix C, but here we explain the main idea behind it. When the norm of the volatility of D_t is exactly zero we have $r\beta_t^\top \sigma(\bar{b}_t) = \gamma(a_t, \bar{b}_t, \phi_t)^\top U'(\phi_t)$, so by (11) and (12) we have

$$a_t \in \arg\max_{a' \in A} \; rg(a', \bar{b}_t) + \underbrace{U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)^\top \sigma^{-1}(\bar{b}_t)}_{r\beta_t^\top} \mu(a', \bar{b}_t),$$
$$b \in \arg\max_{b' \in B} \phi_t h(a^*, b', \bar{b}_t) + (1 - \phi_t) h(a_t, b', \bar{b}_t) \quad \forall b \in \operatorname{supp} \bar{b}_t,$$

and hence $(a_t, \bar{b}_t) = \mathcal{N}(\phi_t, \phi_t(1-\phi_t)U'(\phi_t)/r)$. Then, by (19) the drift of D_t must equal rD_t. The proof of Lemma C.8 uses a continuity argument to show that in order for the drift of D_t to be below $rD_t - \varepsilon$, the volatility of D_t must be uniformly bounded away from 0.

Since we have assumed that $D_0 > 0$ the claim above readily implies that D_t must grow arbitrarily large with positive probability, which is a contradiction since both W_t and $U(\phi_t)$ are bounded processes. This contradiction shows that a public sequential equilibrium that yields the normal type a payoff greater than $U(p)$ cannot exist. Similarly, it can be shown that no equilibrium can yield a payoff less than $U(p)$.

We turn to the construction of a sequential equilibrium that yields a payoff of $U(p)$ to the normal type. Consider the stochastic differential equation (13) with $(a_t, \bar{b}_t)_{t\geq 0}$ defined by (17). Since the function $\phi \mapsto \gamma(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r), \phi)$ is Lipschitz continuous, this equation has a unique solution $(\phi_t)_{t\geq 0}$ with initial condition $\phi_0 = p$. We now show that $(a_t, \bar{b}_t, \phi_t)_{t\geq 0}$ is a public sequential equilibrium in which $W_t = U(\phi_t)$ is the process of continuation values of the normal type. By Proposition 1, the belief process $(\phi_t)_{t\geq 0}$ is consistent with $(a_t, \bar{b}_t)_{t\geq 0}$. Moreover, since $W_t = U(\phi_t)$ is a bounded process with drift $r(W_t - g(a_t, \bar{b}_t))$ by (15) and (19), Proposition 2 implies that $(W_t)_{t\geq 0}$ is the process of continuation values of the normal type under $(a_t, \bar{b}_t)_{t\geq 0}$. Thus, the process $(\beta_t)_{t\geq 0}$, given by the representation of W_t in Proposition 2, must satisfy $r\beta_t^\top \sigma(b_t) = U'(\phi_t)\gamma(a_t, b_t, \phi_t)^\top$. Finally, to see that the strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ is sequentially rational under $(\phi_t)_{t\geq 0}$, recall that $(a_t, \bar{b}_t) = \mathcal{N}(\phi_t, \phi_t(1-\phi_t)U'(\phi_t)/r)$, hence

$$\begin{aligned} a_t &= \arg\max_{a' \in A} \; rg(a', \bar{b}_t) + \underbrace{U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)^\top \sigma(\bar{b}_t)^{-1}}_{r\beta_t^\top} \mu(a', \bar{b}_t)\,, \\ \bar{b}_t &= \arg\max_{b' \in B} \; \phi_t h(a^*, b', \bar{b}_t) + (1-\phi_t)h(a_t, b', \bar{b}_t), \end{aligned} \tag{21}$$

and therefore sequential rationality follows from Proposition 3. We conclude that $(a_t, \bar{b}_t, \phi_t)_{t\geq 0}$ is a public sequential equilibrium that yields a payoff of $U(p)$ to the normal type.

Finally, we show that in any public sequential equilibrium $(a_t, \bar{b}_t, \phi_t)_{t\geq 0}$ the equilibrium actions are uniquely determined by the small players' belief by (17). Indeed, if $(W_t)_{t\geq 0}$ is the process of continuation values of the normal type, then the pair (ϕ_t, W_t) must stay on the graph of U, because there is no public sequential equilibrium with continuation value different from $U(\phi_t)$, as we have shown above. Therefore, the volatility of $D_t = W_t - U(\phi_t)$ must be identically zero, i.e. $r\beta_t^\top \sigma(\bar{b}_t) = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)^\top$. Thus Proposition 3 implies condition (21), and therefore $(a_t, \bar{b}_t) = \mathcal{N}(\phi_t, \phi_t(1-\phi_t)U'(\phi_t)/r)$. ■

Theorem 4 is a striking result because it highlights equilibrium properties that hold in discrete time only in approximation, and also because it summarizes reputational incentives through a single variable $z = \phi(1-\phi)U'(\phi)/r$. By contrast, in the standard discrete-time model equilibrium behavior is not pinned down by the small players' posterior belief and Markov perfect equilibria may fail to exist.[21] However, as we explain in our conclusions in Section 10, we expect the characterization of Theorem 4 to capture the limit equilibrium behavior in games in which players can respond to new information quickly and information arrives in a continuous fashion.[22] The uniqueness and characterization result shows that our

[21] See Mailath and Samuelson (2006, pp. 560–566) for a discrete-time finite-horizon variation of the product-choice game in which no Markov perfect equilibrium exists.

[22] We expect our methods to apply broadly to other continuous-time games with payoff-relevant state variables, such as the Cournot competition with mean-reverting prices of Sannikov and Skrzypacz (2007). In that model the market price is the payoff-relevant state variable.

continuous-time formulation provides a tractable model of reputation phenomena that can be valuable for applications, as we demonstrate by several examples below. The fact that the reputational incentives are summarized by a single variable is particularly useful for applications with multi-dimensional actions.

The characterization of the unique sequential equilibrium by a differential equation, in addition to being tractable for numerical computation, is useful to derive qualitative conclusions about equilibrium behavior, both at the general level and for specific classes of games. First, in addition to the existence, uniqueness and characterization of sequential equilibrium, Theorem 4 provides a simple sufficient condition for the equilibrium value of the normal type to be monotonically increasing in reputation; namely, the condition that the stage-game Bayesian Nash equilibrium payoff of the normal type is increasing in the prior on the behavioral type. By contrast, in the canonical discrete-time model, it is not known under which conditions reputation has a positive value.

Second, Theorem 4 implies that, under Conditions 1 and 2, in equilibrium the normal type never perfectly imitates the behavioral type. Since the equilibrium actions are determined by (17), if the normal type imitated the behavioral type perfectly at time t, then $(a^*, \bar{b}_t)$ would be a Bayesian Nash equilibrium of the static game with prior ϕ_t, and this would violate Condition 1.

Third, the characterization implies the following square root law of substitution between the discount rate and the volatility of the signals:

Corollary 1. *Under Conditions 1 and 2, multiplying the discount rate by a factor of $\alpha > 0$ has the same effect on the equilibrium value of the normal type as rescaling the volatility matrix σ by a factor of $\sqrt{\alpha}$.*

Proof. This follows directly from observing that the discount rate r and the volatility matrix σ enter the optimality equation only through the product $r\sigma\sigma^\top$. ■

Fourth, the differential equation characterization of Theorem 4 is also useful to study reputational incentives in the limit as the large player gets patient. In Section 6.1 below, we use the optimality equation to prove a new general result about reputation effects at the level of equilibrium behavior (as opposed to equilibrium payoffs, as in Fudenberg and Levine (1992) and Faingold (2008)). Under Conditions 1 and 2, if the behavioral type plays the Stackelberg action and the public signals satisfy an identifiability condition, as the large player gets patient the equilibrium action of the normal type approaches the Stackelberg action at every reputation level (Theorem 5).

Finally, to illustrate how our characterization can be useful to study specific classes of games, we discuss a few examples:

(a) *Signal manipulation.* An agent chooses a costly effort $a_1 \in [0, 1]$, which affects two public signals, X_1 and X_2. The agent also chooses a level of signal manipulation $a_2 \in [0, 1]$, which affects only signal X_1. Thus, as in Holmstrom and Migrom's (1991) multitasking agency model, the agent's hidden action is multi-dimensional and the signals are distorted by a Brownian motion. Specifically,

$$
\begin{aligned}
dX_{1,t} &= (a_{1,t} + a_{2,t})\,dt + dZ_{1,t}, \\
dX_{2,t} &= a_{1,t}\,dt + \sigma\,dZ_{2,t},
\end{aligned}
$$

where $\sigma \geq 1$, i.e. the signal that cannot be manipulated is less informative. There is a competitive market of small identical principals, each of whom is willing to hire the agent at a spot wage of

$\bar{b}_t$, which equals the market expectation of the agent's effort given the public observations of the signals. Thus, the principals do not care directly about the agent's signal manipulation activity, but in equilibrium manipulation can affect the principals' behavior through their statistical inference of the agent's effort. The agent's cost of effort and manipulation is quadratic, so that his flow payoff is

$$\bar{b}_t - \frac{1}{2}(a_{1,t}^2 + a_{2,t}^2).$$

Finally, the behavioral type is the Stackelberg type, i.e. he is committed to $a^* = (1, 0)$ (full effort and no signal manipulation).

The signal manipulation game satisfies Conditions 1 and 2 directly (both 2.a and 2.b), and hence its sequential equilibrium is unique and Markovian in reputation, and the value function of the normal type is increasing and characterized by the optimality equation. We can examine the agent's intertemporal incentives using the characterization. In equilibrium, when the reputational weight is $z \geq 0$, the agent will choose (a_1, a_2) to maximize

$$-\frac{1}{2}({a'_1}^2 + {a'_2}^2) + z \begin{bmatrix} 1 - a_1 - a_2 & 1 - a_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} a'_1 \\ a'_2 \end{bmatrix}$$

over all $(a'_1, a'_2) \in [0, 1] \times [0, 1]$. The first-order conditions are

$$\begin{aligned} -a_1 + z(1 - a_1 - a_2) + z\sigma^{-2}(1 - a_1) &= 0, \\ -a_2 + z(1 - a_1 - a_2) &= 0, \end{aligned}$$

which yield

$$\begin{aligned} a_1 &= \frac{(1 + 1/\sigma^2)z + z^2/\sigma^2}{1 + (2 + 1/\sigma^2)z + z^2/\sigma^2}, \\ a_2 &= \frac{z}{1 + (2 + 1/\sigma^2)z + z^2/\sigma^2}. \end{aligned}$$

As shown in Figure 4, when the reputational weight is close to zero (as in equilibrium when ϕ is near 0 or 1), the agent exerts low effort and engages in nearly zero manipulation. As the reputational weight z grows, the agent's effort increases monotonically and converges to maximal effort $a_1^* = 1$, while the manipulation action is single-peaked and approaches zero manipulation. In Figure 4 we can also see the effect of the informativeness of the non-manipulable signal: as σ increases, so that the non-manipulable signal becomes noisier, the agent's equilibrium effort decreases and the amount of signal manipulation increases at every level of the reputational weight $z > 0$. The intuition is that when the volatility of the non-manipulable signal increases, it becomes cheaper to maintain a reputation by engaging in signal manipulation relative to exerting true effort. Indeed, a greater σ implies that the small players update their beliefs by less when they observe unexpected changes in the non-manipulable signal X_2; on the other hand, irrespective of σ the actions a_1 and a_2 are perfect substitutes in terms of their impact in the informativeness of X_1.

Finally, we emphasize that characterizations like this one are impossible to achieve in discrete time, because equilibria typically fail to be Markovian and incentives cannot be characterized by a single reputational weight parameter.

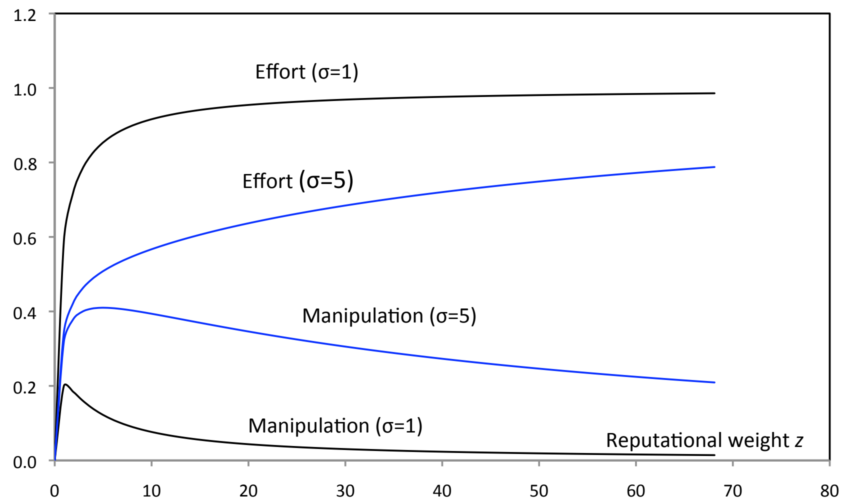

Figure 4: Effort and manipulation in the signal manipulation game.

(b) *Monetary policy.* As in Cukierman and Meltzer (1986), at each time $t \geq 0$ a central bank chooses a costly policy variable $a_t \in [0, 1]$, which is the target rate of monetary growth. This policy variable is unobservable to the "population" and affects the stochastic evolution of the aggregate price level, P_t, as follows:

$$P_t = \exp X_t, \quad \text{where } dX_t = a_t\, dt + \sigma\, dZ_t.$$

At each time t, the population formulates a rational expectation a_t^e of the current rate of money growth, given their past observations of X. In reduced form, the behavior of the population gives rise to a law of motion of the aggregate level of employment n_t (in logarithmic scale), as follows:

$$dn_t = \kappa(\bar{n} - n_t)\, dt + \varsigma(dX_t - a_t^e\, dt), \tag{22}$$

where $\bar{n}$ is the long-run level of employment. Thus, the residual change in employment is proportional to the unanticipated inflation, as in a Phillips curve.

The central bank cares about stimulating the economy but inflation is costly to society; specifically, the average discounted payoff of the central bank is

$$\int_0^\infty re^{-rt}\big(\alpha n_t \;-\; \frac{a_t^2}{2}\big)\, dt, \tag{23}$$

where $\alpha > 0$ is the marginal rate of substitution between stimulus and inflation. To calculate the expected payoff of the central bank, first note that (22) implies

$$n_t = n_0 e^{-\kappa t} + \bar{n}(1 - e^{-\kappa t}) + \varsigma \int_0^t e^{\kappa(s-t)}\, (dX_s - a_s^e\, ds).$$

Plugging into (23) yields the following expression for the central bank's discounted payoff:

$$n_0 \frac{r}{r+\kappa} + \bar{n}\big(1 - \frac{r}{r+\kappa}\big) + \int_0^\infty re^{-rt}\big(\alpha\varsigma \int_0^t e^{\kappa(s-t)}\, (dX_s - a_s^e\, ds) \;-\; \frac{a_t^2}{2}\big)\, dt,$$

which, by integration-by-parts, equals

$$n_0 \frac{r}{r+\kappa} + \bar{n}\big(1 - \frac{r}{r+\kappa}\big) + \int_0^\infty r e^{-rt} \big(\frac{\alpha\varsigma}{r+\kappa}\,(dX_t - a_t^e\,dt) \;-\; \frac{a_t^2}{2}\,dt\big).$$

Thus, modulo a constant, the expected flow payoff of the central bank is

$$\frac{\alpha\varsigma}{r+\kappa}(a_t - a_t^e) - \frac{a_t^2}{2}.$$

In a static world, the unique Nash equilibrium would have the central bank targeting an inflation rate of $\frac{\alpha\varsigma}{r+\kappa} > 0$. By contrast, if the central bank had the credibility to commit to a rule, it would find it optimal to commit to zero inflation target. Indeed, under commitment, the population would anticipate the target, and hence the term $a_t - a_t^e$ would be identically zero and the central bank would be left with minimizing the cost of inflation.[23]

Consider now the reputation game with a behavioral type committed to $a^* = 0$. To study the central bank's incentives, consider his maximization problem in the definition of correspondence $\mathcal{N}$. When the reputational weight is z, the central bank chooses $a \in [0,1]$ to maximize

$$\frac{\alpha\varsigma a'}{r+\kappa} - \frac{a'^2}{2} - \frac{z a a'}{\sigma^2} \qquad \text{over all } a' \in [0,1],$$

and therefore sets

$$a = \frac{\alpha\varsigma}{(r+\kappa)(1+z/\sigma^2)}.$$

Thus, when the reputational weight z is near zero (which happens in equilibrium when ϕ is close to 0 or 1), the central bank succumbs to his myopic incentive to inflate prices and sets the target rate around $\frac{\alpha\varsigma}{r+\kappa}$; as the reputational weight grows, the target inflation decreases monotonically and approaches zero asymptotically. Moreover, in equilibrium the central bank sets a higher target inflation rate at each reputational weight when: (i) it values stimulus more (i.e. α is greater), (ii) the effect of unanticipated inflation on employment takes longer to disappear (i.e. κ is smaller), (iii) changes in employment are more sensitive to unexpected inflation (i.e. ς is greater), or (iv) the fluctuations in the aggregate price level are more volatile (i.e. σ is greater)

(c) *Product choice.* The repeated product choice game of Section 2 satisfies Conditions 1 and 2 (both 2.a and 2.b), and hence its sequential equilibrium is unique, Markovian and the equilibrium value function of the normal type, U, is increasing in reputation. For each $(\phi, z) \in [0,1] \times \mathbb{R}$ the unique profile $(a(\phi,z), b(\phi,z)) \in \mathcal{N}(\phi,z)$ is

$$a(\phi,z) = \begin{cases} 0 & : \quad z \le 1 \\ 1 - 1/z & : \quad z > 1 \end{cases},$$

$$b(\phi,z) = \begin{cases} 0 & : \quad \phi + (1-\phi)a(\phi,z) \le 1/4 \\ 4 - (\phi + (1-\phi)a(\phi,z))^{-1} & : \quad \phi + (1-\phi)a(\phi,z) > 1/4 \end{cases}.$$

[23] This discrepancy between the Nash and the Stackelberg outcomes is an instance of the familiar time consistency problem in macroeconomic policy studied by Kydland and Prescott (1977).

The particular functional form we have chosen for the product choice game is not important to satisfy Conditions 1 and 2. More generally, consider the case in which the quality of the product still follows[24]

$$dX_t = a_t dt + dZ_t,$$

but the flow profit of the firm and the flow surplus of the consumers take the more general form

$$\bar{b}_t - c(a_t) \qquad \text{and} \qquad v(b_t^i, \bar{b}_t)\, dX_t - b_t^i\, dt,$$

respectively, where $c(a_t)$ is the cost of effort and $v(b_t^i, \bar{b}_t)$ is the consumer's valuation of the quality of the product. Assume that c and v are twice continuously differentiable and satisfy the standard conditions $c' > 0$, $c'' \geq 0$, $v_1 > 0$ and $v_{11} \leq 0$, where the subscripts denote partial derivatives. In addition, assume that

$$v_{12} < -v_{11}, \tag{24}$$

so that either there are negative externalities among the consumers, i.e. $v_{12} < 0$ as in the example above, or there are positive, but weak, externalities among them. Then, Conditions 1 and 2.b are satisfied and the conclusion of Theorem 4 still applies. An example in which condition (24) fails is analyzed in Section 7.

Finally, note that condition (24) arises naturally in the important case in which there is a continuous flow of short-lived consumers (as in Fudenberg and Levine (1992), Cripps, Mailath, and Samuelson (2004) and Faingold (2008)) instead of a cross-section population of small long-lived consumers (cf. the discussion in Section 3.1). In fact, when there is a single short-lived consumer at each time t, v is independent of $\bar{b}$ and therefore (24) reduces to the condition $v_{11} < 0$.

6.1 Reputation effects at the behavioral level (as $r \to 0$)

With the equilibrium uniqueness result in place, we can examine the impact of the large player's patience on the equilibrium strategies of the large player and the population of small players. In Theorem 5 below, we present a reputation effects result that is similar in spirit to Theorem 1 but concerns equilibrium *behavior* rather than equilibrium *payoffs*.

Define

$$\mathcal{N}' \stackrel{\text{def}}{=} \{(a, b, \phi) : \exists z \in \mathbb{R} \text{ such that } (a, b) \in \mathcal{N}(\phi, z)\}.$$

We need the folowing assumption:

Condition 3. *The following two conditions are satisfied:*

a. $\exists K > 0$ *such that* $|\mu(a^*, b) - \mu(a, b)| \geq K\,|a^* - a|$, $\forall (a, b)$ *in the range of* $\mathcal{N}$,

b. $\exists R > 0$ *such that* $|b^* - b| \leq R\,(1 - \phi)|a^* - a|$, $\forall (a, b, \phi) \in \mathcal{N}'$, $\forall b^* \in \mathsf{B}(a^*)$.

[24] When the drift is independent of the aggregate strategy of the small players and is monotonic in the large player's action, imposing a linear drift is just a normalization.

Condition 3.a is an identifiability condition: it implies that at every action profile in the range of $\mathcal{N}$, the action of the normal type can be statistically distinguished from a^* by estimating the drift of the public signal. Condition 3.b is a mild technical condition. To wit, the upper hemi-continuity of $\mathcal{N}$ implies that for any convergent sequence $(a_n, b_n, \phi_n)_{n\in\mathbb{N}}$ such that $(a_n, b_n) \in \mathcal{N}(\phi_n, z_n)$ for some z_n, if either $\lim a_n = a^*$ or $\lim \phi_n = 1$ then $\lim b_n = b^* \in \mathsf{B}(a^*)$; Condition 3.b strengthens this continuity property by requiring the modulus of continuity to be proportional to $(1-\phi_n)|a^* - a_n|$. In Section 6.2 below, we provide sufficient conditions for Condition 3 (as well as for Conditions 1 and 2) in terms of the primitives of the game.

Recall that the *Stackelberg payoff* is

$$\bar{g}^{\mathsf{s}} = \max_{a\in A} \max_{b\in\mathsf{B}(a)} g(a, b).$$

Say that a^* is a *Stackelberg action* if it attains the Stackelberg payoff, i.e. $\bar{g}^{\mathsf{s}} = \max_{b\in\mathsf{B}(a^*)} g(a^*, b)$. Finally, under Conditions 1 and 2, we write $(\mathsf{a}_r(\phi), \mathsf{b}_r(\phi))$ to designate the unique Markov perfect equilibrium when the discount rate is r and, likewise, we write $U_r(\phi)$ for the equilibrium value of the normal type.

Theorem 5. *Assume Conditions 1, 2 and 3 and that a^* is the Stackelberg action. Let $\{b^*\} = \mathsf{B}(a^*)$. Then, for each $\phi \in (0, 1)$,*

$$\lim_{r\to 0} U_r(\phi) = \bar{g}^{\mathsf{s}} = g(a^*, b^*) \quad \textit{and} \quad \lim_{r\to 0} (\mathsf{a}_r(\phi), \mathsf{b}_r(\phi)) = (a^*, b^*).$$

The rate of convergence above is not uniform in the reputation level: as ϕ approaches 0 or 1, the convergence gets slower and slower. The proof of Theorem 5, presented in Appendix C.5, uses the differential equation characterization of Theorem 4, as well as the payoff bounds of Theorem 1.

6.2 Primitive sufficient conditions

To illustrate the applicability of Theorems 4 and 5 we provide sufficient conditions for Conditions 1, 2 and 3 in terms of the basic data of the game—stage-game payoff functions, drift and volatility. The conditions are stronger than necessary, but are easy to check and have a transparent interpretation.

To ease the exposition, we focus on the case in which the actions and the public signal are one-dimensional.[25] We use subscripts to denote partial derivatives.

Proposition 4. *Suppose A, $B \subset \mathbb{R}$ are compact intervals, $a^* = \max A$ and the public signal is one-dimensional. If $g : A \times B \to \mathbb{R}$, $h : A \times B \times B \to \mathbb{R}$ and $\mu : A \times B \to \mathbb{R}$ are twice continuously differentiable, $\sigma : B \to \mathbb{R} \setminus \{0\}$ is continuous and for each $(a, b, \bar{b}) \in A \times B \times B$,*

(a) $g_{11}(a, \bar{b}) < 0$, $h_{22}(a, b, \bar{b}) < 0$, $(h_{22} + h_{23})(a, \bar{b}, \bar{b}) < 0$, $\big(g_{11}(h_{22} + h_{23}) - g_{12}h_{12}\big)(a, \bar{b}, \bar{b}) > 0$,

(b) $g_2(a, \bar{b})(h_2(a^*, \bar{b}, \bar{b}) - h_2(a, \bar{b}, \bar{b})) \geq 0$,

[25] In models with multi-dimensional actions / signals, the appropriate versions of conditions (a)–(d) below are significantly more cumbersome. This is reminiscent of the analysis of equilibrium uniqueness in one-shot games and has nothing to do with reputation dynamics.

(c) $\mu_1(a, \bar{b}) > 0$*,* $\mu_{11}(a, \bar{b}) \le 0$*,*

then Conditions 2.b and 3 are satisfied. If, in addition,

(d) $g_1(a^*, b^*) < 0$*, where* $\{b^*\} = \mathsf{B}(a^*)$*,*

then Condition 1 is also satisfied.

Evidently, any conditions implying equilibrium uniqueness in the reputation game ought to be strong conditions, as they must, at least, imply equilibrium uniqueness in the stage game. Interestingly, Proposition 4 shows that in our continuous-time framework such conditions are not much stronger than standard conditions for equilibrium uniqueness in static games. Indeed, (a) and (b) are standard sufficient conditions for the static-game analogue of Condition 2.b, so the only extra condition that comes from the reputation dynamics is (c), a monotonicity and concavity assumption on the drift which is natural in many settings. We view this as the main reason why our continuous-time formulation is attractive from the applied perspective. Many interesting applications, such as the product choice game of Section 2 and the signal manipulation and monetary policy games presented after Theorem 4 satisfy the conditions of Proposition 4.

For Condition 2.b, first we ensure that the best-reply correspondences are single-valued by assuming that the payoffs are strictly concave in own action, i.e. $g_{11} < 0$ and $h_{22} < 0$, and that $(h_{22}+h_{23})(a, b, b) < 0$ for all (a, b). The latter condition guarantees that the payoff complementarity between the actions of the small players is not too strong, ruling out coordination effects which can give rise to multiple equilibria. Next, the assumption that $g_{11}(h_{22} + h_{23}) - g_{12}h_{12} > 0$ (along the diagonal of the small players' actions) implies that the graphs of the best-reply functions intersect only once. This is also a familiar condition: together with $g_{11} < 0$ and $h_{22} < 0$ it implies the negative definiteness of the matrix of second-order derivatives of the payoff functions of the complete information one-shot game. In addition, condition (c) guarantees that the large player's best-reply remains unique when the reputational term $z\sigma(b)^{-2}(\mu(a^*, \bar{b}) - \mu(a, \bar{b}))\mu(a', \bar{b})$, with $z \ge 0$, is added to his static payoff. The monotonicity of the stage-game Bayesian Nash equilibrium payoff of the normal type follows from condition (b), which requires that whenever the small players can hurt the normal type by decreasing their action (so that $g_2 > 0$), their marginal utility cannot be smaller when they are facing the behavioral type than when they are facing the normal type (otherwise they would have an incentive to decrease their action when their beliefs increase, causing a decrease in the payoff of the normal type).

Condition 3.a follows from $\mu_1 > 0$ and Condition 3.b follows from $h_{22} < 0$ and $h_{22} + h_{23} \le 0$ (along the diagonal of the small players' actions). Finally, for Condition 1, observe that under the assumption that $\mu_1 > 0$ there is no $a \ne a^*$ with $\mu(a, b^*) = \mu(a^*, b^*)$. Thus Condition 1 follows directly from assumption (d), which states that a^* is not part of a static Nash equilibrium of the complete information game. The proof of Proposition 4 is presented in Appendix C.6.

Under a set of assumptions similar to those of Proposition 4, Liu and Skrzypacz (2010) prove uniqueness of sequential equilibrium in discrete-time reputation games in which the short-run players observe only the outcomes of a fixed, finite number of past periods. They also find that in the complete information version of their repeated game, the only equilibrium is the repeated play of the stage-game Nash equilibrium, a result similar to our Theorem 3.

Remark 4. The assumption that a^* is the greatest action in A is only for convenience. The conclusion of Proposition 4 remains valid under any $a^* \in A$, provided condition (c) is replaced by

$$\mu_1(a, \bar{b}) > 0, \quad (\mu(a^*, \bar{b}) - \mu(a, \bar{b}))\mu_{11}(a, \bar{b}) \le 0 \quad \forall (a, \bar{b}) \in A \times B,$$

and condition (d) by

$$g_1(a^*, b^*) \begin{cases} < 0 & : \quad a^* = \max A, \\ \neq 0 & : \quad a^* \in (\min A, \max A), \\ > 0 & : \quad a^* = \min A. \end{cases}$$

Remark 5. Among conditions (a)–(d), perhaps the ones that are most difficult to check are

$$\big(g_{11}(h_{22} + h_{23}) - g_{12}h_{12}\big)(a, \bar{b}, \bar{b}) > 0 \quad \text{and} \quad g_2(a, \bar{b})(h_2(a^*, \bar{b}, \bar{b}) - h_2(a, \bar{b}, \bar{b})) \ge 0.$$

But, assuming the other conditions in (a), a simple sufficient condition for the above is that

$$g_{12}(a, \bar{b}) \le 0, \quad h_{12}(a, \bar{b}, \bar{b}) \ge 0 \quad \text{and} \quad g_2(a, \bar{b}) \ge 0,$$

as is the case in the product choice game of Section 2.

7 General characterization

This section extends the analysis of Section 6 to environments with multiple sequential equilibria. When the correspondence $\mathcal{N}$ is not single-valued (so that Condition 2 is violated), the correspondence of sequential equilibrium payoffs, $\mathcal{E}$, may also fail to be single-valued. Theorem 6 below characterizes $\mathcal{E}$ in this general case.

To prove the general characterization, we maintain Condition 1 but replace Condition 2 by:

Condition 4. $\mathcal{N}(\phi, z) \neq \emptyset \ \forall (\phi, z) \in [0, 1] \times \mathbb{R}$.

This assumption is automatically satisfied in the mixed-strategy extension of the game (cf. remark 2). However, to simplify the exposition, we have chosen not to deal with mixed strategies explicitly and impose Condition 4 instead.

Consider the optimality equation of Section 6. When $\mathcal{N}$ is a multi-valued correspondence, there can be multiple bounded functions which solve

$$U''(\phi) = \frac{2U'(\phi)}{1 - \phi} + \frac{2r(U(\phi) - g(\mathsf{a}(\phi), \bar{\mathsf{b}}(\phi)))}{|\gamma(\mathsf{a}(\phi), \bar{\mathsf{b}}(\phi), \phi)|^2}, \quad \phi \in (0, 1) \tag{25}$$

for different selections $\phi \mapsto (\mathsf{a}(\phi), \bar{\mathsf{b}}(\phi)) \in \mathcal{N}(\phi, \phi(1 - \phi)U'(\phi)/r)$. An argument similar to the proof of Theorem 4 can be used to show that for each such solution U and each prior p, there exists a sequential equilibrium that achieves the payoff $U(p)$ for the normal type. Therefore, a natural conjecture is that the correspondence of sequential equilibrium payoffs, $\mathcal{E}$, contains all values between its upper boundary, the greatest solution of (25), and its lower boundary, the least solution of (25). Accordingly, the pair $(\mathsf{a}(\phi), \bar{\mathsf{b}}(\phi)) \in \mathcal{N}(\phi, \phi(1 - \phi)U'(\phi)/r)$ should minimize the right-hand side of (25) for the upper boundary, and maximize it for the lower boundary.

However, the differential equation

$$U''(\phi) \;=\; H(\phi, U(\phi), U'(\phi)), \quad \phi \in (0,1), \tag{26}$$

where $H : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is given by

$$H(\phi, u, u') \overset{\text{def}}{=} \frac{2u'}{1-\phi} + \min\left\{ \frac{2r(u - g(a,\bar{b}))}{|\gamma(a,\bar{b},\phi)|^2} \;:\; (a,\bar{b}) \in \mathcal{N}(\phi, \phi(1-\phi)u'/r) \right\}, \tag{27}$$

may fail to have a solution in the classical sense. In general, $\mathcal{N}$ is upper hemi-continuous but not necessarily lower hemi-continuous, so the right-hand side of (26) can be discontinuous. Due to this technical difficulty, our analysis relies on a generalized notion of solution, called *viscosity solution* (cf. Definition 2 below), which applies to discontinuous equations.[26] We show that the upper boundary $U(\phi) \overset{\text{def}}{=} \sup\, \mathcal{E}(\phi)$ is the greatest viscosity solution of the *upper optimality equation* (26), and that the lower boundary $L(\phi) \overset{\text{def}}{=} \inf\, \mathcal{E}(\phi)$ is the least solution of the *lower optimality equation*, which is defined analogously, by replacing minimum by maximum in the definition of H.

While in general viscosity solutions can fail to be differentiable, we show that the upper boundary U is continuously differentiable and has an absolutely continuous derivative. In particular, when $\mathcal{N}$ is continuous in a neighborhood of $(\phi, \phi(1-\phi)U'(\phi))$ for some $\phi \in (0,1)$, so that H is also continuous in that neighborhood, any viscosity solution of (26) is a classical solution around ϕ. Otherwise, we show that $U''(\phi)$, which exists almost everywhere since U' is absolutely continuous, must take values in the interval between $H(\phi, U(\phi), U'(\phi))$ and $H^*(\phi, U(\phi), U'(\phi))$, where H^* is the *upper semi-continuous envelope* of H, i.e. the least u.s.c. function which is greater than or equal to H everywhere. (Note that H is necessarily lower semi-continuous, i.e. $H = H_*$.)

Definition 2. A bounded function $U : (0,1) \to \mathbb{R}$ is a *viscosity super-solution* of the upper optimality equation if for every $\phi \in (0,1)$ and every twice continuously differentiable *test function* $V : (0,1) \to \mathbb{R}$,

$$U_*(\phi) = V(\phi) \;\text{ and }\; U_* \geq V \quad \Longrightarrow \quad V''(\phi) \leq H^*(\phi, V(\phi), V'(\phi))\,.$$

A bounded function $U : (0,1) \to \mathbb{R}$ is a *viscosity sub-solution* if for every $\phi \in (0,1)$ and every twice continuously differentiable test function $V : (0,1) \to \mathbb{R}$,

$$U^*(\phi) = V(\phi) \;\text{ and }\; U^* \leq V \quad \Longrightarrow \quad V''(\phi) \geq H_*(\phi, V(\phi), V'(\phi)).$$

A bounded function U is a *viscosity solution* if it is both a super-solution and a sub-solution.[27]

Appendix D presents the complete analysis, which we summarize here. Propositions D.1 and D.2 show that U, the upper boundary of $\mathcal{E}$, is a bounded viscosity solution of the upper optimality equation. Lemma D.3 then shows that U must be a continuously differentiable function with absolutely continuous derivative (so its second derivative exists almost everywhere), and hence U must solve the differential inclusion

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi))\,,\; H^*(\phi, U(\phi), U'(\phi))] \quad \text{a.e.} \tag{28}$$

[26]For an introduction to viscosity solutions we refer the reader to Crandall, Ishii, and Lions (1992).
[27]This is equivalent to Definition 2.2 in Crandall, Ishii, and Lions (1992).

In particular, when H is continuous in a neighborhood of $(\phi, U(\phi), U'(\phi))$ then U satisfies equation (26) in the classical sense in a neighborhood of ϕ. Finally, Proposition D.3 shows that U must be the *greatest* bounded solution of (28).

We summarize our characterization in the following theorem.

Theorem 6. *Assume Conditions 1 and 4 and that $\mathcal{E}$ is nonempty-valued. Then, $\mathcal{E}$ is a compact- and convex-valued continuous correspondence whose upper and lower boundaries are continuously differentiable functions with absolutely continuous derivatives. Moreover, the upper boundary of $\mathcal{E}$ is the greatest bounded solution of the differential inclusion (28), and the lower boundary is the least bounded solution of the analogous differential inclusion, where maximum is replaced by minimum in the definition of H.*

To illustrate the case of multiple sequential equilibria, we provide two examples.

(a) *Product choice with positive externalities.* This is a simple variation of the product choice game of Section 2. Suppose the firm chooses effort level $a_t \in [0,1]$, where $a^* = 1$ is the action of the behavioral type, and that each consumer chooses a level of service $b_t^i \in [0,2]$. The public signal about the firm's effort follows

$$dX_t = a_t\,dt + dZ_t.$$

The payoff flow of the normal type is $(\bar{b}_t - a_t)\,dt$ and each consumer $i \in I$ receives the payoff flow $b_t^i \bar{b}_t\,dX_t - b_t^i\,dt$. Thus, the consumers' payoff function features a positive externality: greater usage $\bar{b}_t$ of the service by other consumers allows each individual consumer to enjoy the service more.

The unique Nash equilibrium of the static game is $(0,0)$. The correspondence $\mathcal{N}(\phi, z)$ determines the action of the normal type uniquely by

$$a = \begin{cases} 0 & \text{if } z \leq 1 \\ 1 - 1/z & \text{otherwise.} \end{cases} \tag{29}$$

The consumers' actions are uniquely $\bar{b} = 0$ only when $(1-\phi)a + \phi a^* < 1/2$. When $(1-\phi)a + \phi a^* \geq 1/2$ the partial game amongst the consumers, who face a coordination problem, has two pure Nash equilibria with $\bar{b} = 0$ and $\bar{b} = 2$ (and one mixed equilibrium when $(1-\phi)a + \phi a^* > 1/2$). Thus, the correspondence $\mathcal{N}(\phi, z)$ is single-valued only on a subset of its domain.

How is this multiplicity reflected in the equilibrium correspondence $\mathcal{E}(p)$? Figure 5 displays the upper boundary of $\mathcal{E}(p)$ computed for discount rates $r = 0.1$, 0.2 and 0.5. The lower boundary for this example is identically zero, because the static game among the consumers has an equilibrium with $\bar{b} = 0$. For each discount rate, the upper boundary U is divided into three regions. In the region near $\phi = 0$, where the upper boundary is displayed as a solid line, the correspondence $\phi \mapsto \mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r)$ is single-valued and U satisfies the upper optimality equation in the classical sense. In the region near $\phi = 1$, where the upper boundary is displayed as a dashed line, the correspondence $\mathcal{N}$ is continuous and takes multiple values (two pure and one mixed). There, U also satisfies the upper optimality equation in the classical sense, with the small players' action given by $\bar{b} = 2$. In the middle region, where the upper boundary is shows as a dotted line, we have

$$U''(\phi) \in \left(\frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - 2 + a)}{|\gamma(a, 2, \phi)|^2}, \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - 0 + a)}{|\gamma(a, 0, \phi)|^2} \right),$$

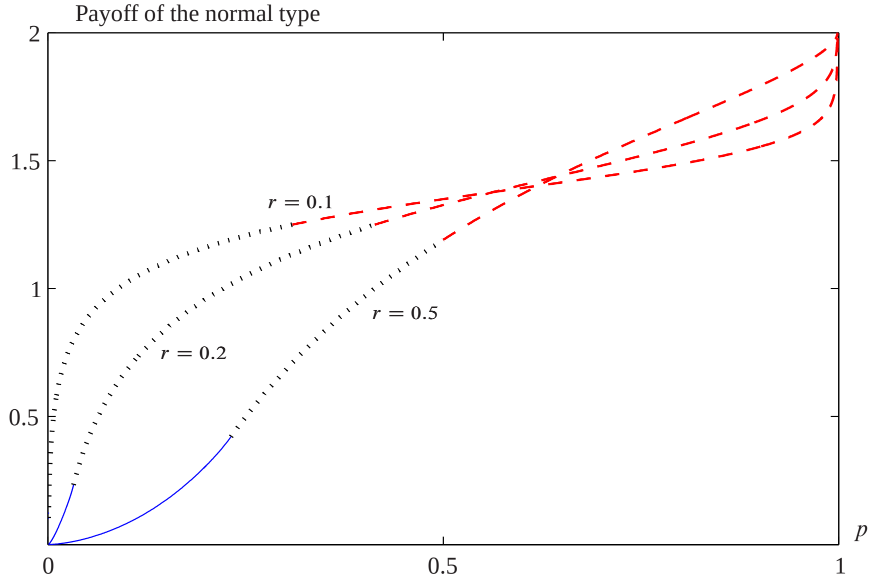

Figure 5: The upper boundary of $\mathcal{E}(p)$.

where a is given by (29) with $z = \phi(1-\phi)U'(\phi)/r$, and 0 and 2 are the two pure values of $\bar{b}$ that the correspondence $\mathcal{N}$ returns. In that range, the correspondence $\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r)$ is discontinuous in its arguments: if we lower $U'(\phi)$ slightly the equilibrium among the consumers with $\bar{b} = 2$ breaks down. These properties of the upper boundary follow from its characterization as the greatest solution of the upper optimality equation.

(b) *Bad reputation.* This is a continuous-time version of the reputation game of Ely and Valimaki (2003) with noisy public signals (see also Ely, Fudenberg, and Levine (2008)). The large player is a car mechanic, who chooses the probability $a_1 \in [0,1]$ with which he replaces the engine on cars that need an engine replacement, and the probability $a_2 \in [0,1]$ with which he replaces the engine on cars that need a mere tune-up. Thus, the stage-game action of the mechanic, $a = (a_1, a_2)$, is two-dimensional.

Each small player (car owner), without the knowledge of whether his car needs an engine replacement or a tune-up, decides on the probability $b \in [0,1]$ with which he brings the car to the mechanic. The behavioral type of mechanic—a bad behavioral type—replaces the engines of all cars, irrespective of which repair is more suitable, i.e. $a^* = (1,1)$. Car owners observe noisy information about the number of engines replaced by the mechanic in the past:

$$dX_t = \bar{b}_t(a_{1,t} + a_{2,t})\,dt + dZ_t.$$

The payoffs are given by

$$g(a, \bar{b}) = \bar{b}(a_1 - a_2) \quad \text{and} \quad h(a, b, \bar{b}) = b(a_1 - a_2 - 1/2).$$

Thus the normal type of mechanic is a good type, in that he prefers to replace the engine only when it is needed. As for the car owners, they prefer to take their car to the mechanic only when they believe the mechanic is sufficiently honest, i.e. $a_1 - a_2 \geq 1/2$; otherwise, they prefer not to take the car to the mechanic at all.

While this game violates Condition 1, we are able to characterize the set of public sequential equilibrium payoffs of the normal type via a natural extension of Theorem 6 (see Remark 6 below). The correspondence $\mathcal{E}$ is illustrated in Figure 6. The lower boundary of this correspondence, L, is identically 0. The upper boundary, U, displayed as a solid line, is characterized by three regions: (i) for $\phi \geq 1/2$, $U(\phi) = 0$, (ii) for $\phi \in [\phi^*, 1/2]$, $U(\phi) = r \log \frac{1-\phi}{\phi}$, and (iii) for $\phi \leq \phi^*$, $U(\phi)$ solves the upper optimality equation in the classical sense:

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi)-1)}{\phi^2(1-\phi)^2},$$

with boundary conditions

$$U(0) = 1, \quad U(\phi^*) = r \log \frac{1-\phi^*}{\phi^*} \quad \text{and} \quad U'(\phi^*) = -\frac{r}{\phi^*(1-\phi^*)},$$

which is attained by the action profile $(a, \bar{b}) = ((1, 0), 1)$.[28] In particular, the upper boundary U is always below $r \log \frac{1-\phi}{\phi}$ for $\phi < 1/2$, as illustrated in Figure 6.

For $\phi \leq 1/2$ the set $\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r)$ includes the profile $((1, 0), 1)$ as well as the profiles $((a_1, a_2), 0)$ with $a_1 - a_2 \leq 1/(2(1-\phi))$, and the former profile must be played on the upper boundary of the equilibrium set in this belief range.[29] Moreover, for each prior $\phi < 1/2$, there is a (non-Markovian) sequential equilibrium attaining the upper boundary in which $((1, 0), 1)$ is played on the equilibrium path so long as the car owners' posterior remains below $1/2$ and the continuation value of the normal type is strictly positive.[30] However, since $r \log \frac{1-\phi}{\phi} \to 0$ as $r \to 0$ for all $\phi > 0$, for all priors the greatest equilibrium payoff of the normal type converges to 0 as $r \to 0$. This result is analogous to Ely and Valimaki (2003, Theorem 1).

Remark 6. The characterization of Theorem 6 can be extended to games in which Condition 1 fails as follows. Denote by

$$\mathcal{V}(\phi) = \text{co}\left\{g(a, \bar{b}) \;:\; (a, \bar{b}) \in \mathcal{N}(\phi, 0) \text{ and } \mu(a, \bar{b}) = \mu(a^*, \bar{b})\right\}.$$

[28] This differential equation turns out to have a closed-form solution:

$$U(\phi) = 1 - \left(1 - r \log \frac{1-\phi^*}{\phi^*}\right)\left[\frac{(1-\phi^*)\phi}{\phi^*(1-\phi)}\right]^{\frac{1}{2}(1+\sqrt{1+8r})}, \quad \phi \in [0, \phi^*].$$

The cutoff ϕ^* is pin down by the condition $U'(\phi^*) = -r/(\phi^*(1-\phi^*))$, which yields $\phi^* = 1/\big(1 + e^{\frac{5-\sqrt{1+8r}}{4r}}\big)$.

[29] For $\phi \in [\phi^*, 1/2]$, the slope $U'(\phi) = -r/(\phi(1-\phi))$ is such that, at the profile $((1, 0), 1) \in \mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r)$, the mechanic is indifferent among all actions $(a_1, 0)$, $a_1 \in [0, 1]$. Moreover, the correspondence $\mathcal{N}$ is discontinuous at $(\phi, \phi(1-\phi)U'(\phi)/r)$ for all $\phi \in [\phi^*, 1/2]$ and U satisfies the differential inclusion (28) with strict inequality.

[30] In this equilibrium, the posterior on the behavioral type, ϕ_t, and the continuation values of the normal type, W_t, follow $d\phi_t = \phi_t(1-\phi_t)\,(dX_t - (\phi_t + 1)\,dt)$ and $dW_t = r(W_t - 1)\,dt + \phi_t(1-\phi_t)U'(\phi_t)\,(dX_t - dt)$ up until the first time when (ϕ_t, W_t) hits the lower boundary; from then on, the equilibrium play follows the static equilibrium $((0, 0), 0)$ and the posterior no longer updates. When ϕ_t is below ϕ^*, the optimality equation implies that the pair (ϕ_t, W_t) remains at the upper boundary; but, for $\phi_t \in [\phi^*, 1/2]$, the differential inclusion, which is satisfied with strict inequality, implies that W_t eventually falls strictly below the upper boundary.

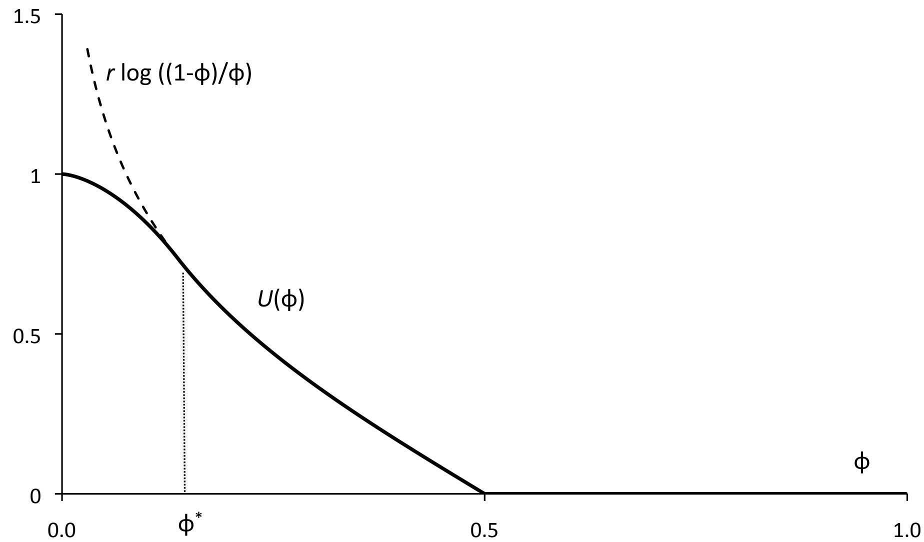

Figure 6: The correspondence $\mathcal{E}$ in the bad reputation game (for $r = 0.4$).

The set $\mathcal{V}(\phi)$ can be attained by equilibria in which the normal type always "looks" like the behavioral type, and his reputation stays fixed. For games in which Condition 1 holds, the correspondence $\mathcal{V}(\phi)$ is empty, and for the bad reputation example above, $\mathcal{V}(\phi) = \{0\}$ for all ϕ. Then, for any belief $\phi \in (0,1)$, the upper boundary, $U(\phi)$, is the maximum between $\max \mathcal{V}(\phi)$ and the greatest solution of the differential inclusion (28) that remains bounded both to the left and to the right of ϕ until the end of the interval $(0,1)$ or until it reaches the correspondence $\mathcal{V}$. The lower boundary is characterized analogously. For our bad reputation example, the upper boundary of $\mathcal{E}$ solves the differential inclusion on $(0, 1/2]$ and reaches the correspondence $\mathcal{V}$ at the belief level $1/2$.

Remark 7. A caveat of Theorem 6 is that it assumes the existence of a public sequential equilibrium, although existence is not guaranteed in general unless we assume the availability of a public randomization device. Standard existence arguments based on fixed-point methods do not apply to our continuous-time games (even assuming finite action sets and allowing mixed strategies), because continuous time renders the set of partial histories at any time t uncountable.[31] This is similar to the existence problem that arises in the context of subgame perfect equilibrium in extensive-form games with continuum action sets, as in Harris, Reny, and Robson (1995). Moreover, while it can be shown that the differential inclusion (28) is guaranteed to have a bounded solution $U : (0,1) \to \mathbb{R}$ under Conditions 1 and 4, this does not imply the existence of a sequential equilibrium, because a selection $\phi \mapsto (\mathsf{a}(\phi), \bar{\mathsf{b}}(\phi))$ satisfying (25) may not exist when $\mathcal{N}$ is not connected-valued. However, if the model is suitably enlarged to allow for public randomization, then existence of sequential equilibrium is restored. In our supplementary appendix, Faingold and Sannikov (2010), we explain the formalism of public randomization in continuous time and demonstrate that a sequential equilibrium in publicly randomized strategies is guaranteed to exist under Conditions 1

[31]At a technical level, the problem is the lack of a topology on strategy sets under which expected discounted payoffs are continuous and strategy sets are compact.

and 4. It is interesting to note that also in Harris, Reny, and Robson (1995) public randomization is the key to obtain existence of subgame perfect equilibrium.

8 Multiple behavioral types

In this section we consider reputation games with multiple behavior types. We extend the recursive characterization of sequential equilibrium of Section 4 and prove an analogue of Cripps, Mailath, and Samuelson's (2004) result that reputation effects are a temporary phenomenon. These results lay the groundwork for future analysis of these games, and we leave the extension of the characterizations of Sections 6 and 7 for future research.

Suppose there are $K < \infty$ behavioral types, where each type $k \in \{1, \dots, K\}$ plays a fixed action $a_k^* \in A$ at all times, after all histories. Initially the small players believe that the large player is behavioral type k with probability $p_k \in [0, 1]$, so that $p_0 = 1 - \sum_{k=1}^K p_k > 0$ is the prior probability on the normal type. All else is exactly as described in Section 3.

To derive the recursive characterization of sequential equilibrium, the main difference from Section 4 is that now the small players' belief is a vector in the K-dimensional simplex, Δ^K. Accordingly, for each $k = 0, \dots, K$ we write $\phi_{k,t}$ to designate the small players' belief that the large player is of type k, and write $\phi_t = (\phi_{0,t}, \dots, \phi_{K,t})$.

Proposition 5 (Belief Consistency)**.** *Fix a prior $p \in \Delta^K$. A belief process $(\phi_t)_{t \geq 0}$ is consistent with a strategy profile $(a_t, \bar{b}_t)_{t \geq 0}$ if and only if $\phi_0 = p$ and for each $k = 0, \dots, K$,*

$$d\phi_{k,t} = \gamma_k(a_t, \bar{b}_t, \phi_t) \cdot \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\, dt), \tag{30}$$

where for each $(a, \bar{b}, \phi) \in A \times \Delta(B) \times \Delta^K$,

$$\gamma_0(a, \bar{b}, \phi) \stackrel{\text{def}}{=} \phi_0 \sigma(\bar{b})^{-1}(\mu(a, \bar{b}) - \mu^\phi(a, \bar{b})),$$

$$\gamma_k(a, \bar{b}, \phi) \stackrel{\text{def}}{=} \phi_k \sigma(\bar{b})^{-1}(\mu(a_k^*, \bar{b}) - \mu^\phi(a, \bar{b})), \quad k = 1, \dots, K,$$

$$\mu^\phi(a, \bar{b}) \stackrel{\text{def}}{=} \phi_0 \mu(a, \bar{b}) + \sum_{k=1}^K \phi_k \mu(a_k^*, \bar{b}).$$

Proof. As in the proof of Proposition 1, the relative likelihood $\xi_{k,t}$ that a signal path arises $(X_s; s \in [0, t])$ from the behavior of type k instead of the normal type is characterized by

$$d\xi_{k,t} = \xi_{k,t}\, \rho_{k,t} \cdot dZ_s^0, \qquad \xi_{k,0} = 1,$$

where $\rho_{k,t} \stackrel{\text{def}}{=} \sigma(\bar{b}_t)^{-1}(\mu(a_k^*, \bar{b}_t) - \mu(a_t, \bar{b}_t))$ and $dZ_t^0 \stackrel{\text{def}}{=} \sigma(\bar{b}_t)^{-1}(dX_t - \mu(a_t, \bar{b}_t)\, dt)$ is a Brownian motion under the normal type. By Bayes' rule,

$$\phi_{k,t} = \frac{p_k \xi_{k,t}}{p_0 + \sum_{k=1}^K p_k \xi_{k,t}}.$$

Applying Itô's formula to this expression yields

$$d\phi_{k,t} = \frac{p_k}{p_0 + \sum_{k=1}^K p_k \xi_{k,t}}\, d\xi_{k,t} - \sum_{k'=1}^K \frac{p_k \xi_{k,t}}{(p_0 + \sum_{k=1}^K p_k \xi_{k,t})^2} p_{k'}\, d\xi_{k',t} + (\dots)\, dt,$$

where we do not need to derive the $(\ldots)\,dt$ term because we know that $(\phi_{k,t})_{t\geq 0}$ is a martingale from the point of view of the small players. Since $dZ_t^{\phi} = \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\,dt)$ is a Brownian motion from the viewpoint of the small players,

$$\begin{aligned} d\phi_{k,t} &= \frac{p_k}{p_0 + \sum_{k=1}^K p_k \xi_{k,t}} \xi_{k,t}\rho_{k,t} \cdot dZ_t^{\phi} - \Big(\sum_{k'=1}^K \frac{p_k \xi_{k,t}}{(p_0 + \sum_{k=1}^K p_k \xi_{k,t})^2} p_{k'}\xi_{k',t}\rho_{k',t} \Big) \cdot dZ_t^{\phi} \\ &= \phi_{k,t}(\rho_{k,t} - \sum_{k'=1}^K \phi_{k',t}\rho_{k',t}) \cdot dZ_t^{\phi} \\ &= \phi_{k,t}\sigma(\bar{b}_t)^{-1}(\mu(a_k^*, \bar{b}_t) - \mu^{\phi_t}(a_t, \bar{b}_t)) \cdot dZ_t^{\phi}, \end{aligned}$$

which is the desired result. ∎

Proceeding towards the recursive characterization of sequential equilibrium, it can be readily verified that the characterization of the continuation value of the normal type, viz. Proposition 2, remains valid under multiple behavior types. Also the characterization of sequential rationality given in Proposition 3 continues to hold, provided the small players' incentive constraint (12) is replaced by

$$\bar{b}_t \in \operatorname*{arg\,max}_{b \in B} \; \phi_{0,t} h(a_t, b, \bar{b}_t) + \sum_{k=1}^K \phi_{k,t} h(a_k^*, b, \bar{b}_t) \quad \forall b \in \operatorname{supp}\; \bar{b}_t. \tag{31}$$

Thus we have the following analogue of Theorem 2:

Theorem 7 (Sequential Equilibrium)**.** *Fix the prior $p \in \Delta^K$. A public strategy profile $(a_t, \bar{b})_{t\geq 0}$ and a belief process $(\phi_t)_{t\geq 0}$ form a sequential equilibrium with continuation values $(W_t)_{t\geq 0}$ for the normal type if and only if there exists a random process $(\beta_t)_{t\geq 0}$ in $\mathcal{L}$ such that*

(a) $(\phi_t)_{t\geq 0}$ *satisfies equation* (30) *with initial condition* $\phi_0 = p$,

(b) $(W_t)_{t\geq 0}$ *is a bounded process satisfying equation* (7)*, given* $(\beta_t)_{t\geq 0}$,

(c) $(a_t, \bar{b}_t)_{t\geq 0}$ *satisfy the incentive constraints* (11) *and* (31)*, given* $(\beta_t)_{t\geq 0}$ *and* $(\phi_t)_{t\geq 0}$.

It is beyond the scope of this paper to apply Theorem 7 to obtain characterizations similar to Theorems 4 and 6 under multiple behavioral types. However, to illustrate the applicability of Theorem 7, we use it to prove an analogue of Cripps, Mailath, and Samuelson's (2004) result that in every equilibrium the reputation of the large player disappears in the long run, when the large player is the normal type.

The following condition extends Condition 1 to the setting with multiple behavioral types:

Condition 1′. *For each $\phi \in \Delta^K$ and each static Bayesian Nash equlibrium $(a, \bar{b})$ of the game with prior ϕ, we have $\mu(a, \bar{b}) \notin \operatorname{co}\{\mu(a_k^*, \bar{b}) : k = 1, \ldots, K\}$.*

Note that when the flow payoff of the small players depends on the actions of the large player only through the public signal (cf. equation (1) and the discussion therein), then Condition 1′ becomes equivalent to a simpler condition: *for each static Nash equilibrium $(a, \bar{b})$ of the complete information game, $\mu(a, \bar{b}) \notin \operatorname{co}\{\mu(a_k^*, \bar{b}) : k = 1, \ldots, K\}$.*

The following result, whose proof is presented in Appendix E, is similar to Cripps, Mailath, and Samuelson (2004, Theorem 4), albeit under a different assumption.

Theorem 8. *Under Condition* $1'$*, in every public sequential equilibrium* $\lim_{t\to\infty}\phi_{0,t} = 1$ *almost surely under the normal type.*

When there is a single behavioral type, Condition $1'$ cannot be dispensed with. If it fails, then for some prior there is a static Bayesian Nash equilibrium (BNE) in which the behavior of the normal type is indistinguishable from the behavioral type. Thus, the repeated play of this BNE is a sequential equilibrium of the reputation game in which the posterior belief never changes.

While the conclusion of Theorem 8 is similar to the conclusion of Cripps, Mailath, and Samuelson (2004, Theorem 4), our assumptions are different. In general, the discrete-time analogue of Condition $1'$—viz. the requirement that in every static Bayesian Nash equilibrium the distribution over signals induced by the normal type is not a convex combination of the distributions induced by the behavioral types—is neither stronger nor weaker than the assumptions of Cripps, Mailath, and Samuelson (2004, Theorem 4), which are:

(i) the stage-game actions of the large player are identifiable;

(ii) no behavioral type plays an action which is part of a static Nash equilibrium of the complete information game;

(iii) the small players' best-reply to each behavioral type is unique.

However, when there is a single behavioral type, Condition $1'$ is implied by conditions (i) and (ii) above. It is therefore surprising that our result does not require condition (iii), which is known to be a necessary assumption in the discrete-time setting of Cripps, Mailath, and Samuelson (2004). The reason why we can dispense with condition (iii) in our continuous-time framework is related to our equilibrium degeneracy result under complete information (Theorem 3). In discrete time, when condition (i) and (ii) hold but condition (iii) fails, it is generally possible to construct sequential equilibria where the normal type plays the action of the behavioral type after every history, in which case the large player's type is not revealed in the long run. In such equilibrium, the incentives of the normal type arise from the threat of a punishment phase in which the small players play the best-reply to the behavioral type that the normal type dislikes the most. However, we know from the analysis of Section 5 that intertemporal incentives of this sort cannot be provided in our continuous-time setting.

9 Poisson signals

This section examines a variation of the model in which the public signals are driven by a Poisson process, instead of a Brownian motion. First, we derive a recursive characterization of public sequential equilibrium akin to Theorem 2. Then, for a class of games in which the signals are "good news," we show that the sequential equilibrium is unique, Markovian and characterized by a functional differential equation.

As discussed in Section 5, our result that the equilibria of the underlying complete information game is degenerate under Brownian signals (Theorem 3) is reminiscent of Abreu, Milgrom, and Pearce's (1991) analysis of a repeated prisoners' dilemma with Poisson signals. This seminal paper shows that when the arrival of a Poisson signal is "good news"—evidence that the players are behaving non-opportunistically—the equilibria of the repeated game must collapse to the static Nash equilibrium in the limit as the period length tends to zero, irrespective of the discount rate. On the other hand, when Poisson arrivals are "bad news"—evidence of non-cooperative behavior—Abreu, Milgrom, and Pearce (1991) show that the greatest symmetric public equilibrium payoff is greater than, and bounded away from, the static Nash equilibrium payoff as the length of the period shrinks, provided the discount rate is low enough and the signal is sufficiently informative. Thus, in complete information games, the structure of equilibrium is qualitatively different under Poisson and Brownian signals.[32] A similar distinction exists in the context of reputation games, and we exploit it below to derive a characterization of sequential equilibrium in the good news case.[33]

Throughout the section we assume that the public signal, now denoted $(N_t)_{t\geq 0}$, is a counting process with Poisson intensity $\lambda(a_t, \bar{b}_t) > 0$, where $\lambda : A \times B \to \mathbb{R}_+$ is a continuous function. This means that $(N_t)_{t\geq 0}$ is increasing and right-continuous, has left limits everywhere, takes values in the non-negative integers and has the property that $N_t - \int_0^t \lambda(a_s, \bar{b}_s)\, ds$ is a martingale. A *public strategy* profile is now a random process $(a_t, \bar{b}_t)_{t\geq 0}$ with values in $A \times \Delta(B)$ which is predictable with respect to the filtration generated by $(N_t)_{t\geq 0}$.[34] Otherwise, the structure of the reputation game is as described in Section 3.

The next proposition, which is analogous to Proposition 1, characterizes the evolution of the small players' posterior beliefs under Poisson signals.

Proposition 6 (Belief Consistency)**.** *Fix a prior probability $p \in [0, 1]$ on the behavioral type. A belief process $(\phi_t)_{t\geq 0}$ is consistent with a public strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ if and only if $\phi_0 = p$ and*

$$d\phi_t = \Delta\phi_{t-}(a_t, \bar{b}_t)\,(dN_t - \lambda^{\phi_{t-}}(a_t, \bar{b}_t)\,dt), \tag{32}$$

where for each $(a, \bar{b}, \phi) \in A \times \Delta B \times [0, 1]$,

$$\Delta\phi(a, \bar{b}) \stackrel{\text{def}}{=} \phi(1-\phi)\frac{(\lambda(a^*, \bar{b}) - \lambda(a, \bar{b}))}{\lambda^{\phi}(a, \bar{b})},$$

$$\lambda^{\phi}(a, \bar{b}) \stackrel{\text{def}}{=} \phi\lambda(a^*, \bar{b}) + (1-\phi)\lambda(a, \bar{b}).$$

Proof. Similar to the proof of Proposition 1, but replaces Girsanov's Theorem and Itô's formula by their appropriate counterparts in the Poisson setting (cf. Brémaud (1981, pp. 165–168, 337–339)). ■

Equation (32) above means that the posterior jumps by $\phi_t - \phi_{t-} = \Delta\phi_{t-}(a_t, \bar{b}_t)$ when a Poisson event arrives, and that between two consecutive arrivals the posterior follows the differential equation

$$d\phi_t/dt = -\phi_t(1-\phi_t)(\lambda(a^*, \bar{b}_t) - \lambda(a_t, \bar{b}_t)),$$

[32] Sannikov and Skrzypacz (2010) examine this difference in detail, showing that long-run players must use the Brownian and Poisson signals in distinct ways to create incentives. Specifically, Brownian signals can be used effectively only through payoff transfers along tangent hyperplanes, as in Fudenberg, Levine, and Maskin (1994), whereas Poisson jumps can also create incentives by "burning value," i.e. moving orthogonally to the tangent hyperplane.

[33] In a recent paper, Board and Meyer-ter-Vehn (2010) also compare the structure of equilibria under Brownian, Poisson good news and Poisson bad news signals in a product choice game in continuous-time. In the class of games they examine, the firm has two strategic types.

[34] For the definition of predictability see (Brémaud, 1981, p. 8). Any real-valued process $(Y_t)_{t\geq 0}$ which is predictable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ must have the property that each Y_t is measurable w.r.t. the σ-field $\mathcal{F}_{t-} \stackrel{\text{def}}{=} \bigvee_{s<t} \mathcal{F}_s$.

which is pathwise deterministic. Thus, coefficient $\Delta\phi(a, \bar{b})$ plays here a role similar to that played by $\gamma(a, \bar{b}, \phi)$ in the Brownian setting, measuring the sensitivity of the belief updating to the fluctuations in the public signal. From the small players' viewpoint the process $N_t - \int_0^t \lambda^{\phi_{s-}}(a_s, \bar{b}_s)\, ds$ is a martingale and so is the belief process $(\phi_t)_{t\geq 0}$.

The following proposition, which is analogous to Proposition 2, characterizes the evolution of the continuation value of the normal type.

Proposition 7 (Continuation Values)**.** *A bounded process $(W_t)_{t\geq 0}$ is the process of continuation values of the normal type under a strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ if and only if there exists a predictable process $(\zeta_t)_{t\geq 0}$ such that*

$$dW_t = r(W_{t-} - g(a_t, \bar{b}_t))\, dt + r\zeta_t(dN_t - \lambda(a_t, \bar{b}_t)\, dt). \tag{33}$$

Proof. Similar to the proof of Proposition 2, but replaces the Martingale Representation Theorem with the analogue result for Poisson-driven martingales (cf. Brémaud (1981, p. 68)). ■

Equation (33) means that the continuation value of the normal type jumps by $W_t - W_{t-} = r\zeta_t$ when a Poisson event arrives, and has drift given by $r(W_t - g(a_t, \bar{b}_t) - \zeta_t\lambda(a_t, \bar{b}_t))$ between two consecutive Poisson arrivals.

Turning to sequential rationality, for each $(\phi, \zeta) \in [0, 1] \times \mathbb{R}$ consider the set $\mathcal{M}(\phi, \zeta)$ of all action profiles $(a, \bar{b}) \in A \times \Delta(B)$ satisfying

$$a \in \operatorname*{arg\,max}_{a' \in A} g(a', \bar{b}) + \zeta\lambda(a', \bar{b}),$$

$$b \in \operatorname*{arg\,max}_{b' \in B} \phi h(a^*, b', \bar{b}) + (1 - \phi)h(a, b', \bar{b}) \quad \forall b \in \operatorname{supp} \bar{b}.$$

The next result, which is analogous to Proposition 3 and has a similar proof, characterizes sequential rationality using these local incentive constraints.

Proposition 8 (Sequential Rationality)**.** *Let $(a_t, \bar{b}_t)_{t\geq 0}$ be a public strategy profile, $(\phi_t)_{t\geq 0}$ a belief process and $(\zeta_t)_{t\geq 0}$ the predictable process from Proposition 7. Then, $(a_t, \bar{b}_t)_{t\geq 0}$ is sequentially rational with respect to $(\phi_t)_{t\geq 0}$ if and only if*

$$(a_t, \bar{b}_t) \in \mathcal{M}(\phi_{t-}, \zeta_t) \quad \textit{almost everywhere}. \tag{34}$$

We can summarize the recursive characterization of sequential equilibria of Poisson reputation games in the following theorem.

Theorem 9 (Sequential Equilibrium)**.** *Fix the prior probability $p \in [0, 1]$ on the behavioral type. A public strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ and a belief process process $(\phi_t)_{t\geq 0}$ form a sequential equilibrium with continuation payoffs $(W_t)_{t\geq 0}$ for the normal type if and only if there exists a predictable process $(\zeta_t)_{t\geq 0}$ such that*

(a) $(\phi_t)_{t\geq 0}$ solves equation (32) with initial condition $\phi_0 = p$;

(b) $(W_t)_{t\geq 0}$ is bounded and satisfies equation (33), given $(\zeta_t)_{t\geq 0}$;

(c) $(a_t, \bar{b}_t)_{t\geq 0}$ satisfies the incentive constraint (34), given $(\zeta_t)_{t\geq 0}$ and $(\phi_t)_{t\geq 0}$.

9.1 Good news and unique sequential equilibrium

In this section we identify a class of Poisson reputation games with a unique sequential equilibrium. We assume that the payoff functions and the signal structure satisfy conditions similar to those of Proposition 4, and, in addition, assume that the Poisson signals are good news. First, we show that under complete information, the unique equilibrium of the continuous-time game is the repeated play of the static equilibrium (Theorem 10), as in Abreu, Milgrom, and Pearce (1991). Second, for reputation games, we show that the sequential equilibrium is unique, Markovian and characterized by a functional differential equation (Theorem 11). Finally, we discuss briefly how the analysis would change when the Poisson signals are bad news and, in particular, why we expect the uniqueness and characterization results to break down in this case.

We impose the following assumptions: A, $B \subset \mathbb{R}$ are compact intervals, $a^* = \max A$, the Poisson intensity depends only on the large player's action—so we write $\lambda(a)$ for each $a \in A$—, the functions g, h and λ are twice continuously differentiable, and for each $(a, b, \bar{b}) \in A \times B \times B$,

(a) $g_{11}(a, \bar{b}) < 0$, $h_{22}(a, b, \bar{b}) < 0$, $(h_{22} + h_{23})(a, \bar{b}, \bar{b}) < 0$, $g_{12}(a, \bar{b}) \le 0$, $h_{12}(a, \bar{b}, \bar{b}) \ge 0$,

(b) $g_2(a, \bar{b}) \ge 0$,

(c) $\lambda'(a) > 0$, $\lambda''(a) \le 0$,

(d) $g_1(a^*, b^*) < 0$, where $\{b^*\} = \mathsf{B}(a^*)$.

For future reference, we call this set of assumptions the *good news model*.

Note that conditions (a), (b) and (d) above are similar to conditions (a), (b) and (d) of Proposition 4, albeit slightly stronger (cf. Remark 5). Condition (c) means that the signals are "good news" for the large player. Indeed, under $\lambda' > 0$ the arrival of a Poisson signal is perceived by the small players as evidence in favor of the behavioral type; this is beneficial for the normal type, since under conditions (a) and (b) the flow payoff of the normal type is increasing in reputation. This is in contrast to the case in which the signals are "bad news," which arises when conditions (a), (b) and (d) hold but condition (c) is replaced by:

(c′) $\lambda'(a) < 0$, $\lambda''(a) \ge 0$.

We discuss the bad news case briefly at the end of the section.

We begin the analysis with the complete information game. In the good news model, the equilibrium of the continuous-time game collapses to the static Nash equilibrium, similar to what happens in the Brownian case (Theorem 3) and in the repeated prisoners' dilemma of Abreu, Milgrom, and Pearce (1991).

Theorem 10. *In the good news model, if the small players are certain that they are facing the normal type, i.e. $p = 0$, then the unique public equilibrium of the continuous-time game is the repeated play of the unique static Nash equilibrium, irrespective of the discount rate.*

Proof. As shown in the first part of the proof of Proposition 4, the static game has a unique Nash equilibrium (a^N, b^N), where b^N is a mass-point distribution. Let $(a_t, \bar{b}_t)_{t \ge 0}$ be an arbitrary public equilibrium with continuation values $(W_t)_{t \ge 0}$ for the normal type. Suppose, towards a contradiction, that

$W_0 > g(a^N, b^N)$. (The proof for the reciprocal case, $W_0 < g(a^N, b^N)$, is similar and therefore omitted.) By Theorem 9, for some predictable process $(\zeta_t)_{t\geq 0}$ the large player's continuation value must follow

$$dW_t = \underbrace{r(W_t - g(a_t, \bar{b}_t) - \zeta_t \lambda(a_t))}_{\text{drift}}\, dt + \underbrace{r\zeta_t}_{\text{jump size}}\, dN_t,$$

where a_t maximizes $g(a', \bar{b}_t) + \zeta_t \lambda(a')$ over all $a' \in A$, and $\bar{b}_t$ maximizes $h(a_t, b', \bar{b}_t)$ over all $b' \in \Delta(B)$. Let $\bar{D} \overset{\text{def}}{=} W_0 - g(a^N, b^N) > 0$.

Claim. *There exists $c > 0$ such that, so long as $W_t \geq g(a^N, b^N) + \bar{D}/2$, either the drift of W_t is greater than $r\bar{D}/4$ or, upon the arrival of a Poisson event, the size of the jump in W_t is greater than c.*

The claim follows from the following lemma, whose proof is in Appendix F.

Lemma 2. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \geq 0$ and after all public histories, $g(a_t, \bar{b}_t) + \zeta_t \lambda(a_t) \geq g(a^N, b^N) + \varepsilon$ implies $\zeta_t \geq \delta$.*

Indeed, letting $\varepsilon = \bar{D}/4$ in this lemma gives a $\delta > 0$ with the property that whenever $g(a_t, \bar{b}_t) + \zeta_t \lambda(a_t) \geq rg(a^N, b^N) + \varepsilon$ the size of the jump in W_t, which equals $r\zeta_t$, must be greater than or equal to $c \overset{\text{def}}{=} r\delta$. Moreover, when $g(a_t, \bar{b}_t) + \zeta_t \lambda(a_t) < rg(a^N, b^N) + \bar{D}/4$ and $W_t \geq g(a^N, b^N) + \bar{D}/2$, then the drift of W_t, which equals $r(W_t - g(a_t, \bar{b}_t) - \zeta_t \lambda(a_t))$, must be greater than $r\bar{D}/4$, and this concludes the proof of the claim.

Since we have assumed that $\bar{D} = W_0 - g(a^N, b^N) > 0$, the claim above readily implies that W_t must grow arbitrarily large with positive probability, and this is a contradiction since $(W_t)_{t\geq 0}$ is bounded. ■

We now turn to the incomplete information case, i.e. $p \in (0, 1)$. Theorem 11 below shows that in the good news model the reputation game has a unique sequential equilibrium, which is Markovian and characterized by a functional differential equation. As in the Brownian case, the equilibrium actions are determined by the posterior on the behavioral type and the equilibrium value function of the normal type. Thus, in order to state our characterization, we first need to examine how a candidate value function for the normal type affects the incentives of the players. In effect, Proposition 9 below—which is the Poisson counterpart of Proposition 4—establishes the existence, uniqueness and Lipschitz continuity of action profiles satisfying the incentive constraint (34), where $r\zeta_t$ is the jump in the continuation value W_t when the pair (ϕ_t, W_t) is constrained to lie in the graph of a candidate value function $V : (0, 1) \to \mathbb{R}$. The proof of Proposition 9 is presented in Appendix F.

We write $\mathcal{C}^{\mathsf{inc}}([0, 1])$ to denote the complete metric space of real-valued continuous increasing functions over the interval $[0, 1]$, equipped with the supremum distance.

Proposition 9. *In the good news model, for each $(\phi, V) \in [0, 1] \times \mathcal{C}^{\mathsf{inc}}([0, 1])$ there is a unique action profile $(\mathsf{a}(\phi, V), \mathsf{b}(\phi, V)) \in A \times B$ satisfying the incentive constraint*

$$(\mathsf{a}(\phi, V), \mathsf{b}(\phi, V)) \in \mathcal{M}\Big(\phi, \big(V(\phi + \Delta\phi(\mathsf{a}(\phi, V))) - V(\phi)\big)/r\Big). \tag{35}$$

Moreover, $(\mathsf{a}, \mathsf{b}) : [0, 1] \times \mathcal{C}^{\mathsf{inc}}([0, 1]) \to A \times B$ is a continuous function, and for each $\phi \in [0, 1]$, $V \mapsto (\mathsf{a}(\phi, V), \mathsf{b}(\phi, V))$ is a Lipschitz continuous function on the metric space $\mathcal{C}^{\mathsf{inc}}([0, 1])$, with a Lipschitz constant that is uniform in ϕ.

The characterization of sequential equilibrium, presented in Theorem 11 below, uses the following differential equation, called the *optimality equation:*

$$U'(\phi) = \frac{rg(\mathsf{a}(\phi,U),\mathsf{b}(\phi,U)) + \lambda(\mathsf{a}(\phi,U))\Delta U(\phi) - rU(\phi)}{\lambda^{\phi}(\mathsf{a}(\phi,U))\Delta\phi(\mathsf{a}(\phi,U))}, \qquad \phi \in (0,1), \tag{36}$$

where

$$\Delta U(\phi) \overset{\text{def}}{=} U(\phi + \Delta\phi(\mathsf{a}(\phi,U))) - U(\phi)$$

and the function $(\mathsf{a}(\cdot),\mathsf{b}(\cdot))$ is defined implicitly by (35). This is a *functional retarded differential equation*, because of the delayed term $U(\phi + \Delta\phi)$ on the right-hand side and the fact that the size of the lag, $\Delta\phi(\mathsf{a}(\phi,U))$, is endogenously determined by the global behavior of the solution U over $[\phi,1)$, rather than by the value of U at a single point.

The main result of this section is:

Theorem 11. *In the good news model, the correspondence of sequential equilibrium payoffs of the normal type, $\mathcal{E} : [0,1] \to \mathbb{R}$, is single-valued and coincides, on the interval $(0,1)$, with the unique bounded increasing solution $U : (0,1) \to \mathbb{R}$ of the optimality equation (36). Moreover, at $p \in \{0,1\}$, $\mathcal{E}(p)$ satisfies the boundary conditions*

$$\lim_{\phi\to p} U(\phi) = \mathcal{E}(p) = g(\mathcal{M}(p,0)) \quad \text{and} \quad \lim_{\phi\to p} \phi(1-\phi)U'(\phi) = 0.$$

Finally, for each prior $p \in [0,1]$ there is a unique public sequential equilibrium, which is Markovian in the small players' posterior belief: at each time t and after each public history, the equilibrium action profile is $(\mathsf{a}(\phi_{t-},U),\mathsf{b}(\phi_{t-},U))$, where $(\mathsf{a}(\cdot),\mathsf{b}(\cdot))$ is the continuous function defined implicitly by condition (35); the small players' posterior $(\phi_t)_{t\geq 1}$ follows equation (32) with initial condition $\phi_0 = p$; and the continuation value of the normal type is $W_t = U(\phi_t)$.

However, equilibrium uniqueness generally breaks down when signals are bad news. Under assumptions (a), (b), (c′) and (d), multiple non-Markovian sequential equilibria may exist, despite the fact that $\mathcal{M}(\phi,\zeta)$ remains a singleton for each $(\phi,\zeta) \in [0,1] \times (-\infty,0]$. Indeed, this multiplicity already arises in the underlying complete information game, as in the repeated prisoner's dilemma of Abreu, Milgrom, and Pearce (1991). Intuitively, the large player can have incentives to play an action different from the static best-reply by a threat that if a bad signal arrives, he will be punished by perpetual reversion to the static Nash equilibrium.[35] In the reputation game, multiple non-Markovian equilibria may exist for a similar reason. Recall that in the Markov perfect equilibrium of a game in which Poisson signals are good news, the reaction of the large player's continuation payoff to the arrival of a signal is completely determined by the updated beliefs and the equilibrium value function. Payoffs above those in the Markov perfect equilibrium cannot be sustained by a threat of reversion to the Markov equilibrium, because those punishments would have to be applied after good news, and therefore they cannot create incentives to play actions closer to a^*. By contrast, when signals are bad news, such punishments can create incentives to play actions closer to a^* effectively.

While we do not characterize sequential equilibria for the case in which the signals are bad news, we conjecture that the upper and lower boundaries of the graph of the correspondence of sequential equilibrium payoffs of the large player solve a pair of (coupled) functional differential equations. Incentive

[35] Naturally, for such profile to be an equilibrium the signals must be sufficiently informative.

provision must satisfy the feasibility condition that after bad news, the belief-continuation value pair transitions to a point between the upper and lower boundaries. Subject to this feasibility condition, payoff maximization implies that when the initial belief-continuation pair is on the upper boundary of the graph of the correspondence of sequential equilibrium payoffs, conditional on the absence of bad news it must stay on the upper boundary. A similar statement must hold for the lower boundary. This observation gives rise to a coupled pair of differential equations that characterize the upper and lower boundaries. We leave the formalization of this conjecture for future research.

10 Conclusion

Our result that many continuous-time reputation games have a unique public sequential equilibrium, which is Markovian in the population's belief, does not have an analogue in discrete time. One may wonder what happens to the equilibria of discrete-time games in the limit as the time between actions Δ shrinks to 0, as in Abreu, Milgrom, and Pearce (1991), Faingold (2008), Fudenberg and Levine (2007), Fudenberg and Levine (2009) , Sannikov and Skrzypacz (2007) and Sannikov and Skrzypacz (2010). While it is beyond the scope of this paper to answer this question rigorously, we will make some guided conjectures and leave the formal analysis to future research.

To be specific, consider what happens as the length of the period of fixed actions, Δ, shrinks to zero in a game with Brownian signals satisfying Conditions 1 and 2. For a fixed value of $\Delta > 0$, the upper and lower boundaries of the set of equilibrium payoffs will generally be different. For a given action profile, it is uniquely determined how the small players' belief responds to the public signal, but there is some room to choose continuation values within the bounds of the equilibrium payoff set. Consider the task of maximizing the expected payoff of the large player by the choice of a current equilibrium action and feasible continuation values. In many instances the solution to this optimization problem would take the form of a tail test: the large player's continuation values are chosen on the upper boundary of the equilibrium payoff set, unless the public signal falls below a cut-off, in which case continuation values are taken from the lower boundary. This way of providing incentives is reminiscent of the use of the cutoff tests to trigger punishments in Sannikov and Skrzypacz (2007). As in the complete-information game of Section 5, it becomes less and less efficient to use such tests to provide incentives as $\Delta \to 0$. Therefore, it is natural to conjecture that as $\Delta \to 0$, the distance between the upper and lower boundaries of the equilibrium value set converges to 0.

Appendix

A Proof of Lemma 1

Fix an arbitrary constant $M > 0$. Consider the set Φ_0 of all tuples $(a, \bar{b}, \beta) \in A \times \Delta(B) \times \mathbb{R}^d$ satisfying

$$a \in \arg\max_{a' \in A} \; g(a', \bar{b}) + \beta \cdot \mu(a', \bar{b}), \quad b \in \arg\max_{b' \in B} h(a, b', \bar{b}), \; \forall b \in \text{supp } \bar{b}, \quad g(a, \bar{b}) \geq \bar{v} + \varepsilon, \tag{37}$$

and $|\beta| \leq M$. Note that Φ_0 is a compact set, as it is a closed subset of the compact space $A \times \Delta(B) \times \{\beta \in \mathbb{R}^d : |\beta| \leq M\}$, where $\Delta(B)$ is equipped with the weak* topology. Therefore, the continuous function $(a, \bar{b}, \beta) \mapsto |\beta|$ achieves its minimum, η , on Φ_0, and we must have $\eta > 0$ because of the condition $g(a, \bar{b}) \geq \bar{v} + \varepsilon$. It follows that $|\beta| \geq \delta \stackrel{\text{def}}{=} \min\{M, \eta\}$ for any (a, b, β) satisfying conditions (37). ■

B Bounds on coefficient $\gamma(a, \bar{b}, \phi)$

This technical appendix proves a bound on $\gamma(a, \bar{b}, \phi)$ which is used in the subsequent analysis.

Lemma B.1. *Assume Condition 1. There exists a constant $C > 0$ such that for all $(a, \bar{b}, \phi, z) \in A \times \Delta(B) \times (0, 1) \times \mathbb{R}$,*

$$(a, \bar{b}) \in \mathcal{N}(\phi, z) \quad \Longrightarrow \quad (1 + |z|) \frac{|\gamma(a, \bar{b}, \phi)|}{\phi(1 - \phi)} \geq C.$$

Proof. If the thesis of the lemma were false, there would be a sequence $(a_n, \bar{b}_n, \phi_n, z_n)_{n \in \mathbb{N}}$ with $\phi_n \in (0, 1)$ and $(a_n, \bar{b}_n) \in \mathcal{N}(\phi_n, z_n)$ for all $n \in \mathbb{N}$, such that both $z_n |\gamma(a_n, \bar{b}_n, \phi_n)|/(\phi_n(1 - \phi_n))$ and $|\gamma(a_n, \bar{b}_n, \phi_n)|/(\phi_n(1 - \phi_n))$ converged to 0. Passing to a sub-sequence if necessary, we can assume that $(a_n, \bar{b}_n, \phi_n)$ converges to some $(a, \bar{b}, \phi) \in A \times \Delta(B) \times [0, 1]$. Then, by the definition of $\mathcal{N}$ and the continuity of g, μ, σ and h, the profile $(a, \bar{b})$ must be a Bayesian Nash equilibrium of the static game with prior ϕ, since $z_n(\mu(a^*, \bar{b}_n) - \mu(a_n, \bar{b}_n))^\top (\sigma(\bar{b}_n)\sigma(\bar{b}_n)^\top)^{-1} = z_n \gamma(a_n, \bar{b}_n, \phi_n)^\top \sigma(\bar{b}_n)^{-1}/(\phi_n(1 - \phi_n)) \to 0$. Hence, $\mu(a, \bar{b}) \neq \mu(a^*, \bar{b})$ by Condition 1, and therefore $\liminf_n |\gamma(a_n, \bar{b}_n, \phi_n)|/(\phi_n(1 - \phi_n)) \geq |\sigma(\bar{b})^{-1}(\mu(a^*, \bar{b}) - \mu(a, \bar{b}))| > 0$, which is a contradiction. ■

The following corollary shows that the right-hand side of the optimality equation satisfies a quadratic growth condition whenever the beliefs are bounded away from 0 and 1. This technical result is used in the existence proofs of Appendices C and D.

Corollary B.1 (Quadratic growth). *Assume Condition 1. For all $M > 0$ and $\varepsilon > 0$ there exists $K > 0$ such that for all $\phi \in [\varepsilon, 1 - \varepsilon]$, $(a, \bar{b}) \in A \times B$, $u \in [-M, M]$ and $u' \in \mathbb{R}$,*

$$(a, \bar{b}) \in \mathcal{N}(\phi, \phi(1 - \phi)u'/r) \quad \Longrightarrow \quad \left| \frac{2u'}{1 - \phi} + \frac{2r(u - g(a, \bar{b}))}{|\gamma(a, \bar{b}, \phi)|^2} \right| \leq K(1 + |u'|^2).$$

Proof. Follows directly from Lemma B.1 and the bounds $u \in [-M, M]$ and $\phi \in [\varepsilon, 1 - \varepsilon]$. ■

C Appendix for Section 6

Throughout this section we maintain Conditions 1 and 2.a. For the case when Condition 2.b holds, all the arguments in this section remain valid provided we change the definition of $\mathcal{N}$ and set $\mathcal{N}(\phi, z)$ equal to $\mathcal{N}(\phi, 0)$ for $z < 0$. Proposition C.4 shows that under Conditions 1 and 2.b the resulting solution U must be increasing, so the values of $\mathcal{N}(\phi, z)$ for $z < 0$ are irrelevant.

C.1 Existence of a bounded solution of the optimality equation

In this subsection we prove the following proposition.

Proposition C.1. *The optimality equation (19) has at least one $\mathcal{C}^2$-solution that takes values in the interval $[\underline{g}, \bar{g}]$ of feasible payoffs of the large player.*

The proof relies on standard results from the theory of boundary-value problems for second-order equations. We now review the part of that theory that is relevant for our existence result. Given a continuous function $H : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ and real numbers c and d, consider the following boundary value problem:

$$\begin{aligned} &U''(x) = H(x, U(x), U'(x)),\ x \in [a,b] \\ &U(a) = c, U(b) = d. \end{aligned} \tag{38}$$

Given real numbers α and β, we are interested in sufficient conditions for (38) to admit a $\mathcal{C}^2$-solution $U : [a,b] \to \mathbb{R}$ with $\alpha \le U(x) \le \beta$ for all $x \in [a,b]$. One such sufficient condition is called *Nagumo condition*, which posits the existence of a positive continuous function $\psi : [0,\infty) \to \mathbb{R}$ satisfying

$$\int_0^\infty \frac{v\,dv}{\psi(v)} = \infty$$

and

$$|H(x,u,u')| \le \psi(|u'|) \quad \forall (x,u,u') \in [a,b] \times [\alpha,\beta] \times \mathbb{R}.$$

In the proof of Proposition C.1 below we use the following standard result, which follows from Theorems II.3.1 and I.4.4 in de Coster and Habets (2006):

Lemma C.1. *Suppose that $\alpha \le c \le \beta$, $\alpha \le d \le \beta$ and that $H : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Nagumo condition relative to α and β. Then:*

(a) the boundary value problem (38) admits a solution satisfying $\alpha \le U(x) \le \beta$ for all $x \in [a,b]$;

(b) there is a constant $R > 0$ such that every $\mathcal{C}^2$-function $U : [a,b] \to \mathbb{R}$ that satisfies $\alpha \le U(x) \le \beta$ for all $x \in [a,b]$ and solves

$$U''(x) = H(x, U(x), U'(x)), \quad x \in [a,b],$$

satisfies $|U'(x)| \le R$ for all $x \in [a,b]$.

We are now ready to prove Proposition C.1.

Proof of Proposition C.1. Since the right-hand side of the optimality equation blows up at $\phi = 0$ and $\phi = 1$, our strategy of proof is to construct the solution as the limit of a sequence of solutions on expanding closed subintervals of $(0, 1)$. Indeed, let $H : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ denote the right-hand side of the optimality equation, i.e.

$$H(\phi, u, u') = \frac{2u'}{1-\phi} + \frac{2r(u - g(\mathcal{N}(\phi, \phi(1-\phi)u'/r)))}{|\gamma(\mathcal{N}(\phi, \phi(1-\phi)u'/r), \phi)|^2},$$

and for each $n \in \mathbb{N}$ consider the boundary value problem:

$$\begin{aligned} &U''(\phi) = H(\phi, U(\phi), U'(\phi)), \quad \phi \in [1/n, 1-1/n], \\ &U(1/n) = \underline{g}, U(1-1/n) = \bar{g}. \end{aligned}$$

By Corollary B.1, there exists a constant $K_n > 0$ such that

$$|H(\phi, u, u')| \leq K_n(1 + |u'|^2) \quad \forall (\phi, u, u') \in [1/n, 1-1/n] \times [\underline{g}, \bar{g}] \times \mathbb{R}.$$

Since $\int_0^\infty K_n^{-1}(1 + v^2)^{-1} v \, dv = \infty$, for each $n \in \mathbb{N}$ the boundary value problem above satisfies the hypothesis of Lemma C.1 relative to $\alpha = \underline{g}$ and $\beta = \bar{g}$. Therefore, for each $n \in \mathbb{N}$ there exists a $\mathcal{C}^2$-function $U_n : [1/n, 1-1/n] \to \mathbb{R}$ which solves the optimality equation on $[1/n, 1-1/n]$ and satisfies $\underline{g} \leq U_n \leq \bar{g}$. Since for $m \geq n$ the restriction of U_m to $[1/n, 1-1/n]$ also solves the optimality equation on $[1/n, 1-1/n]$, by Lemma C.1 and the quadratic growth condition above the first and second derivatives of U_m are uniformly bounded for $m \geq n$, and hence the sequence $(U_m, U_m')_{m \geq n}$ is bounded and equicontinuous over the domain $[1/n, 1-1/n]$. By the Arzelà-Ascoli Theorem, for every $n \in \mathbb{N}$ there exists a subsequence of $(U_m, U_m')_{m \geq n}$ which converges uniformly on $[1/n, 1-1/n]$. Then, using a diagonalization argument, we can find a subsequence of $(U_n)_{n \in \mathbb{N}}$, denoted $(U_{n_k})_{k \in \mathbb{N}}$, which converges pointwise to a continuously differentiable function $U : (0,1) \to [\underline{g}, \bar{g}]$ such that on every closed subinterval of $(0, 1)$ the convergence takes place in $\mathcal{C}^1$.

Finally, U must solve the optimality equation on $(0, 1)$, since $U''_{n_k}(\phi) = H(\phi, U_{n_k}(\phi), U'_{n_k}(\phi))$ converges to $H(\phi, U(\phi), U'(\phi))$ uniformly on every closed subinterval of $(0, 1)$, by the continuity of H and the uniform convergence $(U_{n_k}, U'_{n_k}) \to (U, U')$ on closed subintervals of $(0, 1)$. ■

C.2 Boundary conditions

Proposition C.2. *If U is a bounded solution of the optimality equation* (19) *on* $(0,1)$*, then it satisfies the following boundary conditions at* $p = 0$ *and* 1*:*

$$\lim_{\phi \to p} U(\phi) = g(\mathcal{N}(p, 0)), \quad \lim_{\phi \to p} \phi(1-\phi)U'(\phi) = 0, \quad \lim_{\phi \to p} \phi^2(1-\phi)^2 U''(\phi) = 0. \tag{39}$$

Proof. Directly from Lemmas C.4, C.5 and C.6 below. Lemmas C.2 and C.3 are intermediate steps. ■

Lemma C.2. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the optimality equation, then U has bounded variation.*

Proof. Suppose there exists a bounded solution U of the optimality equation with unbounded variation near $p = 0$ (the case $p = 1$ is similar). Then let $(\phi_n)_{n \in \mathbb{N}}$ be a decreasing sequence of consecutive local maxima and minima of U, such that ϕ_n is a local maximum for n odd and a local minimum for n even.

Thus for n odd we have $U'(\phi_n) = 0$ and $U''(\phi_n) \leq 0$. From the optimality equation it follows that $g(\mathcal{N}(\phi_n, 0)) \geq U(\phi_n)$. Likewise, for n even we have $g(\mathcal{N}(\phi_n, 0)) \leq U(\phi_n)$. Thus, the total variation of $g(\mathcal{N}(\phi, 0))$ on $(0, \phi_1]$ is no smaller than the total variation of U and therefore $g(\mathcal{N}(\phi, 0))$ has unbounded variation near zero. However, this is a contradiction, since $g(\mathcal{N}(\phi, 0))$ is Lipschitz continuous under Condition 2. ■

Lemma C.3. *Let $U : (0,1) \to \mathbb{R}$ be any bounded continuously differentiable function. Then*

$$\liminf_{\phi \to 0} \phi U'(\phi) \ \leq\ 0\ \leq\ \limsup_{\phi \to 0} \phi U'(\phi)\,, \quad \textit{and}$$

$$\liminf_{\phi \to 1} (1-\phi) U'(\phi) \ \leq\ 0\ \leq\ \limsup_{\phi \to 1} (1-\phi) U'(\phi)\,.$$

Proof. Suppose, towards a contradiction, that $\liminf_{\phi \to 0} \phi U'(\phi) > 0$ (the case $\limsup_{\phi \to 0} \phi U'(\phi) < 0$ is analogous). Then, for some $c > 0$ and $\bar{\phi} > 0$, we must have $\phi U'(\phi) \geq c$ for all $\phi \in (0, \bar{\phi}]$, which implies $U'(\phi) \geq c/\phi$ for all $\phi \in (0, \bar{\phi}]$. But then U cannot be bounded, since the anti-derivative of $1/\phi$, which is $\log \phi$, tends to ∞ as $\phi \to 0$, a contradiction. The proof for the case $\phi \to 1$ is analogous. ■

Lemma C.4. *If U is a bounded solution of the optimality equation, then $\lim_{\phi \to p} \phi(1-\phi) U'(\phi) = 0$ for $p \in \{0, 1\}$.*

Proof. Suppose, towards a contradiction, that $\phi U'(\phi) \nrightarrow 0$ as $\phi \to 0$. Then, by Lemma C.3,

$$\liminf_{\phi \to 0} \phi U'(\phi) \ \leq\ 0\ \leq\ \limsup_{\phi \to 0} \phi U'(\phi)\,,$$

with at least one strict inequality. Without loss of generality, assume $\limsup_{\phi \to 0} \phi U'(\phi) > 0$. Hence there exist constants $0 < k < K$, such that $\phi U'(\phi)$ crosses levels k and K infinitely many times in a neighborhood of 0. Thus, by Lemma B.1, there exists $C > 0$ such that $|\gamma(a, \bar{b}, \phi)| \geq C\,\phi$ whenever $\phi U'(\phi) \in (k, K)$ and $\phi \in (0, \frac{1}{2})$. On the other hand, by the optimality equation, for some constant $L > 0$ we have $|U''(\phi)| \leq \frac{L}{\phi^2}$. This bound implies that for all $\phi \in (0, \frac{1}{2})$ such that $\phi U'(\phi) \in (k, K)$,

$$|(\phi U'(\phi))'| \ \leq\ |\phi U''(\phi)| + |U'(\phi)| \ =\ (1 + \frac{|\phi U''(\phi)|}{|U'(\phi)|})\,|U'(\phi)| \ \leq\ (1 + \frac{L}{k})\,|U'(\phi)|\,,$$

which implies

$$|U'(\phi)| \geq \frac{|(\phi U'(\phi))'|}{1 + L/k}\,.$$

It follows that on every interval where $\phi U'(\phi)$ crosses k and stays in (k, K) until crossing K, the total variation of U is at least $(K-k)/(1+L/k)$. Since this happens infinitely many times in a neighborhood of $\phi = 0$, function U must have unbounded variation in that neighborhood, and this is a contradiction (by virtue of Lemma C.2.) The proof that $\lim_{\phi \to 1} (1-\phi) U'(\phi) = 0$ is analogous. ■

Lemma C.5. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the optimality equation, then for $p \in \{0, 1\}$,*

$$\lim_{\phi \to p} U(\phi) = g(\mathcal{N}(p, 0))\,.$$

Proof. First, by Lemma C.2, U must have bounded variation and so the $\lim_{\phi\to p} U(\phi)$ exists. Consider $p = 0$ and assume, towards a contradiction, that $\lim_{\phi\to 0} U(\phi) = U_0 < g(a^N, b^N)$, where $(a^N, b^N) = \mathcal{N}(0,0)$ is the Nash equilibrium of the stage game (the proof for the reciprocal case is similar). By Lemma C.4, $\lim_{\phi\to 0} \phi U'(\phi) = 0$, which implies that $\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r)$ converges to (a^N, b^N) as $\phi \to 0$. Recall the optimality equation:

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r))}{|\gamma(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r), \phi)|^2} = \frac{2U'(\phi)}{1-\phi} + \frac{h(\phi)}{\phi^2},$$

where $h(\phi)$ is a continuous function that converges to

$$\frac{2r(U_0 - g(a^N, b^N))}{|\sigma(b^N)^{-1}(\mu(a^*, b^N) - \mu(a^N, b^N))|^2} < 0.$$

as $\phi \to 0$. Since $U'(\phi) = o(1/\phi)$ by Lemma C.3, it follows that for some $\bar{\phi} > 0$ there exists a constant $K > 0$ such that $U''(\phi) < -K/\phi^2$ for all $\phi \in (0, \bar{\phi})$. But then U cannot be bounded, since the second-order anti-derivative of $-1/\phi^2$, which is $\log \phi$, tends to $-\infty$ as $\phi \to 0$. The proof for the case $p = 1$ is similar. ■

Lemma C.6. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the optimality equation, then*

$$\lim_{\phi\to p} \phi^2(1-\phi)^2 U''(\phi) = 0\,, \quad \text{for } p \in \{0,1\}\,.$$

Proof. Consider $p = 1$. Fix an arbitrary $M > 0$ and choose $\underline{\phi} \in (0,1)$ so that $(1-\phi)|U'(\phi)| < M$ for all $\phi \in (\underline{\phi}, 1)$. By Lemma B.1 there exists $C > 0$ such that $|\gamma(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r), \phi)| \geq C(1-\phi)$ for all $\phi \in (\underline{\phi}, 1)$. Hence, by the optimality equation, for all $\phi \in (\underline{\phi}, 1)$ we have

$$\begin{aligned}(1-\phi)^2|U''(\phi)| \;&\leq\; 2(1-\phi)|U'(\phi)| + (1-\phi)^2 \frac{2r|U(\phi) - g(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r))|}{|\gamma(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r), \phi)|^2} \\ &\leq\; 2(1-\phi)|U'(\phi)| + 2rC^{-2}|U(\phi) - g(\mathcal{N}(\phi, \phi(1-\phi)U'(\phi)/r))| \;\longrightarrow\; 0 \quad \text{as} \quad \phi \to 1,\end{aligned}$$

by Lemmas C.4 and C.5. The case $p = 0$ is analogous. ■

C.3 Uniqueness

Lemma C.7. *If two bounded solutions of the optimality equation, U and V, satisfy $U(\phi_0) \leq V(\phi_0)$ and $U'(\phi_0) \leq V'(\phi_0)$ with at least one strict inequality, then $U(\phi) < V(\phi)$ and $U'(\phi) < V'(\phi)$ for all $\phi > \phi_0$. Similarly, if $U(\phi_0) \leq V(\phi_0)$ and $U'(\phi_0) \geq V'(\phi_0)$ with at least one strict inequality, then $U(\phi) < V(\phi)$ and $U'(\phi) > V'(\phi)$ for all $\phi < \phi_0$.*

Proof. Suppose that $U(\phi_0) \leq V(\phi_0)$ and $U'(\phi_0) < V'(\phi_0)$. If $U'(\phi) < V'(\phi)$ for all $\phi > \phi_0$ then we must also have $U(\phi) < V(\phi)$ on that range. Otherwise, let

$$\phi_1 \stackrel{\text{def}}{=} \inf\{\phi \in [\phi_0, 1) : U'(\phi) \geq V'(\phi)\}.$$

Then, $U'(\phi_1) = V'(\phi_1)$ by continuity, and $U(\phi_1) < V(\phi_1)$ since $U(\phi_0) \leq V(\phi_0)$ and $U'(\phi) < V'(\phi)$ on $[\phi_0, \phi_1)$. By the optimality equation, it follows that $U''(\phi_1) < V''(\phi_1)$, and hence $U'(\phi_1 - \varepsilon) > V'(\phi_1 - \varepsilon)$ for sufficiently small $\varepsilon > 0$, and this contradicts the definition of ϕ_1.

For the case when $U(\phi_0) < V(\phi_0)$ and $U'(\phi_0) = V'(\phi_0)$ the optimality equation implies that $U''(\phi_0) < V''(\phi_0)$. Therefore, $U'(\phi) < V'(\phi)$ on $(\phi_0, \phi_0 + \varepsilon)$, and the argument proceeds as above.

Finally, the argument for $\phi < \phi_0$ when $U(\phi_0) \leq V(\phi_0)$ and $U'(\phi_0) \geq V'(\phi_0)$ with at least one strict inequality is similar. ■

Proposition C.3. *The optimality equation has a unique bounded solution.*

Proof. By Proposition C.1 a bounded solution of the optimality equation exists. Suppose U and V are two such bounded solutions. Assuming that $V(\phi) > U(\phi)$ for some $\phi \in (0, 1)$, let $\phi_0 \in (0, 1)$ be the point where the difference $V - U$ is maximized, which is well-defined because $\lim_{\phi \to p} U(\phi) = \lim_{\phi \to p} V(\phi)$ for $p \in \{0, 1\}$ by Proposition C.2. Thus we have $V(\phi_0) - U(\phi_0) > 0$ and $V'(\phi_0) - U'(\phi_0) = 0$. But then, by Lemma C.7, the difference $V(\phi) - U(\phi)$ must be strictly increasing for $\phi > \phi_0$, a contradiction. ■

Finally, the following proposition shows that under Conditions 1 and 2.b, the unique solution $U : (0, 1) \to \mathbb{R}$ of the modified optimality equation in which $\mathcal{N}(\phi, z)$ is set equal to $\mathcal{N}(\phi, 0)$ when $z < 0$ must be an increasing function. In particular, U must be the unique bounded increasing solution of the optimality equation.

Proposition C.4. *Under Conditions 1 and 2.b, $U : (0, 1) \to \mathbb{R}$ is an increasing function, and is therefore the unique bounded increasing solution of the optimality equation.*

Proof. By Proposition C.3, the modified optimality equation—with $\mathcal{N}(\phi, z)$ set equal to $\mathcal{N}(\phi, 0)$ when $z < 0$—has a unique bounded solution $U : (0, 1) \to \mathbb{R}$, which satisfies the boundary conditions $\lim_{\phi \to 0} U(\phi) = g(\mathcal{N}(0, 0))$ and $\lim_{\phi \to 1} U(\phi) = g(\mathcal{N}(1, 0))$ by Lemma C.5.

Towards a contradiction, suppose that U is not increasing, so that $U'(\phi) < 0$ for some $\phi \in (0, 1)$. Take a maximal subinterval $(\phi_0, \phi_1) \subseteq (0, 1)$ on which U is strictly decreasing. Since $g(\mathcal{N}(\phi, 0))$ is increasing in ϕ, we have $\lim_{\phi \to 0} U(\phi) = g(\mathcal{N}(0, 0)) \leq g(\mathcal{N}(1, 0)) = \lim_{\phi \to 1} U(\phi)$, hence $(\phi_0, \phi_1) \neq (0, 1)$. Without loss of generality, assume $\phi_1 < 1$. Then, ϕ_1 must be an interior local minimum, so $U'(\phi_1) = 0$ and $U''(\phi_1) \geq 0$. Also, we must have $U(\phi_1) \geq g(\mathcal{N}(\phi_1, 0))$, for otherwise

$$U''(\phi_1) = \frac{2r(U(\phi_1) - g(\mathcal{N}(\phi_1, 0)))}{|\gamma(\mathcal{N}(\phi_1, 0), \phi_1)|^2} < 0.$$

But then, since

$$\lim_{\phi \to \phi_0} U(\phi) > U(\phi_1) \geq g(\mathcal{N}(\phi_1, 0)) \geq g(\mathcal{N}(0, 0)) = \lim_{\phi \to 0} U(\phi),$$

it follows that $\phi_0 > 0$. Therefore, $U'(\phi_0) = 0$ and

$$U''(\phi_0) = \frac{2r(U(\phi_0) - g(\mathcal{N}(\phi_0, 0)))}{|\gamma(\mathcal{N}(\phi_0, 0), \phi_0)|^2} \geq \frac{2r(U(\phi_0) - g(\mathcal{N}(\phi_1, 0)))}{|\gamma(\mathcal{N}(\phi_0, 0), \phi_0)|^2} > 0,$$

so ϕ_0 is a strict local minimum, a contradiction. ■

C.4 The continuity lemma used in the proof of Theorem 4

Lemma C.8. *Let* $U : (0,1) \to \mathbb{R}$ *be the unique bounded solution of the optimality equation and let* $d : A \times \Delta(B) \times [0,1] \to \mathbb{R}$ *and* $f : A \times \Delta(B) \times [0,1] \times \mathbb{R} \to \mathbb{R}$ *be the continuous functions defined by*

$$d(a,\bar{b},\phi) \stackrel{\text{def}}{=} \begin{cases} r(U(\phi) - g(a,\bar{b})) - \frac{|\gamma(a,b,\phi)|^2}{1-\phi} U'(\phi) - \frac{1}{2}|\gamma(a,b,\phi)|^2 U''(\phi) & : \quad \phi \in (0,1) \\ r(g(\mathcal{N}(\phi,0)) - g(a,\bar{b})) & : \quad \phi = 0 \text{ or } 1 \end{cases} \tag{40}$$

and

$$f(a,\bar{b},\phi,\beta) \stackrel{\text{def}}{=} r\beta^\top \sigma(\bar{b}) - \underbrace{\phi(1-\phi)(\mu(a^*,\bar{b}) - \mu(a,\bar{b}))^\top (\sigma(\bar{b})^\top)^{-1}}_{\gamma(a,b,\phi)^\top} U'(\phi). \tag{41}$$

For every $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $(a,\bar{b},\phi,\beta)$ *satisfying*

$$\begin{aligned} a &\in \arg\max_{a' \in A} \; g(a',\bar{b}) + \beta \cdot \mu(a',\bar{b}) \\ \bar{b} &\in \arg\max_{b' \in B} \; \phi h(a^*,b',\bar{b}) + (1-\phi)h(a,b',\bar{b}) \quad \forall b \in \operatorname{supp} \bar{b}, \end{aligned} \tag{42}$$

either $d(a,\bar{b},\phi) > -\varepsilon$ *or* $f(a,\bar{b},\phi,\beta) \geq \delta$.

Proof. Since $\phi(1-\phi)U'(\phi)$ is bounded (by Proposition C.2) and there exists $c > 0$ such that $|\sigma(\bar{b}) \cdot y| \geq c|y|$ for all $y \in \mathbb{R}^d$ and $\bar{b} \in \Delta(B)$, there exist constants $M > 0$ and $m > 0$ such that $|f(a,\bar{b},\phi,\beta)| > m$ for all $\beta \in \mathbb{R}^d$ with $|\beta| > M$.

Consider the set Φ of all tuples $(a,b,\phi,\beta) \in A \times \Delta(B) \times [0,1] \times \mathbb{R}^d$ with $|\beta| \leq M$ that satisfy (42) and $d(a,\bar{b},\phi) \leq -\varepsilon$. Since U satisfies the boundary conditions (39) by Proposition C.2, d is a continuous function and hence Φ is a closed subset of the compact set $\{(a,b,\phi,\beta) \in A \times \Delta(B) \times [0,1] \times \mathbb{R}^d : |\beta| \leq M\}$, and therefore Φ is a compact set.[36] The boundary conditions (39) also imply that the function $|f|$ is also continuous, so it achieves its minimum, denoted η, on Φ. We must have $\eta > 0$, since for all $(a,\bar{b},\phi,\beta) \in \Phi$ we have $d(a,\bar{b},\phi) = 0$ whenever $f(a,\bar{b},\phi,\beta) = 0$ by the optimality equation, as we argued in the proof of Theorem 4. It follows that for all $(a,\bar{b},\phi,\beta)$ satisfying (42), either $d(a,\bar{b},\phi) > -\varepsilon$ or $|f(a,\bar{b},\phi,\beta)| \geq \delta \stackrel{\text{def}}{=} \min\{m,\eta\}$, as required. ∎

C.5 Proof of Theorem 5

First, we need the following lemma:

Lemma C.9. *Under Conditions 1, 2 and 3, for any* $\phi_0 \in (0,1)$ *and* $k > 0$ *the initial value problem*

$$v'(\phi) = \frac{2v(\phi)}{1-\phi} + \frac{2(g(a^*,b^*) - g(\widetilde{\mathcal{N}}(\phi,\phi(1-\phi)v(\phi)))}{|\gamma(\widetilde{\mathcal{N}}(\phi,\phi(1-\phi)v(\phi)),\phi)|^2}, \quad v(\phi_0) = k, \tag{43}$$

where

$$\widetilde{\mathcal{N}}(\phi,z) \stackrel{\text{def}}{=} \begin{cases} \mathcal{N}(\phi,z), & \text{if Condition 2.a holds,} \\ \mathcal{N}(\phi,\max\{z,0\}), & \text{if Condition 2.b holds,} \end{cases}$$

has a unique solution on the interval $(0,1)$. *Moreover, the solution satisfies* $\liminf_{\phi \to 1} v(\phi) < 0$.

[36] Recall that $\Delta(B)$ is compact in the topology of weak convergence of probability measures.

Proof. Assume Conditions 1, 2.a and 3. (The proof for the case when Condition 2.b holds is very similar and thus ommitted.) Fix $\phi_0 \in (0,1)$ and $k > 0$. First, note that on every compact subset of $(0,1)\times\mathbb{R}$, the right-hand side of (43) is Lipschitz continuous in (ϕ, v) by Conditions 1 and 2.a and Lemma B.1. This implies that a unique solution exists on a neighborhood of ϕ_0. Let J_{ϕ_0} denote the maximal interval of existence / uniqueness of the solution of (43), and let us show that $J_{\phi_0} = (0,1)$. Indeed, the Lipschitz continuity of g, Condition 3 and the bound from Lemma B.1 imply that the solution of (43) satisfies:

$$\forall \varepsilon > 0\ \exists K_\varepsilon > 0 \text{ such that } |v'(\phi)| \ \leq\ K_\varepsilon(1 + |v(\phi)|),\ \forall \phi \in J_{\phi_0} \cap [\varepsilon, 1-\varepsilon].$$

Given this linear growth condition, a standard argument shows that $|v|$ and $|v'|$ cannot blow-up in any closed sub-interval of $(0,1)$. Thus, we must have $J_{\phi_0} = (0,1)$, i.e. the solution v is well-defined on the whole interval $(0,1)$.

It remains to show that $\liminf_{\phi\to 1} v(\phi) < 0$, but first we will prove the intermediate result that $\limsup_{\phi\to 1}(1-\phi)v(\phi) \leq 0$. By the large player's incentive constraint in the definition of $\mathcal{N}$, for each $(a,b,\phi,z) \in A \times B \times (0,1) \times \mathbb{R}$ with $(a,b) \in \mathcal{N}(\phi, z)$,

$$\begin{aligned} g(a,b) - g(a^*,b) \ &\geq\ z(\mu(a^*,b) - \mu(a,b))^\top(\sigma(b)\sigma(b)^\top)^{-1}(\mu(a^*,b) - \mu(a,b)) \ = \\ &=\ z\,|\sigma(b)^{-1}(\mu(a^*,b) - \mu(a,b))|^2 \ =\ z|\gamma(a,b,\phi)|^2/(\phi(1-\phi))^2. \end{aligned} \tag{44}$$

This implies that, for each $\phi \in (0,1)$ and $(a,b) \in \mathcal{N}(\phi, \phi(1-\phi)v(\phi))$,

$$\begin{aligned} \frac{2(g(a^*,b^*) - g(a,b))}{|\gamma(a,b,\phi)|^2} &= \frac{2(g(a^*,b^*) - g(a^*,b))}{|\gamma(a,b,\phi)|^2} + \frac{2(g(a^*,b) - g(a,b))}{|\gamma(a,b,\phi)|^2} \\ &\leq \frac{2|g(a^*,b^*) - g(a^*,b)|}{|\gamma(a,b,\phi)|^2} - \frac{2v(\phi)}{\phi(1-\phi)}. \end{aligned} \tag{45}$$

Moreover, by the Lipschitz continuity of g and Condition 3, there is a constant $K_1 > 0$ such that for each $\phi \in (0,1)$ and $(a,b) \in \mathcal{N}(\phi, \phi(1-\phi)v(\phi))$,

$$|g(a^*,b^*) - g(a^*,b)| \leq K_1\phi^{-1}|\gamma(a,b,\phi)|.$$

Plugging this inequality in (45) yields, for each $\phi \in (0,1)$ and $(a,b) \in \mathcal{N}(\phi, \phi(1-\phi)v(\phi))$,

$$\frac{2(g(a^*,b^*) - g(a,b))}{|\gamma(a,b,\phi)|^2} \leq -\frac{2v(\phi)}{\phi(1-\phi)} + \frac{2K_1}{\phi|\gamma(a,b,\phi)|}.$$

But since by Lemma B.1 there is a constant $C > 0$ such that for each $\phi \in (0,1)$ and $(a,b) \in \mathcal{N}(\phi, \phi(1-\phi)v(\phi))$,

$$|\gamma(a,b,\phi)| \geq \frac{C\phi(1-\phi)}{1 + \phi(1-\phi)|v(\phi)|},$$

it follows that for each $\phi \in (0,1)$ and $(a,b) \in \mathcal{N}(\phi, \phi(1-\phi)v(\phi))$,

$$\frac{2(g(a^*,b^*) - g(a,b))}{|\gamma(a,b,\phi)|^2} \leq -\frac{2v(\phi)}{\phi(1-\phi)} + \frac{2K_1(1 + \phi(1-\phi)|v(\phi)|)}{C\phi^2(1-\phi)}.$$

Plugging this inequality in (43) and simplifying yields

$$v'(\phi) \leq C_1|v(\phi)| + \frac{C_2}{1-\phi}, \quad \forall \phi \in [\phi_0, 1),$$

where $C_1 = 2(K_1/C - 1)/\phi_0$ and $C_2 = 2K_1/(C\phi_0^2)$. This differential inequality implies

$$v(\phi) \leq ke^{C_1(\phi-\phi_0)} + C_2 e^{C_1(\phi-\phi_0)} \int_{\phi_0}^{\phi} \frac{e^{-C_1(x-\phi_0)}}{1-x}\, dx, \quad \forall \phi \in [\phi_0, 1),$$

and hence,

$$v(\phi) \leq C_3 - C_4 \log(1-\phi), \quad \forall \phi \in [\phi_0, 1),$$

where C_3 and C_4 are positive constants. But since $\lim_{x\to 0} x \log x = 0$, it follows that $\limsup_{\phi\to 1} (1-\phi)v(\phi) \leq 0$, as was to be proved.

Finally, let us show that $\liminf_{\phi\to 1} v(\phi) < 0$. Suppose not, i.e. suppose $\liminf_{\phi\to 1} v(\phi) \geq 0$. Then we must have $\lim_{\phi\to 1}(1-\phi)v(\phi) = 0$, since we have already shown that $\limsup_{\phi\to 1}(1-\phi)v(\phi) \leq 0$ above. Hence, $\lim_{\phi\to 1} \mathcal{N}(\phi, \phi(1-\phi)v(\phi)) = \mathcal{N}(1,0) = (\tilde{a}, b^*)$, where $\tilde{a} \in \arg\max_{a\in A} g(a, b^*)$. But since $g(a^*, b^*) - g(\tilde{a}, b^*) < 0$ by Condition 1, and $\mu(a^*, b^*) \neq \mu(\tilde{a}, b^*)$ by Condition 3.a, there is a constant $K_2 > 0$ such that

$$\frac{2(g(a^*, b^*) - g(\mathcal{N}(\phi, \phi(1-\phi)v(\phi)))}{|\gamma(\mathcal{N}(\phi, \phi(1-\phi)v(\phi)), \phi)|^2} \leq -\frac{2K_2}{(1-\phi)^2}, \quad \forall \phi \approx 1.$$

Also, since $\lim_{\phi\to 1}(1-\phi)v(\phi) = 0$, we have

$$\frac{2v(\phi)}{1-\phi} \leq \frac{K_2}{(1-\phi)^2}, \quad \forall \phi \approx 1.$$

Plugging these two inequalities in (43) and simplifying yields

$$v'(\phi) \leq -\frac{K_2}{(1-\phi)^2}, \quad \forall \phi \approx 1,$$

which implies $\lim_{\phi\to 1} v(\phi) = -\infty$. But this is a contradiction, since we have assumed $\liminf_{\phi\to 1} v(\phi) \geq 0$. ■

We are now ready to prove Theorem 5.

Proof of Theorem 5. Fix an arbitrary $\phi_0 \in (0,1)$. If we show that $\lim_{r\to 0} U_r'(\phi_0)/r = \infty$ it will follow that $\lim_{r\to 0} \mathsf{a}_r(\phi_0) = a^*$, since there is a constant $K_0 > 0$ such that for each $r > 0$,

$$\bar{g} - \underline{g} \;\geq\; g(\mathsf{a}_r(\phi_0), \mathsf{b}_r(\phi_0)) - g(a^*, \mathsf{b}_r(\phi_0)) \;\geq\; K_0|a^* - \mathsf{a}_r(\phi_0)|^2\, U_r'(\phi_0)/r,$$

by the large player's incentive constraint (44) and Condition 3.a. Since $\lim_{r\to 0} \mathsf{a}_r(\phi_0) = a^*$ implies $\lim_{r\to 0} \mathsf{b}_r(\phi_0) = b^*$, to conclude the proof we need only show that $\lim_{r\to 0} U_r'(\phi_0)/r = \infty$.

En route to a contradiction, suppose there is some $k > 0$ such that $\liminf_{r\to 0} U_r'(\phi_0)/r \leq k$.

Claim. $\forall \varepsilon > 0$, $\forall \phi_1 \in (\phi_0, 1)$ $\exists \bar{r} > 0$ *such that* $\forall r \in (0, \bar{r}]$,

$$U_r'(\phi_0) \leq kr \quad \Rightarrow \quad U_r'(\phi) \leq r(v(\phi) + \varepsilon) \quad \forall \phi \in [\phi_0, \phi_1],$$

where v *is the unique solution of the initial value problem* (43).

To prove this claim, fix $\varepsilon > 0$ and $\phi_1 \in (\phi_0, 1)$ and recall that, by Theorem 1, for each $\delta > 0$ there exists $r_\delta > 0$ such that for each $0 < r < r_\delta$ and $\phi \in [\phi_0, \phi_1]$ we have $U_r(\phi) < g(a^*, b^*) + \delta$, and hence

$$U_r''(\phi) < \frac{2U_r'(\phi)}{1-\phi} + \frac{2r(\delta + g(a^*, b^*) - g(\mathcal{N}(\phi, \phi(1-\phi)U_r'(\phi)/r)))}{|\gamma(\mathcal{N}(\phi, \phi(1-\phi)U_r'(\phi)/r), \phi)|^2}.$$

Thus, for each $\delta > 0$ and $0 < r < r_\delta$,

$$U_r'(\phi_0) \leq kr \quad \Rightarrow \quad U_r'(\phi) \leq V_{r,\delta}(\phi) \quad \forall \phi \in [\phi_0, \phi_1], \tag{46}$$

where $V_{r,\delta}$ solves the initial value problem[37]

$$V_{r,\delta}'(\phi) = \frac{2V_{r,\delta}(\phi)}{1-\phi} + \frac{2r(\delta + g(a^*, b^*) - g(\widetilde{\mathcal{N}}(\phi, \phi(1-\phi)V_{r,\delta}(\phi)/r)))}{|\gamma(\widetilde{\mathcal{N}}(\phi, \phi(1-\phi)V_{r,\delta}(\phi)/r), \phi)|^2}, \quad V_{r,\delta}(\phi_0) = kr.$$

where

$$\widetilde{\mathcal{N}}(\phi, z) = \begin{cases} \mathcal{N}(\phi, z), & \text{if Condition 2.a holds} \\ \mathcal{N}(\phi, \max\{z, 0\}), & \text{if Condition 2.b holds.} \end{cases}$$

Clearly, $V_{r,\delta}$ must be homegeneous of degree one in r, so it must be of the form $V_{r,\delta}(\phi) = rv_\delta(\phi)$, where v_δ is independent of r and solves the initial value problem

$$v_\delta'(\phi) = \frac{2v_\delta(\phi)}{1-\phi} + \frac{2(\delta + g(a^*, b^*) - g(\widetilde{\mathcal{N}}(\phi, \phi(1-\phi)v_\delta(\phi))))}{|\gamma(\widetilde{\mathcal{N}}(\phi, \phi(1-\phi)v_\delta(\phi)), \phi)|^2}, \quad v_\delta(\phi_0) = k, \tag{47}$$

which coincides with (43) when $\delta = 0$. Thus, by (46), it suffices to show that for some $\delta > 0$ we have $v_\delta(\phi) \leq v(\phi) + \varepsilon$ for all $\phi \in [\phi_0, \phi_1]$, where v is the unique solution of (43). In effect, over the domain $[\phi_0, \phi_1]$, the right-hand side of (47) is jointly continuous in (δ, ϕ, v_δ) and Lipschitz continuous in v_δ uniformly in (ϕ, δ). Therefore, by standard results on existence, uniqueness and continuity of solutions to ordinary differential equations, for every $\delta > 0$ small enough, a unique solution, v_δ, exists on the interval $[\phi_0, \phi_1]$ and the mapping $\delta \mapsto v_\delta$ is continuous in the sup-norm. Hence, for some $\delta_0 > 0$ small enough, $v_{\delta_0}(\phi) \leq v(\phi) + \varepsilon$ for all $\phi \in [\phi_0, \phi_1]$. Letting $\bar{r} = r_{\delta_0}$ thus concludes the proof of the claim.

The claim above implies

$$\liminf_{\phi \to 1} \liminf_{r \to 0} U_r'(\phi)/r < 0, \tag{48}$$

since $\liminf_{\phi \to 1} v(\phi) < 0$ by Lemma C.9, and $\liminf_{r \to 0} U_r'(\phi_0)/r \leq k$ by assumption. Thus, under Condition 2.b we readily get a contradiction, since in this case $U_r(\phi)$ must be increasing in ϕ for each $r > 0$, by Theorem 4.

Now suppose Condition 2.a holds. Since $g(\mathcal{N}(1, 0)) > g(a^*, b^*)$ by Condition 1, there is some $\eta > 0$ such that

$$g(\mathcal{N}(\phi, 0)) > g(a^*, b^*) + \eta, \quad \forall \phi \approx 1.$$

This fact, combined with (48) and the upper bound from Theorem 1, implies that there is some $\phi_1 \in (\phi_0, 1)$ and $r > 0$ such that

$$U_r'(\phi_1) < 0 \quad \text{and} \quad U_r(\phi_1) < g(a^*, b^*) + \eta < g(\mathcal{N}(\phi, 0)), \quad \forall \phi \in [\phi_1, 1). \tag{49}$$

[37] Implication (46) follows from the fact that $U_r''(\phi) < V_{r,\delta}'(\phi)$ whenever $U_r'(\phi) = V_{r,\delta}(\phi)$.

We claim that $U_r(\phi) < g(a^*, b^*) + \eta$ for all $\phi \in [\phi_1, 1)$. Otherwise, U_r must have a local minimum at some point $\phi_2 \in (\phi_1, 1)$ where

$$U_r(\phi_2) < g(a^*, b^*) + \eta, \tag{50}$$

since $U_r(\phi_1) < g(a^*, b^*) + \eta$ and $U_r'(\phi_1) < 0$. Since at the local minimum ϕ_2 we must have $U_r'(\phi_2) = 0$ and $U_r''(\phi_2) \geq 0$, the optimality equation implies

$$0 \leq U''(\phi_2) = \frac{2r(U_r(\phi_2) - g(\mathcal{N}(\phi_2, 0)))}{|\gamma(\mathcal{N}(\phi_2, 0), \phi)|^2},$$

and hence $U_r(\phi_2) - g(\mathcal{N}(\phi_2, 0)) \geq 0$, which is impossible by (49) and (50). We have thus proved that $U_r(\phi) < g(a^*, b^*) + \eta$ for all $\phi \in [\phi_1, 1)$. But this is a contradiction, since U_r must satisfy the boundary condition $\lim_{\phi \to 1} U_r(\phi) = g(\mathcal{N}(1, 0))$ by Theorem 4. ■

C.6 Proof of Proposition 4

The proof relies on two lemmas, presented below. Throughout this section we maintain all the assumptions of Proposition 4.

Lemma C.10. $\mathcal{N}(\phi, z) \neq \emptyset, \quad \forall (\phi, z) \in [0, 1] \times [0, \infty)$.

Proof. Fix $(\phi, z) \in [0, 1] \times [0, \infty)$ and consider the correspondence $\Gamma : A \times B \rightrightarrows A \times B$,

$$\Gamma(a, b) = \left\{ (\hat{a}, \hat{b}) : \begin{array}{l} \hat{a} \in \arg\max_{a'} \ g(a', b) + z\sigma(b)^{-2}(\mu(a^*, b) - \mu(a, b))\mu(a', b) \\ \hat{b} \in \arg\max_{b'} \ \phi h(a^*, b', b) + (1 - \phi) h(a, b', b) \end{array} \right\}.$$

Thus, an action profile $(a, b) \in A \times B$ belongs to $\mathcal{N}(\phi, z)$ if and only if it is a fixed point of Γ. By Brouwer's fixed point theorem it is enough to show that Γ is single-valued and continuous. Indeed, since g, h, μ and σ are continuous, Γ is non-empty-valued and upper hemi-continuous. To see that Γ is actually single-valued (and hence continuous), fix $(a, b) \in A \times B$ and note that the assumptions $g_{11} < 0$, $h_{22} < 0$, $\mu_1 > 0$ and $\mu_{11} \leq 0$ imply that $g(\cdot, b) + z\sigma(b)^{-2}(\mu(a^*, b) - \mu(a, b))\mu(\cdot, b)$ and $\phi h(a^*, \cdot, b) + (1 - \phi) h(a, \cdot, b)$ are strictly concave and hence that $\Gamma(a, b)$ is a singleton. ■

The proof of Proposition 4 below uses a first-order condition to characterize the action profile $(a, b) \in \mathcal{N}(\phi, z)$. To express this condition, define a function $F : A \times B \times [0, 1] \times [0, \infty) \to \mathbb{R}^2$ as follows:

$$F(a, b, \phi, z) \stackrel{\text{def}}{=} \left(g_1(a, b) + z\left(\frac{\mu(a^*, b) - \mu(a, b)}{\sigma(b)^2}\right)\mu_1(a, b), \ \phi\, h_2(a^*, b, b) + (1 - \phi)\, h_2(a, b, b) \right). \tag{51}$$

Thus, the first-order condition for $(a, b) \in \mathcal{N}(\phi, z)$ can be written as:

$$F(a, b, \phi, z) \cdot (\hat{a} - a, \hat{b} - b) \leq 0 \quad \forall (\hat{a}, \hat{b}) \in A \times B, \tag{52}$$

where "·" designates inner product.

Lemma C.11. *The function $F : A \times B \times [0, 1] \times [0, \infty) \to \mathbb{R}^2$, defined by (51), satisfies the following conditions:*

(A) $\exists L > 0$ *such that* $\forall (a,b) \in A \times B$ *and* $\forall (\phi, z)$, $(\hat{\phi}, \hat{z}) \in [0,1] \times [0,\infty)$,

$$|F(a,b,\hat{\phi},\hat{z}) - F(a,b,\phi,z)| \ \leq \ L\,|(\hat{\phi},\hat{z}) - (\phi,z)|;$$

(B) $\forall K > 0\, \exists M > 0$ *such that* $\forall (\phi, z) \in [0,1] \times [0,K]$ *and* $\forall (a,b)$, $(\hat{a},\hat{b}) \in A \times B$,

$$\big(F(\hat{a},\hat{b},\phi,z) - F(a,b,\phi,z)\big) \cdot (\hat{a}-a, \hat{b}-b) \ \leq \ -M\,(|\hat{a}-a|^2 + |\hat{b}-b|^2).$$

Proof. For each $(a,b) \in A \times B$ and (ϕ, z), $(\hat{\phi},\hat{z}) \in [0,1] \times [0,\infty)$,

$$F(a,b,\hat{\phi},\hat{z}) - F(a,b,\phi,z) = \begin{bmatrix} \sigma(b)^{-2}(\mu(a^*,b) - \mu(a,b))\mu_1(a,b)(\hat{z}-z) \\ (h_2(a^*,b,b) - h_2(a,b,b))(\hat{\phi}-\phi) \end{bmatrix},$$

which yields condition *(A)* with $L = 2\max_{(a,b)}\ \sigma(b)^{-2}|\mu(a,b)||\mu_1(a,b)| + |h_2(a,b,b)|$.

Turning to condition *(B)*, fix $K > 0$. By the mean value theorem, for each (a,b), $(\hat{a},\hat{b}) \in A \times B$ and $(\phi,z) \in [0,1] \times [0,K]$ there exists some $(\bar{a},\bar{b})$ such that

$$\big(F(\hat{a},\hat{b},\phi,z) - F(a,b,\phi,z)\big) \cdot (\hat{a}-a,\hat{b}-b) = \begin{bmatrix} \hat{a}-a & \hat{b}-b \end{bmatrix} D_{(a,b)}F(\bar{a},\bar{b},\phi,z) \begin{bmatrix} \hat{a}-a \\ \hat{b}-b \end{bmatrix},$$

where $D_{(a,b)}F(\bar{a},\bar{b},\phi,z)$ is the derivative of $F(\cdot,\cdot,\phi,z)$ at $(\bar{a},\bar{b})$. Thus,

$$(F(\hat{a},\hat{b},\phi) - F(a,b,\phi,z)) \cdot (\hat{a}-a,\hat{b}-b) \leq -M(|\hat{a}-a|^2 + |\hat{b}-b|^2),$$

where

$$M \stackrel{\text{def}}{=} -\sup\Big\{ \mathrm{x}^\top D_{(a,b)}F(a,b,\phi,z)\,\mathrm{x} \,:\, (a,b,\phi,z,\mathrm{x}) \in A \times B \times [0,1] \times [0,K] \times \mathbb{R}^2 \text{ with } |\mathrm{x}| = 1 \Big\}.$$

It remains to show that $M > 0$. Since g, h and μ are twice continuously differentiable and σ is continuous, the supremum above is attained at some point in $A \times B \times [0,1] \times [0,K] \times (\mathbb{R}^2 \setminus \{0\})$. Thus, it suffices to prove that $D_{(a,b)}F(a,b,\phi,z)$ is negative definite for each $(a,b,\phi,z) \in A \times B \times [0,1] \times [0,K]$. Fix an arbitrary $(a,b,\phi,z) \in A \times B \times [0,1] \times [0,K]$. We have

$$D_{(a,b)}F(a,b,\phi,z) = \\ = \begin{bmatrix} g_{11} + z\sigma^{-2}((\mu^*-\mu)\mu_{11} - \mu_1^2) & g_{12} + z\sigma^{-2}((\mu^*-\mu)\mu_{12} + (\mu_2^* - \mu_2)\mu_1) \\ (1-\phi)h_{12} & \phi(h_{22}^* + h_{23}^*) + (1-\phi)(h_{22}+h_{23}) \end{bmatrix},$$

where, to save on notation, we omit the arguments of the functions on the right-hand side and use superscript * to indicate when a function is evaluated at (a^*,b,b) or (a^*,b) rather than at (a,b,b) or (a,b). Thus, for each $\varepsilon \geq 0$,

$$D_{(a,b)}F(a,b,\phi,z) = \Psi(\varepsilon) + \Lambda(\varepsilon),$$

where

$$\Psi(\varepsilon) \stackrel{\text{def}}{=} \begin{bmatrix} g_{11} & g_{12} \\ (1-\phi)h_{12} & \phi(h_{22}^* + h_{23}^*) + (1-\phi)(h_{22}+h_{23}) + \varepsilon \end{bmatrix}$$

and

$$\Lambda(\varepsilon) \stackrel{\text{def}}{=} \begin{bmatrix} z\sigma^{-2}((\mu^* - \mu)\mu_{11} - \mu_1^2) & z\sigma^{-2}((\mu^* - \mu)\cdot\mu_{12} + (\mu_2^* - \mu_2)\mu_1) \\ 0 & -\varepsilon \end{bmatrix}.$$

The matrix $\Psi(0)$ is negative definite by condition (a), hence there is $\varepsilon_0 > 0$ small enough such that $\Psi(\varepsilon_0)$ remains negative definite. Moreover, $\Lambda(\varepsilon_0)$ is negative semi-definite by condition (c), since $\varepsilon_0 > 0$ and $z \geq 0$. It follows that $D_{(a,b)}F(a,b,\phi,z) = \Psi(\varepsilon_0) + \Lambda(\varepsilon_0)$ is negative definite, and hence $M > 0$. ■

We are now ready to prove Proposition 4.

Proof of Proposition 4. *(a), (b) & (c) ⇒ Condition 2.b.* First, $\mathcal{N}(\phi, z)$ is nonempty for each $(\phi, z) \in [0,1]\times[0,\infty)$ by Lemma C.10. Moreover, $(a,b) \in \mathcal{N}(\phi,z)$ if and only if (a,b) satisfies the first-order condition (52), because the functions $g(\cdot,b) + z(\mu(a^*,b) - \mu(a,b))\mu(\cdot,b)$ and $\phi h(a^*,\cdot,b) + (1-\phi)h(a,\cdot,b)$ are differentiable and concave for each fixed (a,b). We will now show that $\mathcal{N}(\phi,z)$ is a singleton for each $(\phi,z)\in[0,1]\times[0,\infty)$. Let $(\phi,z)\in[0,1]\times[0,\infty)$ and pick $(a,b), (\hat{a},\hat{b}) \in \mathcal{N}(\phi,z)$. Thus,

$$F(a,b,\phi,z)\cdot(\hat{a}-a,\ \hat{b}-b) \leq 0 \quad \text{and} \quad F(\hat{a},\hat{b},\phi,z)\cdot(a-\hat{a},\ b-\hat{b}) \leq 0,$$

by the first-order conditions. Subtracting the former inequality from the latter yields

$$(F(\hat{a},\hat{b},\phi,z) - F(a,b,\phi,z))\cdot(\hat{a}-a,\ \hat{b}-b) \geq 0,$$

which is possible only if $(a,b) = (\hat{a},\hat{b})$, by part (B) of Lemma C.11. We have thus shown that $\mathcal{N}(\phi,z)$ contains a unique action profile for each $(\phi,z)\in[0,1]\times[0,\infty)$.

Turning to Lipschitz continuity, fix $K > 0$ and let $L > 0$ and $M > 0$ designate the constants from Lemma C.11. Fix $\phi, \hat{\phi} \in [0,1]$ and $z, \hat{z} \in [0,K]$ and let $(a,b)\in\mathcal{N}(\phi,z)$ and $(\hat{a},\hat{b}) \in \mathcal{N}(\hat{\phi},\hat{z})$. By the first-order conditions,

$$\begin{aligned} 0 \ \leq\ & (F(\hat{a},\hat{b},\hat{\phi},\hat{z}) - F(a,b,\phi,z))\cdot(\hat{a}-a,\hat{b}-b) \\ =\ & (F(\hat{a},\hat{b},\hat{\phi},\hat{z}) - F(a,b,\hat{\phi},\hat{z}))\cdot(\hat{a}-a,\hat{b}-b) \\ & + (F(a,b,\hat{\phi},\hat{z}) - F(a,b,\phi,z))\cdot(\hat{a}-a,\hat{b}-b) \\ \leq\ & -M(|\hat{a}-a|^2 + |\hat{b}-b|^2) + L\sqrt{|\hat{\phi}-\phi|^2 + |\hat{z}-z|^2}\sqrt{|\hat{a}-a|^2+|\hat{b}-b|^2}, \end{aligned}$$

where the last inequality follows from Lemma C.11 and the Cauchy-Schwarz inequality. Therefore,

$$\sqrt{|\hat{a}-a|^2+|\hat{b}-b|^2} \leq \frac{L}{M}\sqrt{|\hat{\phi}-\phi|^2+|\hat{z}-z|^2}$$

and we have thus shown that $\mathcal{N}$ is Lipschitz continuous over $[0,1]\times[0,K]$.

It remains to show that $g(\mathcal{N}(\phi,0))$ is increasing in ϕ. Let $(\mathsf{a}(\phi),\mathsf{b}(\phi))$ designate the unique static Bayesian Nash equilibrium when the prior is ϕ. Since a and b are Lipschitz continuous, and hence absolutely continuous, it suffices to show that $0 \leq \frac{d}{d\phi}g(\mathsf{a}(\phi),\mathsf{b}(\phi)) = g_1\frac{d\mathsf{a}}{d\phi} + g_2\frac{d\mathsf{b}}{d\phi}$ almost everywhere, which is equivalent to showing that $g_2\frac{d\mathsf{b}}{d\phi} \geq 0$ a.e. by the first-order condition of the large player.[38] Then, by condition (b), it is enough to show that $d\mathsf{b}(\phi)/d\phi$ has the same sign as $h_2(a^*,\mathsf{b}(\phi),\mathsf{b}(\phi)) -$

[38]Indeed, if $g_1(\mathsf{a}(\phi),\mathsf{b}(\phi)) \neq 0$ then a must be constant in a neighborhood of ϕ, in which case $(d\mathsf{a}/d\phi)(\phi) = 0$.

$h_2(\mathsf{a}(\phi), \mathsf{b}(\phi), \mathsf{b}(\phi))$ almost everywhere. First, consider the case in which $\mathsf{a}(\phi)$ and $\mathsf{b}(\phi)$ are interior solutions, so the first-order conditions are satisfied with equality. By the implicit function theorem,

$$\begin{bmatrix} d\mathsf{a}/d\phi \\ d\mathsf{b}/d\phi \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ (1-\phi)h_{12} & \phi(h_{22}^* + h_{23}^*) + (1-\phi)(h_{22} + h_{23}) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ -\phi(h_2^* - h_2) \end{bmatrix},$$

which implies

$$d\mathsf{b}/d\phi = \frac{\begin{vmatrix} g_{11} & 0 \\ (1-\phi)h_{12} & -\phi(h_2^* - h_2) \end{vmatrix}}{\begin{vmatrix} g_{11} & g_{12} \\ (1-\phi)h_{12} & \phi(h_{22}^* + h_{23}^*) + (1-\phi)(h_{22} + h_{23}) \end{vmatrix}}.$$

By condition (a) the denominator is positive and the numerator has the same sign as $h_2^* - h_2$. Thus, $\frac{d}{d\phi} g(\mathsf{a}(\phi), \mathsf{b}(\phi)) \geq 0$ for almost every ϕ such that $(\mathsf{a}(\phi), \mathsf{b}(\phi))$ is in the interior of $A \times B$. Also, for almost every ϕ, if either $\mathsf{a}(\phi)$ or $\mathsf{b}(\phi)$ is a corner solution, then either (i) b is constant in a neighborhood of ϕ, and hence $g_2 d\mathsf{b}/d\phi = 0$ trivially, or (ii) the first-order condition for b holds with equality and a is constant in a neighborhood of ϕ, in which case differentiating the first-order condition of the small players (with $d\mathsf{a}/d\phi = 0$) yields the desired result.

(a) & (c) ⇒ Condition 3. Condition 3.a follows directly from $\mu_1 > 0$. Turning to Condition 3.b, for each $(a, \phi) \in A \times [0, 1]$ let $b^{\mathsf{BR}}(a, \phi) \in B$ designate the joint best-reply of the small players to (a, ϕ), i.e.

$$b^{\mathsf{BR}}(a, \phi) \in \operatorname*{arg\,max}_b \; \phi h(a^*, b, b^{\mathsf{BR}}(a, \phi)) + (1-\phi) h(a, b, b^{\mathsf{BR}}(a, \phi)). \tag{53}$$

To see why such $b^{\mathsf{BR}}(a, \phi)$ must be unique, note that a necessary (and sufficient) condition for b to be a best-reply of the small players to (a, ϕ) is the first-order condition

$$G(a, b, \phi) \stackrel{\text{def}}{=} \phi h_2(a^*, b, b) + (1-\phi) h_2(a, b, b) \begin{cases} \leq 0 & : \quad b = \min B \\ = 0 & : \quad b \in (\min B, \max B) \\ \geq 0 & : \quad b = \max B. \end{cases}$$

Since $h_{22} + h_{23} < 0$ along the diagonal of the small players' actions, we have $G_2 < 0$ and hence $G(a, b, \phi)$ is strictly decreasing in b for each (a, ϕ). This implies that $b^{\mathsf{BR}}(a, \phi)$ is unique for each (a, ϕ). Next, note that for each $(a, b, \phi) \in A \times B \times [0, 1]$,

$$(G(a, b^*, \phi) - G(a, b, \phi))(b^* - b) \leq -K_1 |b^* - b|^2, \tag{54}$$

where $K_1 = \min_{(a', b', \phi)} |G_2(a', b', \phi')| > 0$. Moreover, G is $\mathcal{C}^1$, and hence for each $(a, \phi) \in A \times [0, 1]$ there is some $\bar{a} \in A$ such that

$$G(a^*, b, \phi) - G(a, b, \phi) = G_1(\bar{a}, b, \phi)(a^* - a) = (1-\phi) h_{12}(\bar{a}, b, b)(a^* - a),$$

and so there is a constant $K_2 > 0$ such that for each $(a, b, \phi) \in A \times B \times [0, 1]$,

$$|G(a^*, b, \phi) - G(a, b, \phi)| \leq K_2 (1-\phi) |a^* - a|. \tag{55}$$

Noting that $b^* = b^{\mathsf{BR}}(a^*, \phi)$ and letting $b = b^{\mathsf{BR}}(a, \phi)$, the first-order conditions yield

$$G(a^*, b^*, \phi)(b - b^*) \leq 0 \quad \text{and} \quad G(a, b, \phi)(b^* - b) \leq 0,$$

and hence

$$(G(a^*, b^*, \phi) - G(a, b, \phi))(b^* - b) \geq 0.$$

This inequality, together with (54) and (55), implies

$$\begin{aligned}
0 &\leq (G(a^*, b^*, \phi) - G(a, b, \phi))(b^* - b) \\
&= (G(a^*, b^*, \phi) - G(a, b^*, \phi))(b^* - b) + (G(a, b^*, \phi) - G(a, b, \phi))(b^* - b) \\
&\leq K_2(1-\phi)|a^* - a||b^* - b| - K_1|b^* - b|^2,
\end{aligned}$$

and therefore,

$$|b^* - b| \leq (K_2/K_1)\,(1-\phi)|a^* - a|.$$

(a), (c) & (d) ⇒ Condition 1. Under assumptions (a) and (c), there is a unique $b^* \in \mathsf{B}(a^*)$. Then, assumption (d) implies that a^* is not a best-reply to b^*. Moreover, under assumption (c) there is no $a \neq a^*$ with $\mu(a, b^*) = \mu(a^*, b^*)$, hence Condition 1 follows. ■

D Appendix for Section 7

Throughout Appendix D, we maintain Conditions 1 and 4. We write U and L to designate the upper and lower boundaries of the correspondence $\mathcal{E}$, respectively, i.e.

$$U(\phi) \stackrel{\text{def}}{=} \sup \mathcal{E}(\phi)\,, \quad L(\phi) \stackrel{\text{def}}{=} \inf \mathcal{E}(\phi), \quad \text{for all } \phi \in [0,1].$$

Proposition D.1. *The upper boundary U is a viscosity sub-solution of the upper optimality equation on* $(0, 1)$.

Proof. If U is not a sub-solution, there exists $q \in (0,1)$ and a $\mathcal{C}^2$-function $V : (0,1) \to \mathbb{R}$ such that $0 = (V - U^*)(q) < (V - U^*)(\phi)$ for all $\phi \in (0,1) \setminus \{q\}$, and

$$H_*(q, V(q), V'(q)) = H_*(q, U^*(q), V'(q)) > V''(q)\,.$$

Since H_* is lower semi-continuous, U^* is upper semi-continuous and $V > U^*$ on $(0,1) \setminus \{q\}$, there exist ε and $\delta > 0$ small enough such that for all $\phi \in [q - \varepsilon, q + \varepsilon]$,

$$H(\phi, V(\phi) - \delta, V'(\phi)) > V''(\phi), \tag{56}$$

$$V(q-\varepsilon) - \delta > U^*(q-\varepsilon) \geq U(q-\varepsilon) \quad \text{and} \quad V(q+\varepsilon) - \delta > U^*(q+\varepsilon) \geq U(q+\varepsilon). \tag{57}$$

Figure 7 displays the configuration of functions U^* and $V - \delta$. Fix a pair $(\phi_0, W_0) \in \mathsf{Graph}\,\mathcal{E}$ with $\phi_0 \in (q - \varepsilon, q + \varepsilon)$ and $W_0 > V(\phi_0) - \delta$. (Such (ϕ_0, W_0) exists because $V(q) = U^*(q)$ and U^* is u.s.c..) Let $(a_t, \bar{b}_t, \phi_t)$ be a sequential equilibrium that attains the pair (ϕ_0, W_0). Denoting by (W_t) the continuation value of the normal type, we have

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\,dt + r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\,dt)$$

for some $\beta \in \mathcal{L}$. Next, we will show that, with positive probability, eventually W_t becomes greater than $U(\phi_t)$, leading to a contradiction since U is the upper boundary of $\mathcal{E}$.

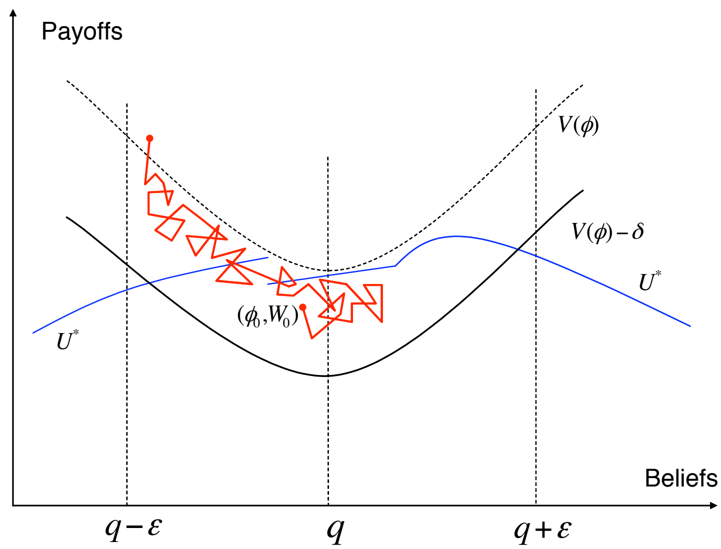

Figure 7: A viscosity sub-solution.

Let $D_t = W_t - (V(\phi_t) - \delta)$. By Itô's formula,

$$dV(\phi_t) = |\gamma_t|^2 \left(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t}\right) dt \;+\; \gamma_t V'(\phi_t)\, dZ_t^{\mathsf{n}},$$

where $\gamma_t = \gamma(a_t, \bar{b}_t, \phi_t)$, and hence,

$$dD_t = \big(rD_t + r(V(\phi_t)-\delta) - rg(a_t,\bar{b}_t) - |\gamma_t|^2 \big(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t}\big)\big)\, dt \;+\; \big(r\beta_t\sigma(\bar{b}_t) - \gamma_t V'(\phi_t)\big)\, dZ_t^{\mathsf{n}}.$$

Therefore, so long as $D_t \geq D_0/2$,

(a) ϕ_t cannot exit the interval $[q-\varepsilon, q+\varepsilon]$ by (57), and

(b) there exists $\eta > 0$ such that either the drift of D_t is greater than $rD_0/2$ or the norm of the volatility of D_t is greater than η, because of inequality (56) using an argument similar to the proof of Lemma C.8.[39]

This implies that with positive probability D_t stays above $D_0/2$ and eventually reaches any arbitrarily large level. Since payoffs are bounded, this leads to a contradiction. We conclude that U must be a sub-solution of the upper optimality equation. ■

[39] While the required argument is similar to the one used in the proof of Lemma C.8, there are important differences, so we outline them here. First, the functions d and f from Lemma C.8 must be re-defined, with the test function V replacing U in the new definition. Second, the definition of the compact set Φ also requires change: Φ would now be the set of all $(a, \bar{b}, \phi, \beta)$ with $\phi \in [q-\varepsilon, q+\varepsilon]$ and $|\beta| \leq M$ such that the incentive constraints (42) are satisfied and $d(a,\bar{b},\phi) \leq 0$. Since ϕ is bounded away from 0 and 1, the boundary conditions will play no role here. Finally, note that U is assumed to satisfy the optimality equation in the lemma, while here V satisfies the strict inequality (56). Accordingly, we need to modify the last part of that proof as follows: $f(a,\bar{b},\phi,\beta) = 0$ implies $d(a,\bar{b},\phi) > 0$, and therefore we have $\eta > 0$.

The next lemma is an auxiliary result used in the proof of Proposition D.2 below.

Lemma D.1. *The correspondence $\mathcal{E}$ of public sequential equilibrium payoffs is convex-valued and has an arc-connected graph.*

Proof. First, we show that $\mathcal{E}$ is convex-valued. Fix $p \in (0,1)$, $\bar{w}, \underline{w} \in \mathcal{E}(p)$ with $\bar{w} > \underline{w}$, and $v \in (\underline{w}, \bar{w})$. Let us show that $v \in \mathcal{E}(p)$. Consider the set $\mathsf{V} = \{(\phi, w) \,|\, w = \alpha|\phi - p| + v\}$ where $\alpha > 0$ is chosen large enough so that $\alpha|\phi - p| + v > U(\phi)$ for all ϕ sufficiently close to 0 and 1. Let $(\phi_t, W_t)_{t\geq 0}$ be the belief / continuation value process of a public sequential equilibrium that yields the normal type a payoff of $\bar{w}$. Let $\tau \overset{\text{def}}{=} \inf\{t > 0 \,|\, (\phi_t, W_t) \in \mathsf{V}\}$. By Theorem 8 specialized to the case with a single behavioral type, we have $\lim_{t\to\infty} \phi_t = 0$ with probability $1-p$, and $\lim_{t\to\infty}\phi_t = 1$ with probability p, hence $\tau < \infty$ almost surely.[40] If with positive probability $\phi_\tau = p$, then $W_\tau = v$ and hence $v \in \mathcal{E}(p)$, which is the desired result. For the case when $\phi_\tau \neq p$ almost surely, the proof proceeds is two steps. In *Step 1*, we construct a pair of continuous curves $\bar{\mathcal{C}}, \underline{\mathcal{C}} \subset \mathsf{Graph}\, \mathcal{E}|_{(0,1)}$ such that their projection on the ϕ-coordinate is the whole $(0,1)$ and

$$\inf\{w \,|\, (p,w) \in \bar{\mathcal{C}}\} > v > \sup\{w \,|\, (p,w) \in \underline{\mathcal{C}}\}.$$

In *Step 2*, we use these curves to construct a sequential equilibrium for prior p under which the normal type receives a payoff of v, concluding the proof that $\mathcal{E}$ is convex valued.

Step 1. If $\phi_\tau \neq p$ almost surely, then both $\phi_\tau < p$ and $\phi_\tau > p$ must happen with positive probability by the martingale property. Hence, there exists a continuous curve $\mathcal{C} \subset \mathsf{graph}\ \mathcal{E}$ with endpoints (p_1, w_1) and (p_2, w_2), with $p_1 < p < p_2$, such that for all $(\phi, w) \in \mathcal{C}$ we have $w > v$ and $\phi \in (p_1, p_2)$. Fix $0 < \varepsilon < p - p_1$. We will now construct a continuous curve $\mathcal{C}' \subset \mathsf{graph}\ \mathcal{E}|_{(0,\, p_1+\varepsilon]}$ that has (p_1, w_1) as an endpoint and satisfies $\inf\{\phi \,|\, \exists w \text{ s.t. } (\phi, w) \in \mathcal{C}'\} = 0$. Fix a public sequential equilibrium of the dynamic game with prior p_1 that yields the normal type a payoff of w_1. Let P^{n} denote the probability measure over the sample paths of X induced by the strategy of the normal type. By Theorem 8 we have $\phi_t \to 0$ P^{n}-almost surely. Moreover, since (ϕ_t) is a supermartingale under P^{n}, the maximal inequality for non-negative supermartingales yields:

$$\mathsf{P}^{\mathsf{n}}\left[\sup_{t\geq 0} \phi_t \leq p_1 + \varepsilon\right] \;\geq\; 1 - \frac{p_1}{p_1+\varepsilon} > 0\,.$$

Therefore, there exists a sample path $(\bar{\phi}_t, \bar{W}_t)_{t\geq 0}$ with the property that $\bar{\phi}_t \leq p_1 + \varepsilon < p$ for all t and $\bar{\phi}_t \to 0$ as $t \to \infty$. Thus, the curve $\mathcal{C}' \subset \mathsf{graph}\ \mathcal{E}$, defined as the image of the path $t \mapsto (\bar{\phi}_t, \bar{W}_t)$, has (p_1, w_1) as an end-point and satisfies $\inf\{\phi \,|\, \exists w \text{ s.t. } (\phi, w) \in \mathcal{C}'\} = 0$. Similarly, we can construct a continuous curve $\mathcal{C}'' \subset \mathsf{Graph}\ \mathcal{E}|_{[p_2-\varepsilon, 1)}$ that has (p_2, w_2) as an endpoint and satisfies $\sup\{\phi \,|\, \exists w \text{ s.t. } (\phi, w) \in \mathcal{C}''\} = 1$.

We have thus constructed a continuous curve $\bar{\mathcal{C}} \overset{\text{def}}{=} \mathcal{C}' \cup \mathcal{C} \cup C'' \subset \mathsf{Graph}\, \mathcal{E}|_{(0,1)}$ which projects onto $(0,1)$ and satisfies $\inf\{w \,|\, (p,w) \in \bar{\mathcal{C}}\} > v$. A similar construction yields a continuous curve $\underline{\mathcal{C}} \subset \mathsf{Graph}\, \mathcal{E}|_{(0,1)}$ which projects onto $(0,1)$ and satisfies $\sup\{w \,|\, (p,w) \in \underline{\mathcal{C}}\} < v$.

Step 2. We will now construct a sequential equilibrium for prior p under which the normal type receives a payoff of v. Let $\phi \mapsto (a(\phi), \bar{b}(\phi))$ be a measurable selection from the correspondence of static

[40] To be precise, Theorem 8 does not state what happens conditional on the behavioral type. However, in the particular case of a single behavioral type, it is easy to adapt the proof of that theorem (cf. Appendix E) to show that $\lim_{t\to\infty} \phi_t = 1$ under the behavioral type.

Bayesian Nash equilibrium. Define $(\phi_t)_{t\geq 0}$ as the unique weak solution of

$$d\phi_t = -|\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2/(1-\phi_t) + \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t) \cdot dZ_t^{\mathsf{n}}$$

with initial condition $\phi_0 = p$.[41] Next, let $(W_t)_{t\geq 0}$ be the unique solution of the pathwise deterministic ODE

$$dW_t = r(W_t - g(a(\phi_t), \bar{b}(\phi_t)))\, dt$$

with initial condition $W_0 = v$, up to the stopping time $T > 0$ when (ϕ_t, W_t) first hits either $\bar{\mathcal{C}}$ or $\underline{\mathcal{C}}$. Define the strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ as follows: for $t < T$, set $(a_t, \bar{b}_t) = (a(\phi_t), \bar{b}(\phi_t))$; from $t = T$ onwards $(a_t, \bar{b}_t)$ follows a sequential equilibrium of the game with prior ϕ_T. By Theorem 2 $(a_t, \bar{b}_t, \phi_t)_{t\geq 0}$ must be a sequential equilibrium that yields the normal type a payoff of v. We have thus shown that $v \in \mathcal{E}(p)$, concluding the proof that $\mathcal{E}$ is convex-valued.

We now show that $\mathsf{Graph}(\mathcal{E})$ is arc-connected. Fix $p < q$, $v \in \mathcal{E}(p)$ and $w \in \mathcal{E}(q)$. Consider a sequential equilibrium of the game with prior q that yields the normal type a payoff of w. Since $\phi_t \to 0$ under the normal type, there exists a continuous curve $\mathcal{C} \subset \mathsf{graph}\ \mathcal{E}$ with endpoints (q, w) and (p, v') for some $v' \in \mathcal{E}(p)$. Since $\mathcal{E}$ is convex-valued, the straight line $\mathcal{C}'$ connecting (p, v) to (p, v') is contained in the graph of $\mathcal{E}$. Hence $\mathcal{C} \cup \mathcal{C}'$ is a continuous curve connecting (q, w) and (p, v) which is contained in the graph of $\mathcal{E}$, concluding the proof that $\mathcal{E}$ has an arc-connected graph. ■

Proposition D.2. *The upper boundary U is a viscosity super-solution of the upper optimality equation on* $(0, 1)$.

Proof. If U is not a super-solution, there exists $q \in (0, 1)$ and a $\mathcal{C}^2$-function $V : (0, 1) \to \mathbb{R}$ such that $0 = (U_* - V)(q) < (U_* - V)(\phi)$ for all $\phi \in (0, 1) \setminus \{q\}$, and

$$H^*(q, V(q), V'(q)) = H^*(q, U_*(q), V'(q)) < V''(q)\,.$$

Since H^* is upper semi-continuous, U_* is lower semi-continuous and $U_* > V$ on $(0, 1) \setminus \{q\}$, there exist $\varepsilon, \delta > 0$ small enough such that for all $\phi \in [q - \varepsilon, q + \varepsilon]$,

$$H(\phi, V(\phi) + \delta, V'(\phi)) < V''(\phi), \tag{58}$$

$$V(q - \varepsilon) + \delta < U(q - \varepsilon) \quad \text{and} \quad V(q + \varepsilon) + \delta < U(q + \varepsilon). \tag{59}$$

Figure 8 displays the configuration of functions U_* and $V + \delta$. Fix a pair (ϕ_0, W_0) with $\phi_0 \in (q-\varepsilon, q+\varepsilon)$ and $U(\phi_0) < W_0 < V(\phi_0) + \delta$.

We will now construct a sequential equilibrium that attains (ϕ_0, W_0), and this will lead to a contradiction since $U(\phi_0) < W_0$ and U is the upper boundary of $\mathcal{E}$.

Let $\phi \mapsto (a(\phi), \bar{b}(\phi)) \in \mathcal{N}(\phi, \phi(1-\phi)V'(\phi)/r)$ be a measurable selection of action profiles that minimize

$$rV(\phi) - rg(a, \bar{b}) - |\gamma(a, \bar{b}, \phi)|^2 (\frac{V''(\phi)}{2} - \frac{V'(\phi)}{1-\phi}) \tag{60}$$

[41] Condition 1 ensures that $\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)$ is bounded away from zero when $(a(\phi_t), \bar{b}(\phi_t)) \in \mathcal{N}(\phi_t, 0)$, hence standard results for existence and uniqueness of weak solutions apply (Karatzas and Shreve, 1991, p. 327).

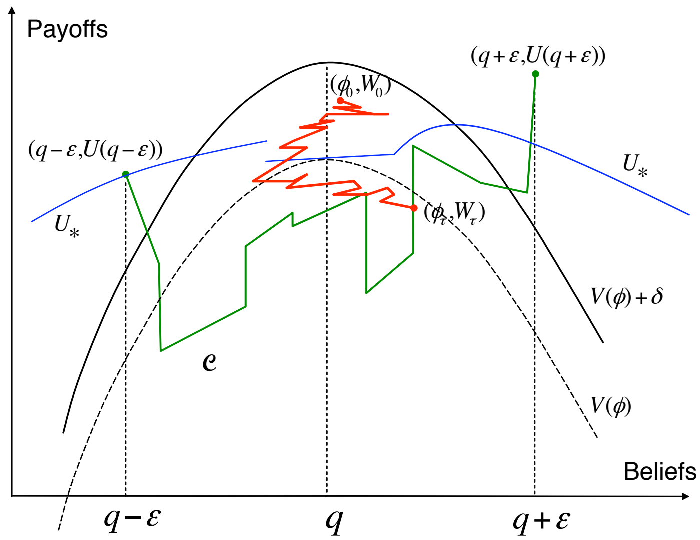

Figure 8: A viscosity super-solution.

over all $(a, \bar{b}) \in \mathcal{N}(\phi, \phi(1-\phi)V'(\phi)/r)$, for each $\phi \in (0,1)$. Define $(\phi_t)_{t\geq 0}$ as the unique weak solution of

$$d\phi_t = -|\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2/(1-\phi_t)\,dt \;+\; \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t) \cdot dZ_t^{\mathsf{n}} \tag{61}$$

on the interval $[q-\varepsilon, q+\varepsilon]$, with initial condition ϕ_0.[42] Next, let $(W_t)_{t\geq 0}$ be the unique strong solution of

$$dW_t = r(W_t - g(a(\phi_t), \bar{b}(\phi_t)))\,dt + \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)V'(\phi_t) \cdot dZ_t^{\mathsf{n}}, \tag{62}$$

with initial condition W_0, until the stopping time when ϕ_t first exits $[q-\varepsilon, q+\varepsilon]$.[43] Then, by Itô's formula, the process $D_t = W_t - V(\phi_t) - \delta$ has zero volatility and drift given by

$$rD_t + rV(\phi_t) - rg(a(\phi_t), \bar{b}(\phi_t)) - |\gamma(a(\phi_t), \bar{b}(\phi_t)), \phi_t)|^2 \left(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t}\right).$$

By (58) and the definition of $(a(\phi), \bar{b}(\phi))$, the drift of D_t is strictly negative so long as $D_t \leq 0$ and $\phi_t \in [q-\varepsilon, q+\varepsilon]$. (Note that $D_0 < 0$.) Therefore, the process $(\phi_t, W_t)_{t\geq 0}$ remains under the curve $(\phi_t, V(\phi_t)+\delta)$ from time zero onwards so long as $\phi_t \in [q-\varepsilon, q+\varepsilon]$.

By Lemma D.1 there exists a continuous curve $\mathcal{C} \subset \mathcal{E}|_{[q-\varepsilon, q+\varepsilon]}$ that connects the points $(q-\varepsilon, U(q-\varepsilon))$ and $(q+\varepsilon, U(q+\varepsilon))$. By (59), the path $\mathcal{C}$ and the function $V+\delta$ bound a connected region in $[q-\varepsilon, q+\varepsilon] \times \mathbb{R}$ that contains (ϕ_0, W_0), as shown in Figure 8. Since the drift of D_t is strictly negative while $\phi_t \in [q-\varepsilon, q+\varepsilon]$, the pair (ϕ_t, W_t) eventually hits the path $\mathcal{C}$ at a stopping time $\tau < \infty$ before ϕ_t exits the interval $[q-\varepsilon, q+\varepsilon]$.

[42] Existence and uniqueness of a weak solution on a closed sub-interval of $(0,1)$ follows from the fact that V' must be bounded on such subinterval, and therefore γ must be bounded away from zero by Lemma B.1. See Karatzas and Shreve (1991, p. 327).

[43] Existence and uniqueness of a strong solution follows from the Lipschitz and linear growth conditions in W, and the boundedness of $\gamma(a(\phi), \bar{b}(\phi), \phi)V'(\phi)$ on $[q-\varepsilon, q+\varepsilon]$.

We will now construct a sequential equilibrium for prior ϕ_0 under which the normal type receives a payoff of W_0. Consider the strategy profile and belief process that coincides with $(a(\phi_t), \bar{b}(\phi_t), \phi_t)$ up to time τ, and follows a sequential equilibrium of the game with prior ϕ_τ at all times after τ. Since W_t is bounded, $(a(\phi_t), \bar{b}(\phi_t)) \in \mathcal{N}(\phi_t, \phi_t(1-\phi_t)V'(\phi_t)/r)$ and the processes $(\phi_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ follow (61) and (62) respectively, Theorem 2 implies that the strategy profile $(a(\phi_t), \bar{b}(\phi_t))_{t\geq 0}$ and the belief process $(\phi_t)_{t\geq 0}$ form a sequential equilibrium of the game with prior ϕ_0. It follows that $W_0 \in \mathcal{E}(\phi_0)$, and this is a contradiction since $W_0 > U(\phi_0)$. Thus, U must be a super-solution of the upper optimality equation. ■

Lemma D.2. *Every bounded viscosity solution of the upper optimality equation is locally Lipschitz continuous.*

Proof. En route to a contradiction, suppose that U is a bounded viscosity solution that is not locally Lipschitz. That is, for some $p \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2})$ satisfying $[p-2\varepsilon, p+2\varepsilon] \subset (0,1)$ the restriction of U to $[p-\varepsilon, p+\varepsilon]$ is not Lipschitz continuous. Let $M = \sup |U|$. By Corollary B.1 there exists $K > 0$ such that for all $(\phi, u, u') \in [p-2\varepsilon, p+2\varepsilon] \times [-M, M] \times \mathbb{R}$,

$$|H^*(\phi, u, u')| \leq K(1 + |u'|^2)\,, \tag{63}$$

Since the restriction of U to $[p-\varepsilon, p+\varepsilon]$ is not Lipschitz continuous, there exist $\phi_0, \phi_1 \in [p-\varepsilon, p+\varepsilon]$ such that

$$\frac{|U_*(\phi_1) - U_*(\phi_0)|}{|\phi_1 - \phi_0|} \geq \max\{1, \exp(2M(4K + 1/\varepsilon))\}\,. \tag{64}$$

Hereafter we assume that $\phi_1 > \phi_0$ and $U_*(\phi_1) > U_*(\phi_0)$. The proof for the reciprocal case is similar and will be omitted for brevity. Let $V : J \to \mathbb{R}$ be the solution of the differential equation

$$V''(\phi) = 2K(1 + V'(\phi)^2)\,, \tag{65}$$

with initial conditions given by

$$V(\phi_1) = U_*(\phi_1) \text{ and } V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0}\,, \tag{66}$$

where J is the maximal interval of existence around ϕ_1.

We claim that V has the following two properties:

(a) There exists $\phi^* \in J \cap (p-2\varepsilon, p+2\varepsilon)$ such that $V(\phi^*) = -M$ and $\phi^* < \phi_0$. In particular, $\phi_0 \in J$.

(b) $V(\phi_0) > U_*(\phi_0)$.

We first prove property (a). For all $\phi \in J$ such that $V'(\phi) > 1$, we have $V''(\phi) < 4KV'(\phi)^2$ or, equivalently, $(\log V')'(\phi) < 4KV'(\phi)$, which implies

$$V(\hat{\phi}) - V(\tilde{\phi}) > \frac{1}{4K}(\log(V'(\hat{\phi})) - \log(V'(\tilde{\phi})))\,, \quad \forall \hat{\phi}, \tilde{\phi} \in J \text{ such that } V'(\hat{\phi}) > V'(\tilde{\phi}) > 1\,. \tag{67}$$

By (64) and (66), we have $\frac{1}{4K}\log(V'(\phi_1)) > 2M$, and therefore a unique $\tilde{\phi} \in J$ exists such that

$$\frac{1}{4K}(\log(V'(\phi_1)) - \log(V'(\tilde{\phi}))) = 2M\,. \tag{68}$$

Since $V'(\tilde{\phi}) > 1$, it follows from (67) that $V(\phi_1) - V(\tilde{\phi}) > 2M$ and so $V(\tilde{\phi}) < -M$. Since $V(\phi_1) > U(\phi_0) \geq -M$, there exists some $\phi^* \in (\tilde{\phi}, \phi_1)$ such that $V(\phi^*) = -M$. Moreover ϕ^* must belong to $(p - 2\varepsilon, p + 2\varepsilon)$, because the strict convexity of V implies

$$\phi_1 - \phi^* < \frac{V(\phi_1) - V(\phi^*)}{V'(\phi^*)} < \frac{2M}{V'(\hat{\phi})} < \frac{2M}{\log(V'(\hat{\phi}))} = \frac{2M}{\log(V'(\phi_1)) - 8KM} < \varepsilon \, ,$$

where the equality follows from (68) and the rightmost inequality follows from (64). Finally, we have $\phi^* < \phi_0$, otherwise the inequality $V(\phi^*) \leq U_*(\phi_0)$ and the initial conditions (66) would imply

$$V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0} \leq \frac{V(\phi_1) - V(\phi^*)}{\phi_1 - \phi^*} \, ,$$

which would violate the strict convexity of V. This concludes the proof of property (a).

Turning to property (b), the strict convexity of V and the initial conditions (66) imply

$$\frac{U_*(\phi_1) - V(\phi_0)}{\phi_1 - \phi_0} = \frac{V(\phi_1) - V(\phi_0)}{\phi_1 - \phi_0} < V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0} \, ,$$

and therefore $V(\phi_0) > U_*(\phi_0)$, as claimed.

Now define $L = \max \{V(\phi) - U_*(\phi) \,|\, \phi \in [\phi^*, \phi_1]\}$. By property (b), we must have $L > 0$. Let $\hat{\phi}$ be a point at which the maximum L is attained. Since $V(\phi_*) = -M$ and $V(\phi_1) = U_*(\phi_1)$, we must have $\hat{\phi} \in (\phi^*, \phi_1)$ and therefore $V - L$ is a test function satisfying

$$U_*(\hat{\phi}) = V(\hat{\phi}) - L \;\text{ and }\; U_*(\phi) \geq V(\phi) - L \;\text{ for each }\; \phi \in (\phi^*, \phi_1) \, .$$

Since U is a viscosity supersolution,

$$V''(\hat{\phi}) \leq H^*(\hat{\phi}, V(\hat{\phi}) - L, V'(\hat{\phi})) \, ,$$

hence, by (63),

$$V''(\hat{\phi}) \leq K(1 + V'(\hat{\phi})^2) < 2K(1 + V'(\hat{\phi})^2)),$$

and this is a contradiction, since by construction V satisfies equation (65). ■

Lemma D.3. *Every bounded viscosity solution of the upper optimality equation is continuously differentiable with absolutely continuous derivatives.*

Proof. Let $U : (0, 1) \to \mathbb{R}$ be a bounded solution of the upper optimality equation. By Lemma D.2, U is locally Lipschitz and hence differentiable almost everywhere. We will now show that U is differentiable everywhere. Fix $\phi \in (0, 1)$. Since U is locally Lipschitz, there exist $\delta > 0$ and $k > 0$ such that for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function $V : (\phi - \delta, \phi + \delta) \to \mathbb{R}$ satisfying $V(p) = U(p)$ and $V \geq U$ we have

$$|V'(p)| \leq k \, .$$

It follows from Corollary B.1 that there exists some $M > 0$ such that

$$|H(p, U(p), V'(p))| \leq M$$

for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function V satisfying $V \geq U$ and $V(p) = U(p)$.

Let us now show that for all $\varepsilon \in (0,\delta)$ and $\varepsilon' \in (0,\varepsilon)$

$$-M\varepsilon'(\varepsilon-\varepsilon') < U(\phi+\varepsilon') - \left(\frac{\varepsilon'}{\varepsilon}U(\phi+\varepsilon) + \frac{\varepsilon-\varepsilon'}{\varepsilon}U(\phi)\right) < M\varepsilon'(\varepsilon-\varepsilon'). \tag{69}$$

If not, for example if the second inequality fails, then we can choose $K > 0$ such that the C^2 function (a parabola)

$$\varepsilon' \mapsto f(\phi+\varepsilon') = \left(\frac{\varepsilon'}{\varepsilon}U(\phi+\varepsilon) + \frac{\varepsilon-\varepsilon'}{\varepsilon}U(\phi)\right) + M\varepsilon'(\varepsilon-\varepsilon') + K$$

is strictly above $U(\phi+\varepsilon')$ over $(0,\varepsilon)$, except for a tangency point at some $\varepsilon'' \in (0,\varepsilon)$. But this contradicts the fact that U is a viscosity subsolution, since $f''(\phi+\varepsilon'') = -2M < H(\phi+\varepsilon'', U(p+\varepsilon''), U'(\phi+\varepsilon''))$. This contradictions proves inequalities (69).

It follows from (69) that for all $0 < \varepsilon' < \varepsilon < \delta$,

$$\left|\frac{U(\phi+\varepsilon')-U(\phi)}{\varepsilon'} - \frac{U(\phi+\varepsilon)-U(\phi)}{\varepsilon}\right| \le M\varepsilon.$$

Thus, as ε converges to 0 from above, $(U(\phi+\varepsilon)-U(\phi))/\varepsilon$ converges to a limit $U'(\phi+)$. Similarly, if ε converges to 0 from below, the quotient above converges to a limit $U'(\phi-)$. We claim that $U'(\phi+) = U'(\phi-)$. Otherwise, if for example $U'(\phi+) > U'(\phi-)$, then the function

$$\varepsilon' \mapsto f_1(\phi+\varepsilon') = U(\phi) + \varepsilon'\frac{U'(\phi-)+U'(\phi+)}{2} + M\varepsilon'^2$$

is below U in a neighborhood of ϕ, except for a tangency point at ϕ. But this leads to a contradiction, because $f_1''(\phi) = 2M > H(\phi, U(\phi), U'(\phi))$ and U is a super-solution. Therefore $U'(\phi+) = U'(\phi-)$ and we conclude that U is differentiable at every $\phi \in (0,1)$.

It remains to show that U' is locally Lipschitz. Fix $\phi \in (0,1)$ and, arguing just as above, choose $\delta > 0$ and $M > 0$ so that

$$|H(p, U(p), V'(p))| \le M$$

for every $p \in (\phi-\delta, \phi+\delta)$ and every smooth test function V satisfying $V(p) = U(p)$ and either $V \ge U$ or $V \le U$. We affirm that for any $p \in (\phi-\delta,\phi+\delta)$ and $\varepsilon \in (0,\delta)$

$$|U'(p) - U'(p+\varepsilon)| \le 2M\varepsilon.$$

Otherwise, e.g. if $U'(p+\varepsilon) > U'(p) + 2M\varepsilon$ for some $p \in (\phi-\delta,\phi+\delta)$ and $\varepsilon \in (0,\delta)$, then the test function

$$\varepsilon' \mapsto f_2(p+\varepsilon') = \frac{\varepsilon'}{\varepsilon}U(p+\varepsilon) + \frac{\varepsilon-\varepsilon'}{\varepsilon}U(p) - M\varepsilon'(\varepsilon-\varepsilon')$$

must be above U at some $\varepsilon' \in (0,\varepsilon)$ (since $f_2'(p+\varepsilon) - f_2'(p) = 2M\varepsilon$.) Therefore, there exists a constant $K > 0$ such that $f_2(p+\varepsilon') - K$ stays below U for $\varepsilon' \in [0,\varepsilon]$, except for a tangency at some $\varepsilon'' \in (0,\varepsilon)$. But then

$$f_2''(\phi+\varepsilon'') = 2M > H(\phi+\varepsilon'', U(\phi+\varepsilon''), U'(\phi+\varepsilon'')),$$

contradicting the assumption that U is a viscosity super-solution. ■

Proposition D.3. *The upper boundary U is a continuously differentiable function, with absolutely continuous derivatives. Moreover, U is the greatest bounded solution of the differential inclusion*

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi)),\ H^*(\phi, U(\phi), U'(\phi))] \text{ a.e.}. \tag{70}$$

Proof. First, by Propositions D.1, and D.2 and Lemma D.3, the upper boundary U is a differentiable function with absolutely continuous derivative that solves the differential inclusion (70). If U is not the greatest bounded solution of (70), then there exists another bounded solution V which is strictly greater than U at some $p \in (0, 1)$. Choose $\varepsilon > 0$ such that $V(p) - \varepsilon > U(p)$. We will show that $V(p) - \varepsilon$ is the payoff of a public sequential equilibrium, which is a contradiction since U is the upper boundary.

From the inequality

$$V''(\phi) \geq H(\phi, V(\phi), V'(\phi)) \text{ a.e.}$$

it follows that a measurable selection $\phi \mapsto (a(\phi), \bar{b}(\phi)) \in \mathcal{N}(\phi, \phi(1-\phi)V'(\phi)/r)$ exists such that

$$rV(\phi) - rg(a(\phi), \bar{b}(\phi), \phi) - |\gamma(a(\phi), \bar{b}(\phi), \phi)|^2 (\frac{V''(\phi)}{2} - \frac{V'(\phi)}{1-\phi}) \leq 0\,, \tag{71}$$

for almost every $\phi \in (0, 1)$. Let (ϕ_t) be the unique weak solution of

$$d\phi_t = -|\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2/(1-\phi_t) + \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)\, dZ_t^{\mathfrak{n}}\,,$$

with initial condition $\phi_0 = p$. Let (W_t) be the unique strong solution of

$$dW_t = r(W_t - g(a(\phi_t), \bar{b}(\phi_t)))\, dt + V'(\phi_t)\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)\, dZ_t^{\mathfrak{n}}\,,$$

with initial condition $W_0 = V(p) - \varepsilon$. Consider the process $D_t = W_t - V(\phi_t)$. It follows from Itô's formula for differentiable functions with absolutely continuous derivatives that:

$$\frac{dD_t}{dt} = rD_t + rV(\phi_t) - rg(a(\phi_t), \bar{b}(\phi_t), \phi_t) - |\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2 (\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t})\,.$$

Therefore, by (71) we have

$$\frac{dD_t}{dt} \leq rD_t\,,$$

and since $D_0 = -\varepsilon < 0$ it follows that $W_t \searrow -\infty$. Let τ be the first time that (ϕ_t, W_t) hits the graph of U. Consider a strategy profile / belief process that coincides with $(a_t, \bar{b}_t, \phi_t)$ up to time τ and, after that, follows a public sequential equilibrium of the game with prior ϕ_τ with value $U(\phi_\tau)$. Theorem 2 implies that the strategy profile / belief process thus constructed is a sequential equilibrium that yields the large player payoff $V(p) - \varepsilon > U(p)$, a contradiction. ■

E Proof of Theorem 8

Consider the closed set

$$IC = \{(a, \bar{b}, \phi, \beta) \in A \times \Delta(B) \times \Delta^K \times \mathbb{R}^d : \text{conditions (11) and (12) are satisfied}\}.$$

Then, the image of this set under

$$(a, \bar{b}, \phi, \beta) \mapsto (\phi_0, \phi_0 \sigma(\bar{b})^{-1}(\mu(a, \bar{b}) - \mu^{\phi}(a, \bar{b})), r\beta^{\top}\sigma(\bar{b}))$$

is a closed set that does not intersect the line segment $(0, 1) \times \{0\} \times \{0\}$ by Condition $1'$. Thus, for any $\varepsilon > 0$ there exist constants $C > 0$ and $M > 0$ such that for all $(a, \bar{b}, \phi, \beta) \in IC$ with $\phi_0 \in [\varepsilon, 1 - \varepsilon]$, either

$$|\phi_0 \sigma(\bar{b})^{-1}(\mu(a, \bar{b}) - \mu^{\phi}(a, \bar{b}))| \geq C \quad \text{or} \quad |r\beta^{\top}\sigma(\bar{b})| \geq M.$$

Fix a public sequential equilibrium $(a_t, \bar{b}_t, \phi_t)_{t \geq 0}$ with continuation values $(W_t)_{t \geq 0}$ for the normal type and consider the evolution of

$$\exp(K_1(W_t - \underline{g})) + K_2 \phi_{0,t}^2$$

while $\phi_{0,t} \in [\varepsilon, 1 - \varepsilon]$, where the constants K_1 and $K_2 > 0$ will be determined later. By Itô's formula, under the probability measure generated by the normal type the process $\exp(K_1(W_t - \underline{g}))$ has drift

$$K_1 \exp(K_1(W_t - \underline{g}))r(W_t - g(a_t, \bar{b}_t)) + K_1^2 \exp(K_1(W_t - \underline{g}))r^2|\beta_t^{\top}\sigma(\bar{b}_t)|^2/2.$$

Thus, we can guarantee that the drift of $\exp(K_1(W_t - \underline{g}))$ is greater than or equal to 1 whenever $|r\beta_t^{\top}\sigma(\bar{b}_t)| \geq M$ by choosing $K_1 > 0$ such that

$$-K_1 r(\bar{g} - \underline{g}) + K_1^2 M^2/2 \geq 1.$$

Moreover, the drift of $K_2 \phi_{0,t}^2$ must always be non-negative, since under the normal type $\phi_{0,t}$ is a submartingale, and hence $K_2 \phi_{0,t}^2$ is also a submartingale. Now, even when $|r\beta_t^{\top}\sigma(\bar{b}_t)| < M$, the drift of $\exp(K_1(W_t - \underline{g}))$ is still greater than or equal to

$$-K_1 \exp(K_1(\bar{g} - \underline{g}))r(\bar{g} - \underline{g}).$$

But in this case we have $|\phi_{0,t}\sigma(\bar{b}_t)^{-1}(\mu(a_t, \bar{b}_t) - \mu^{\phi}(a_t, \bar{b}_t))| \geq C$, so the drift of $K_2 \phi_{0,t}^2$ is greater than or equal to $K_2 C^2$. Thus, by choosing K_2 large enough so that

$$K_2 C^2 - K_1 \exp(K_1(\bar{g} - \underline{g}))r(\bar{g} - \underline{g}) \geq 1,$$

we can ensure that the drift of $\exp(K_1(W_t - \underline{g})) + K_2 \phi_{0,t}^2$ is always greater than 1 while $\phi_{0,t} \in [\varepsilon, 1 - \varepsilon]$. But since $\exp(K_1(W_t - \underline{g})) + K_2 \phi_{0,t}^2$ must be bounded in a sequential equilibrium, it follows that $\phi_{0,t}$ must eventually exit the interval $[\varepsilon, 1 - \varepsilon]$ with probability 1 in any sequential equilibrium. Since $\varepsilon > 0$ is arbitrary, it follows that the bounded submartingale $(\phi_{0,t})_{t \geq 0}$ must converge to 0 or 1 almost surely, and it cannot converge to 0 with positive probability under the probability measure generated by the normal type. ■

F Appendix for Section 9

We begin with the following monotonicity lemma, which will be used throughout this appendix.

Lemma F.1. *Fix (ϕ, ζ), $(\phi', \zeta') \in [0, 1] \times \mathbb{R}$, $(a, b) \in \mathcal{M}(\phi, \zeta)$ and $(a', b') \in \mathcal{M}(\phi', \zeta')$. If $\phi' \leq \phi$, $\zeta' \geq \zeta$ and $\zeta' \geq 0$ then $a' \geq a$.*

Proof. In three steps:

Step 1. The best-reply b^{BR} *of the small players, which is single-valued and defined by the fixed point condition*

$$b^{\mathsf{BR}}(a,\phi) = \operatorname*{arg\,max}_{b'} \phi h(a^*, b', b^{\mathsf{BR}}(a,\phi)) + (1-\phi) h(a^*, b', b^{\mathsf{BR}}(a,\phi)),$$

is increasing in a *and* ϕ*.*

For each $(a,\phi) \in A \times [0,1]$, any pure action $b \in B$ which is a best-reply for the small players to (a,ϕ) must satisfy the first-order condition

$$(\phi h_2(a^*, b, b) + (1-\phi) h_2(a, b, b))(\hat{b} - b) \leq 0 \quad \forall \hat{b} \in B.$$

Since $h_{22} + h_{23} < 0$ along the diagonal of the small players' actions, for each fixed (a,ϕ) the function $b \mapsto \phi h_2(a^*, b, b) + (1-\phi) h_2(a, b, b)$ is strictly decreasing, hence the best-reply of the small players is unique for all (a,ϕ), as claimed.

To see that b^{BR} is increasing, let $a' \geq a$, $\phi' \geq \phi$ and suppose, towards a contradiction, that $b' = b^{\mathsf{BR}}(a',\phi') < b^{\mathsf{BR}}(a,\phi) = b$. By the first-order conditions,

$$(\phi h_2(a^*, b, b) + (1-\phi) h_2(a, b, b))\underbrace{(b' - b)}_{<0} \ \leq\ 0$$

and

$$(\phi' h_2(a^*, b', b') + (1-\phi') h_2(a, b', b'))\underbrace{(b - b')}_{>0} \ \leq\ 0,$$

hence

$$\phi h_2(a^*, b, b) + (1-\phi) h_2(a, b, b) \ \geq\ 0 \ \geq\ \phi' h_2(a^*, b', b') + (1-\phi') h_2(a, b', b'),$$

and this is a contradiction since we have assumed $h_{12} \geq 0$ and $h_{22} + h_{23} < 0$ along the diagonal of small players' actions.

Step 2. For each $(b,\zeta) \in B \times \mathbb{R}$ *define* $\mathsf{BR}(b,\zeta) \stackrel{\text{def}}{=} \operatorname{arg\,max}_{a'} g(a', b) + \zeta\lambda(a')$.[44] *Then, for all* (a,b,ζ) *and* $(a',b',\zeta') \in A \times B \times \mathbb{R}$ *with* $a \in \mathsf{BR}(b,\zeta)$ *and* $a' \in \mathsf{BR}(b',\zeta')$*,*

$$[b' \leq b, \quad \zeta' \geq \zeta \quad \text{and} \quad \zeta' \geq 0] \quad \Longrightarrow \quad a' \geq a.$$

If $a \in \mathsf{BR}(b,\zeta)$ and $a' \in \mathsf{BR}(b',\zeta')$ then the first-order conditions imply

$$(g_1(a,b) + \zeta\lambda'(a))\,(a' - a) \leq 0 \quad \text{and} \quad (g_1(a',b') + \zeta'\lambda'(a'))(a - a') \leq 0.$$

Towards a contradiction, suppose $b' \leq b$, $\zeta' \geq \max\{0,\zeta\}$ and $a' < a$. Then, the inequalities above imply

$$g_1(a,b) + \zeta\lambda'(a) \ \geq\ 0 \ \geq g_1(a',b') + \zeta'\lambda'(a').$$

But this is a contradiction, since we have $g_1(a',b') > g_1(a,b)$ by $g_{11} < 0$ and $g_{12} \leq 0$, and also $\zeta'\lambda'(a') \geq \zeta'\lambda'(a) \geq \zeta\lambda'(a)$ by $\lambda'' \leq 0$, $\lambda' > 0$ and $\zeta' \geq 0$.

Step 3. If $(a,b) \in \mathcal{M}(\phi,\zeta)$*,* $(a',b') \in \mathcal{M}(\phi',\zeta')$*,* $\phi' \leq \phi$ *and* $\zeta' \geq \max\{0,\zeta\}$ *then* $a' \geq a$*.*

Suppose not, i.e. $(a,b) \in \mathcal{M}(\phi,\zeta)$, $(a',b') \in \mathcal{M}(\phi',\zeta')$, $\phi' \leq \phi$, $\zeta' \geq \max\{0,\zeta\}$ and $a' < a$. Then, we must have $b' \leq b$ by Step 1 and $b' > b$ by Step 2, a contradiction. Therefore $a' \geq a$, as claimed. ■

[44] Note that BR may not be single-valued when $\zeta < 0$.

We are now ready to prove Lemma 2, the continuity lemma used in the proof of Theorem 10.

Proof of Lemma 2. Fix an arbitrary constant $M > 0$. Consider the set Φ of all tuples $(a, b, \zeta) \in A \times B \times \mathbb{R}$ satisfying

$$a \in \operatorname*{arg\,max}_{a' \in A} \; g(a', b) + \zeta\lambda(a'), \quad b \in \operatorname*{arg\,max}_{b' \in B} \; h(a, b', b), \quad g(a, b) + \zeta\lambda(a) \geq g(a^N, b^N) + \varepsilon, \tag{72}$$

and $\zeta \leq M$.

We claim that for all $(a, b, \zeta) \in \Phi$ we have $\zeta > 0$. Otherwise, if $\zeta \leq 0$ for some $(a, b, \zeta) \in \Phi$, then $a \leq a^N$ by Lemma F.1 and therefore $b \leq b^N$, since the small players' best reply is increasing in the large player's action, as shown in Step 1 of the proof of Lemma F.1. But since (a^N, b^N) is a Nash equilibrium and $g_2 \geq 0$, it follows that $g(a, b) + \zeta\lambda(a) \leq g(a, b) \leq g(a, b^N) \leq g(a^N, b^N) < g(a^N, b^N) + \varepsilon$, and this is a contradiction since we have assumed that (a, b, ζ) satisfy (72).

Thus, Φ is a compact set and the continuous function $(a, b, \zeta) \mapsto \zeta$ achieves its minimum, ζ_0, on Φ. Moreover, we must have $\zeta_0 > 0$ by the argument in the previous paragraph. It follows that $\zeta \geq \delta \stackrel{\text{def}}{=} \min\{M, \zeta_0\}$ for any (a, b, ζ) satisfying conditions (72). ■

Turning to the proof of Proposition 9, since it is is similar to the proof of Proposition 4, we only provide a sketch.

Proof of Proposition 9 (sketch). As in Appendix C.6, we use a first-order condition to characterize the action profile $(a, b) \in \mathcal{M}(\phi, (V(\phi + \Delta\phi(a)) - V(\phi))/r)$ for each $(\phi, V) \in (0, 1) \times \mathcal{C}^{\text{inc}}([0, 1])$. To express this condition, we define a function $G : A \times B \times [0, 1] \times \mathcal{C}^{\text{inc}}([0, 1]) \to \mathbb{R}^2$ as follows:

$$G(a, b, \phi, V) \stackrel{\text{def}}{=} \Big(g_1(a, b) + \lambda'(a)\big(V(\phi + \Delta\phi(a)) - V(\phi)\big)/r,\; \phi\, h_2(a^*, b, b) + (1-\phi)\, h_2(a, b, b)\Big). \tag{73}$$

Thus, the first-order necessary and sufficient condition for $(a, b) \in \mathcal{M}(\phi, (V(\phi + \Delta\phi(a)) - V(\phi))/r)$ can be stated as:

$$G(a, b, \phi, V) \cdot (\hat{a} - a, \hat{b} - b) \leq 0 \quad \forall (\hat{a}, \hat{b}) \in A \times B.$$

Next, using an argument very similar to the proof of Lemma C.11, we can show:

(A) $\exists L > 0$ such that $\forall (a, b, \phi) \in A \times B \in (0, 1)$ and $V, \hat{V} \in \mathcal{C}^{\text{inc}}([0, 1])$,

$$|G(a, b, \phi, \hat{V}) - G(a, b, \phi, V)| \;\leq\; L\, d_\infty(\hat{V}, V),$$

where d_∞ is the supremum distance on $\mathcal{C}^{\text{inc}}([0, 1])$;

(B) $\exists M > 0$ such that $\forall (\phi, V) \in (0, 1) \times \mathcal{C}^{\text{inc}}([0, 1])$ and $(a, b), (\hat{a}, \hat{b}) \in A \times B$,

$$\big(G(\hat{a}, \hat{b}, \phi, V) - G(a, b, \phi, V)\big) \cdot (\hat{a} - a, \hat{b} - b) \;\leq\; -M\,(|\hat{a} - a|^2 + |\hat{b} - b|^2).$$

Finally, with conditions (A) and (B) above in place, we can follow the steps of the proof of Proposition 4 (with the mapping G replacing F) and prove the desired result. ■

Next, towards the proof of Theorem 11, we first show that the optimality equation has a bounded increasing solution.

Proposition F.1. *There exists a bounded increasing continuous function $U : (0,1) \to \mathbb{R}$ which solves the optimality equation* (36) *on* $(0,1)$.

The proof of Proposition F.1 relies on a series of lemmas.

Lemma F.2. *For every $\varepsilon > 0$ and every compact set $K \subset \mathcal{C}^{\mathsf{inc}}([\varepsilon, 1])$ there exists $c > 0$ such that for all $(\phi, V) \in [\varepsilon, 1-\varepsilon] \times K$,*

$$\Delta\phi(\mathsf{a}(\phi, V)) \geq c.$$

Proof. By Proposition 9 the function $(\phi, V) \mapsto \Delta\phi(\mathsf{a}(\phi, V))$ is continuous on the domain $[\varepsilon, 1-\varepsilon] \times \mathcal{C}^{\mathsf{inc}}([\varepsilon, 1])$, and hence it achieves its minimum on the compact set $[\varepsilon, 1-\varepsilon] \times K$ at some point (ϕ_0, V_0). Thus, letting $c \stackrel{\text{def}}{=} \Delta\phi_0(\mathsf{a}(\phi_0, V_0))$ yields the desired result. Indeed, we must have $c \geq 0$ since λ is increasing. Moreover, c could be zero only if $\mathsf{a}(\phi_0, V_0) = a^*$, in which case $\Delta V_0(\phi_0) = 0$ and hence a^* would have to be part of a static Nash equilibrium of the complete information game, which is ruled out by assumption (c). ∎

For ease of notation, for each continuous and bounded function $V : (0,1) \to \mathbb{R}$ and $\phi \in (0,1)$ let $H(\phi, V)$ denote the right-hand side of the optimality equation, i.e.

$$H(\phi, V) \stackrel{\text{def}}{=} \frac{rg\big(\mathsf{a}(\phi, V), \mathsf{b}(\phi, V)\big) + \lambda\big(\mathsf{a}(\phi, V)\big)\Delta V(\phi) - rV(\phi)}{\lambda^{\phi}(\mathsf{a}(\phi, V))\Delta\phi(\mathsf{a}(\phi, V))}.$$

Next, for each $\alpha \in [\underline{g}, \bar{g}]$ and $\varepsilon > 0$ consider the following initial value problem for a suitably modified version of the optimality equation:

Problem $\mathsf{IVP}(\varepsilon, \alpha)$. *Find a real-valued, continuous function U defined on an interval $[\phi_\alpha, 1]$, with $\phi_\alpha < 1 - \varepsilon$, such that*[45]

$$\begin{aligned} &U'(\phi) = \max\{0, H(\phi, U)\} \quad \forall \phi \in [\phi_\alpha, 1-\varepsilon), \\ &U(\phi) = \alpha \quad \forall \phi \in [1-\varepsilon, 1]. \end{aligned} \tag{74}$$

With this definition in place we have:

Lemma F.3. *For every $\varepsilon > 0$ and $\alpha \in [\underline{g}, \bar{g}]$ a unique solution of the $\mathsf{IVP}(\varepsilon, \alpha)$ exists on an interval $[\phi_\alpha, 1]$, with $\phi_\alpha < 1 - \varepsilon$. Moreover, if for some $\alpha_0 \in [\underline{g}, \bar{g}]$ a unique solution of the $\mathsf{IVP}(\varepsilon, \alpha_0)$ exists on $[\phi_0, 1]$, then for every α in a neighborhood of α_0 a unique solution $U_{\varepsilon,\alpha}$ of the $\mathsf{IVP}(\varepsilon, \alpha)$ exists on the same interval $[\phi_0, 1]$ and $\alpha \mapsto U_{\varepsilon,\alpha}$ is a continuous function under the supremum metric.*

Proof. We will apply Theorems 2.2 and 2.3 from Hale and Lunel (1993, pp. 43–44), which provide sufficient conditions under which an initial value problem for a retarded functional differential equation locally admits a unique solution, which is continuous in initial conditions.

[45] Here our convention is that $U'(\phi)$ is the left-derivative of U at ϕ. But since the right-hand side is continuous in ϕ, a solution can fail to be differentiable only at $\phi = 1 - \varepsilon$, at which point the left-derivative can be different from the right-derivative, which is identically zero given the initial condition.

By Proposition 9 and the fact that $\Delta\phi(\mathsf{a}(\phi, V)) > 0$ for all $(\phi, V) \in [\varepsilon, 1-\varepsilon] \times \mathscr{C}^{\mathsf{inc}}([\varepsilon, 1])$ the function $\max\{0, H\}$ is continuous on $[\varepsilon, 1-\varepsilon] \times \mathscr{C}^{\mathsf{inc}}([\varepsilon, 1])$. Moreover, by Proposition 9 and Lemma F.2, for every compact set $K \subset \mathscr{C}^{\mathsf{inc}}([\varepsilon, 1])$ and every $\phi \in [\varepsilon, 1-\varepsilon]$ the function $V \mapsto \max\{0, H(\phi, V)\}$ is Lipschitz continuous on K with a Lipschitz constant which is uniform in $\phi \in [\varepsilon, 1-\varepsilon]$. We have thus shown that $\max\{0, H\}$, the right-hand side of the retarded functional differential equation (74), satisfies all the conditions of Theorems 2.2 and 2.3 from Hale and Lunel (1993, pg. 43-44), and therefore the $\mathsf{IVP}(\varepsilon, \alpha)$ admits a local solution which is unique and continuous in the initial condition α. ■

Next, we show that for a suitable choice of the initial condition α, the unique solution to the $\mathsf{IVP}(\varepsilon, \alpha)$ exists on the entire interval $[\varepsilon, 1]$, takes values in the set $[\underline{g}, \bar{g}]$ and solves the optimality equation on $[\varepsilon, 1-\varepsilon]$.

Lemma F.4. *For every $\varepsilon > 0$ there exists a continuous increasing function $U : [\varepsilon, 1] \to [\underline{g}, \bar{g}]$ that solves the optimality equation (36) on $[\varepsilon, 1-\varepsilon]$ and is constant on $[1-\varepsilon, 1]$.*

Proof. Fix $\varepsilon > 0$. For each $\alpha \in [\underline{g}, \bar{g}]$ write U_α to designate the unique solution of the $\mathsf{IVP}(\varepsilon, \alpha)$, and let $I_\alpha \subset (0, 1]$ be its maximal interval of existence. Consider the following disjoint sets:

$$J_1 = \left\{\alpha \in [\underline{g}, \bar{g}] \,:\, U_\alpha(\phi) \leq \underline{g} \text{ for some } \phi \in I_\alpha \cap (\frac{\varepsilon}{2}, 1-\varepsilon]\right\}$$

and

$$J_2 = \left\{\alpha \in [\underline{g}, \bar{g}] \,:\, \begin{array}{l} I_\alpha \supset (\frac{\varepsilon}{2}, 1], \\ U_\alpha(\phi) > \underline{g} \text{ for all } \phi \in (\frac{\varepsilon}{2}, 1], \\ H(\phi, U_\alpha) < 0 \text{ for some } \phi \in (\frac{\varepsilon}{2}, 1-\varepsilon] \end{array}\right\}.$$

We will show that J_1 and J_2 are nonempty open sets relative to $[\underline{g}, \bar{g}]$. Since J_1 and J_2 are disjoint we will conclude that there exists some $\alpha \in [\underline{g}, \bar{g}] \setminus (J_1 \cup J_2)$. For this α, the solution to the $\mathsf{IVP}(\varepsilon, \alpha)$, U_α, is well-defined on $(\frac{\varepsilon}{2}, 1]$, takes values in $[\underline{g}, \bar{g}]$ and satisfies $H(\phi, U_\alpha) \geq 0$ for all $\phi \in (\frac{\varepsilon}{2}, 1-\varepsilon]$. The latter implies that U_α solves the optimality equation on $(\frac{\varepsilon}{2}, 1-\varepsilon]$.

Let us show that J_1 and J_2 are open relative to $[\underline{g}, \bar{g}]$. Clearly, J_2 is open in $[\underline{g}, \bar{g}]$ because $\alpha \mapsto U_\alpha$ and $\alpha \mapsto H(\cdot, U_\alpha)$ are continuous functions by Proposition 9 and Lemma F.3. Also J_1 is open in $[\underline{g}, \bar{g}]$, because for each $\alpha \in J_1$ we have $U_\alpha(\phi_0) \leq \underline{g}$ for some $\phi_0 \in I_\alpha \cap (\varepsilon, 1-\varepsilon]$, and therefore

$$H(\phi_0, U_\alpha) = \frac{r\overbrace{\left(g(\mathsf{a}(\phi_0, U_\alpha), \mathsf{b}(\phi_0, U_\alpha)) - U_\alpha(\phi_0)\right)}^{>0} + \lambda\left(\mathsf{a}(\phi_0, U_\alpha)\right)\overbrace{\Delta U_\alpha(\phi_0)}^{\geq 0}}{\lambda^\phi(\mathsf{a}(\phi_0, U_\alpha))\underbrace{\Delta\phi(\mathsf{a}(\phi_0, U_\alpha))}_{>0}} > 0,$$

which implies $U'_\alpha(\phi_0) > 0$ by (74). Hence, there exists $\phi_1 \in (\frac{\varepsilon}{2}, \phi_0)$ with $U_\alpha(\phi_1) < \underline{g}$, and so, by the continuity of $\tilde{\alpha} \mapsto U_{\tilde{\alpha}}$ we must have $\tilde{\alpha} \in J_1$ for all $\tilde{\alpha}$ in a neighborhood of α, as was to be shown.

It remains to show that J_1 and J_2 are non-empty sets. It is clear that $\underline{g} \in J_1$. As for J_2, note that the constant function $\bar{U} = \bar{g}$ solves the equation with initial condition $\bar{g}$. To see this, observe that $\Delta\bar{U} = 0$ identically and that $g(\mathsf{a}(\phi, \bar{U}), \mathsf{b}(\phi, \bar{U})) < \bar{g}$, so $H(\phi, \bar{U}) < 0$, hence $\bar{U}'(\phi) = 0 = \max\{0, H(\phi, \bar{U})\}$. Therefore, $\bar{g} \in J_2$. ■

Lemma F.5. *For every $V \in \mathcal{C}^{\mathsf{inc}}([0,1])$ and every ϕ and $\phi' \in (0,1)$ with $\phi' < \phi$,*

$$\Delta\phi'(\mathsf{a}(\phi', V)) < \Delta\phi(\mathsf{a}(\phi, V)) + \phi - \phi'.$$

Proof. Fix $V \in \mathcal{C}^{\mathsf{inc}}([0,1])$ and ϕ, $\phi' \in (0,1)$ with $\phi' < \phi$. Suppose, towards a contradiction, that $\phi' + \Delta\phi'(\mathsf{a}(\phi', V)) \geq \phi + \Delta\phi(\mathsf{a}(\phi, V))$. Since λ is strictly increasing, the function $(\hat{\phi}, \hat{a}) \mapsto \hat{\phi} + \Delta\hat{\phi}(\hat{a})$ is strictly increasing in $\hat{\phi}$ and strictly decreasing in $\hat{a}$, and therefore we must have $\mathsf{a}(\phi', V) < \mathsf{a}(\phi, V)$. It follows from Lemma F.1 that $\Delta V(\phi') < \Delta V(\phi)$. But since V is increasing we have $V(\phi') \leq V(\phi)$ and $V(\phi' + \Delta\phi'(\mathsf{a}(\phi', V))) \geq V(\phi + \Delta\phi(\mathsf{a}(\phi, V)))$, and therefore $\Delta V(\phi') \geq \Delta V(\phi)$, a contradiction. ■

Lemma F.6. *For every $\varepsilon > 0$ there exists $K > 0$ such that for every continuous, increasing function $V : [\varepsilon, 1] \to [\underline{g}, \bar{g}]$ and every ϕ, $\phi' \in [\varepsilon, 1-\varepsilon]$ with $\phi' < \phi$, if $\mathsf{a}(\phi, V) \neq \min A$ and $\mathsf{a}(\phi', V) \neq \min A$ then*

$$\begin{aligned} &rg(\mathsf{a}(\phi', V), \mathsf{b}(\phi', V)) + \lambda(\mathsf{a}(\phi', V))\Delta V(\phi') \\ &\qquad\qquad \geq rg(\mathsf{a}(\phi, V), \mathsf{b}(\phi, V)) + \lambda(\mathsf{a}(\phi, V))\Delta V(\phi) - K(\phi - \phi'). \end{aligned}$$

Proof. Let $\varepsilon > 0$ be fixed. The proof proceeds in four steps:

Step 1. There exists a constant $K_a > 0$ such that for all $V \in \mathcal{C}^{\mathsf{inc}}([0,1])$ and all ϕ, $\phi' \in [\varepsilon, 1-\varepsilon]$ with $\phi' < \phi$,

$$\mathsf{a}(\phi', V) \geq \mathsf{a}(\phi, V) - K_a(\phi - \phi').$$

By the definition of $\Delta\phi(a)$ and Lemma F.5, for all $\varepsilon \leq \phi < \phi' \leq 1 - \varepsilon$,

$$\begin{aligned} \lambda(\mathsf{a}(\phi, V)) - \lambda(\mathsf{a}(\phi', V)) &= \frac{\lambda^{\phi'}(\mathsf{a}(\phi', V))\Delta\phi'(\mathsf{a}(\phi', V))}{\phi'(1-\phi')} - \frac{\lambda^{\phi}(\mathsf{a}(\phi, V))\Delta\phi(\mathsf{a}(\phi, V))}{\phi(1-\phi)} \\ &= \frac{\overbrace{\phi(1-\phi)\lambda^{\phi'}(\mathsf{a}(\phi', V))}^{\leq \bar{\lambda}}\overbrace{(\Delta\phi'(\mathsf{a}(\phi', V)) - \Delta\phi(\mathsf{a}(\phi, V)))}^{\leq \phi - \phi' \text{ by Lemma F.5}}}{\underbrace{\phi'(1-\phi')\phi(1-\phi)}_{\geq \varepsilon^4}} + \\ &\quad + \frac{\overbrace{\lambda^{\phi}(\mathsf{a}(\phi, V))\Delta\phi(\mathsf{a}(\phi, V))}^{\leq \bar{\lambda}}\overbrace{(\phi(1-\phi) - \phi'(1-\phi'))}^{\leq \phi - \phi'}}{\underbrace{\phi'(1-\phi')\phi(1-\phi)}_{\geq \varepsilon^4}} \\ &\quad + \frac{\overbrace{\phi(1-\phi)\Delta\phi(\mathsf{a}(\phi, V))}^{\leq 1}\overbrace{(\lambda^{\phi'}(\mathsf{a}(\phi', V)) - \lambda^{\phi}(\mathsf{a}(\phi, V)))}^{\leq \lambda(\mathsf{a}(\phi', V)) - \lambda(\mathsf{a}(\phi, V))}}{\underbrace{\phi'(1-\phi')\phi(1-\phi)}_{\geq \varepsilon^4}} \\ &\leq 2\bar{\lambda}\varepsilon^{-4}(\phi - \phi') - \varepsilon^{-4}(\lambda(\mathsf{a}(\phi, V)) - \lambda(\mathsf{a}(\phi', V))), \end{aligned}$$

where $\bar{\lambda} \overset{\text{def}}{=} \max_{a \in A} \lambda(a)$. Hence,

$$\lambda(\mathsf{a}(\phi, V)) - \lambda(\mathsf{a}(\phi', V)) \leq 2\bar{\lambda}(1 + \varepsilon^4)^{-1}(\phi - \phi'),$$

and therefore, by the mean value theorem,

$$\mathsf{a}(\phi, V) - \mathsf{a}(\phi', V) \quad \le \quad \frac{2\bar{\lambda}\,(\phi - \phi')}{\min_a \lambda'(a)(1 + \varepsilon^4)},$$

which proves the desired inequality with $K_a \overset{\text{def}}{=} \frac{2\bar{\lambda}}{\min_a \lambda'(a)(1+\varepsilon^4)}$.

Step 2. There exists a constant $K_b > 0$ such that for all $V \in \mathcal{C}^{\mathsf{inc}}([0,1])$ and all ϕ, $\phi' \in [\varepsilon, 1-\varepsilon]$ with $\phi' < \phi$,

$$\mathsf{b}(\phi', V) \ge \mathsf{b}(\phi, V) - K_b(\phi - \phi').$$

First, note that the best-reply of the small players, $(a, \phi) \mapsto b^{\mathsf{BR}}(a, \phi)$, is increasing and Lipschitz continuous.[46] Therefore, there exist constants c_1 and $c_2 > 0$ such that for all $\phi, \phi' \in (0, 1)$ with $\phi' \le \phi$ and all $a, a' \in A$ with $a' \le a$,

$$0 \le b^{\mathsf{BR}}(a, \phi) - b^{\mathsf{BR}}(a', \phi') \le c_1(a - a') + c_2(\phi - \phi').$$

Hence, for all $V \in \mathcal{C}^{\mathsf{inc}}([0,1])$ and all $\phi, \phi' \in [\varepsilon, 1-\varepsilon]$ with $\phi' < \phi$, if $\mathsf{a}(\phi, V) - \mathsf{a}(\phi', V) \ge 0$ then

$$\begin{aligned} \mathsf{b}(\phi, V) - \mathsf{b}(\phi', V) \quad &\le \quad c_1(\mathsf{a}(\phi, V) - \mathsf{a}(\phi', V)) + c_2(\phi - \phi') \\ &\le \quad c_1 K_a(\phi - \phi') + c_2(\phi - \phi') \\ &= \quad (c_1 K_a + c_2)(\phi - \phi'), \end{aligned}$$

where K_a is the constant from Step 1. Moreover, if $\mathsf{a}(\phi, V) - \mathsf{a}(\phi', V) < 0$ then

$$\begin{aligned} \mathsf{b}(\phi, V) - \mathsf{b}(\phi', V) \quad &\le \quad b^{\mathsf{BR}}(\mathsf{a}(\phi', V), \phi) - b^{\mathsf{BR}}(\mathsf{a}(\phi', V), \phi') \\ &\le \quad c_2(\phi - \phi') \ \le \ (c_1 K_a + c_2)(\phi - \phi'), \end{aligned}$$

which thus proves the desired inequality with $K_b \overset{\text{def}}{=} c_1 K_a + c_2$.

Step 3. For each $(a, b) \in A \times B$ and each $\zeta \ge 0$, if $a = \arg\max_{a'}\ g(a', b) + \zeta\lambda(a') \in (\min A, \max A)$ then $\zeta = -g_1(a, b)/\lambda'(a)$.

This follows directly from the first-order condition when $a \in (\min A, \max A)$.

Step 4. There exists $K > 0$ such that for every ϕ and $\phi' \in [\varepsilon, 1-\varepsilon]$ with $\phi' < \phi$ and every $V \in \mathcal{C}^{\mathsf{inc}}([\varepsilon, 1])$, if $(a, b) = (\mathsf{a}(\phi, V), \mathsf{b}(\phi, V))$ with $a \ne \min A$, and $(a', b') = (\mathsf{a}(\phi', V), \mathsf{b}(\phi', V))$ with $a' \ne \min A$, then

$$rg(a, b) + \lambda(a)\Delta V(\phi) \ \ge \ rg(a', b') + \lambda(a')\Delta V(\phi') \ - \ K(\phi - \phi').$$

First note that $a \ne a^* = \max A$, otherwise we would have $\Delta V(\phi) = 0$, and therefore (a^*, b) would be a static Nash equilibrium of the complete information game, which is ruled out by condition (d). Likewise,

[46] The monotonicity of b^{BR} is Step 2 from the proof of Lemma F.1. The Lipschitz continuity is a straightforward implication of the first-order condition and the facts that $h_{12}(a, b, b)$ is bounded from above and $(h_{22} + h_{23})(a, b, b)$ is bounded away from zero.

$a' \neq a^*$. Since by assumption we also have $a \neq \min A$ and $a' \neq \min A$, we can apply Step 3 to conclude that

$$\Delta V(\phi)/r = \zeta(a,b) \quad \text{and} \quad \Delta V(\phi')/r = \zeta(a',b'),$$

where the function $\zeta : A \times B \to (0,\infty)$ is such that

$$\zeta(\hat{a},\hat{b}) = -g_1(\hat{a},\hat{b})/\lambda'(\hat{a}) \quad \forall(\hat{a},\hat{b}) \in A \times B,$$

so it satisfies $\zeta_1 > 0$ and $\zeta_2 \geq 0$, since $g_{11} < 0$, $g_{12} \leq 0$, $\lambda' > 0$ and $\lambda'' \leq 0$.

Since $(a',b') \in \mathcal{M}(\phi',\zeta(a',b'))$ we have

$$g(a,b') + \lambda(a)\zeta(a',b') \leq g(a',b') + \lambda(a')\zeta(a',b'),$$

and therefore,

$$\begin{aligned}
g(a,b) + \lambda(a)\zeta(a,b) \;-\; &\big(g(a',b') + \lambda(a')\zeta(a',b')\big) \\
&= \quad g(a,b) + \lambda(a)\zeta(a,b) \;-\; \big(g(a,b') + \lambda(a)\zeta(a',b')\big) \\
&\qquad\qquad + \quad g(a,b') + \lambda(a)\zeta(a',b') \;-\; \big(g(a',b') + \lambda(a')\zeta(a',b')\big) \\
&\leq \quad g(a,b) + \lambda(a)\zeta(a,b) \;-\; \big(g(a,b') + \lambda(a)\zeta(a',b')\big) \\
&= \quad g(a,b) + \lambda(a)\zeta(a,b) \;-\; \big(g(a,b') + \lambda(a)\zeta(a,b')\big) \\
&\qquad\qquad\qquad\qquad + \quad \lambda(a)(\zeta(a,b') - \zeta(a',b')).
\end{aligned}$$

Hence, for some $(\bar{a},\bar{b})$ we have

$$\begin{aligned}
g(a,b) + \lambda(a)\zeta(a,b) \;-\; &\big(g(a',b') + \lambda(a')\zeta(a',b')\big) \\
&\leq \quad (g_2(a,\bar{b}) + \lambda(a)\zeta_2(a,\bar{b}))\,(b-b') \;+\; \lambda(a)\zeta_1(\bar{a},b')\,(a-a') \\
&\qquad\qquad\qquad\qquad \leq \quad c_3\,(b-b') + c_4\,(a-a'),
\end{aligned}$$

where $c_3 \overset{\text{def}}{=} \max_{(\hat{a},\hat{b})}\; g_2(\hat{a},\hat{b}) + \lambda(\hat{a})\zeta_2(\hat{a},\hat{b}) \;>\; 0$ and $c_4 \overset{\text{def}}{=} \max_{(\hat{a},\hat{b})}\; \lambda(\hat{a})\zeta_1(\hat{a},\hat{b}) \;>\; 0$. It follows from Steps 1 and 2 that

$$g(a,b) + \lambda(a)\zeta(a,b) \;\leq\; g(a',b') + \lambda(a')\zeta(a',b') \;+\; (c_3K_b + c_4K_a)(\phi - \phi'),$$

which is the desired result with $K \overset{\text{def}}{=} r(c_3K_b + c_4K_a)$. ■

Lemma F.7. *For every $\varepsilon > 0$ there exists $R > 0$ such that for every continuous, increasing function $U : [\varepsilon, 1] \to [\underline{g}, \bar{g}]$ that solves the optimality equation (36) on $[\varepsilon, 1-\varepsilon]$ and is constant on $[1-\varepsilon, 1]$,*

$$|U'(\phi)| \leq R \quad \forall\phi \in [\varepsilon, 1-\varepsilon].$$

Proof. Fix $\varepsilon > 0$. We begin with the definition of the upper bound R. Since $(a^*, b) \notin \mathcal{M}(\phi, 0)$ for all $(b,\phi) \in B \times [0,1]$, there exists $0 < \eta < a^* - \min A$ such that for all $(a,b,\phi,\zeta) \in A \times B \times [0,1] \times [0, \bar{g} - \underline{g}]$ with $(a,b) \in \mathcal{M}(\phi,\zeta)$,

$$a \geq a^* - \eta \quad \Longrightarrow \quad \zeta \geq \eta. \tag{75}$$

Let $K > 0$ designate the constant from Lemma F.6 and define $\underline{\lambda} \stackrel{\text{def}}{=} \min_a \lambda(a)$. Next choose $0 < \delta < \frac{\underline{\lambda}\eta}{2K}$ small enough that for all $\phi \in [\varepsilon, 1-\varepsilon]$ and $a \in A$,

$$\Delta\phi(a) \le \delta \quad \Longrightarrow \quad a \ge a^* - \eta, \tag{76}$$

Let $C \stackrel{\text{def}}{=} (r + \bar{\lambda})(\bar{g} - \underline{g})/\underline{\lambda}$ and choose $R > 2C/\delta$ large enough that

$$\frac{\underline{\lambda}\eta}{2\bar{\lambda}}(\log\delta - \log(\frac{C}{R})) > \bar{g} - \underline{g}. \tag{77}$$

Now suppose the thesis of the lemma were false, i.e. $\max_{\phi\in[\varepsilon,1-\varepsilon]} U'(\phi) > R$ for some continuous, increasing function $U : [\varepsilon, 1] \to [\underline{g}, \bar{g}]$ which solves the optimality equation on $[\varepsilon, 1-\varepsilon]$ and is constant on $[1-\varepsilon, 1]$. Let $\phi_0 \in \arg\max_{\phi\in[\varepsilon,1-\varepsilon]} U'(\phi)$. By the optimality equation,

$$R < U'(\phi_0) \le \frac{(r+\bar{\lambda})(\bar{g}-\underline{g})}{\underline{\lambda}\,\Delta\phi_0(\mathsf{a}(\phi_0, U))} = \frac{C}{\Delta\phi_0(\mathsf{a}(\phi_0, U))},$$

which yields

$$\Delta\phi_0(\mathsf{a}(\phi_0, U)) < \frac{C}{R} < \frac{\delta}{2}, \tag{78}$$

hence $\mathsf{a}(\phi_0, U) > a^* - \eta$ by (76), which implies

$$\Delta U(\phi_0) \ge \eta, \tag{79}$$

by (75).

To conclude the proof it is enough to show that

$$U'(\phi) \ge \frac{\underline{\lambda}\eta}{2\bar{\lambda}(\frac{C}{R} + \phi_0 - \phi)} \qquad \forall \phi \in [\phi_0 - \delta, \phi_0], \tag{80}$$

for this implies

$$U(\phi_0) - U(\phi_0 - \delta) \ge \int_{\phi_0-\delta}^{\phi_0} \frac{\underline{\lambda}\eta\, d\phi}{2\bar{\lambda}(\frac{C}{R} + \phi_0 - \phi)} = \frac{\underline{\lambda}\eta}{2\bar{\lambda}}(\log(\frac{C}{R} + \delta) - \log(\frac{C}{R})) > \bar{g} - \underline{g},$$

by (77), and this is a contradiction since U takes values in $[\underline{g}, \bar{g}]$.

To prove inequality (80) first recall that U solves the optimality equation on $[\varepsilon, 1-\varepsilon]$, that is,

$$U'(\phi) \;=\; \frac{rg(\mathsf{a}(\phi, U), \mathsf{b}(\phi, U)) + \lambda(\mathsf{a}(\phi, U))\Delta U(\phi) - rU(\phi)}{\lambda^{\phi}(\mathsf{a}(\phi, U))\Delta\phi(\mathsf{a}(\phi, U))}, \qquad \forall \phi \in [\varepsilon, 1-\varepsilon]. \tag{81}$$

Hence, by Lemmas F.5 and F.6 and inequality (78), for all $\phi \in [\phi_0 - \delta, \phi_0]$,

$$\begin{aligned} U'(\phi) \;&\ge\; \frac{\overbrace{rg(\mathsf{a}(\phi_0, U), \mathsf{b}(\phi_0, U)) + \lambda(\mathsf{a}(\phi_0, U))\Delta U(\phi_0) - rU(\phi_0)}^{\ge\, \underline{\lambda}\,\Delta\phi_0(\mathsf{a}(\phi_0,U))\,U'(\phi_0),\ \text{by (81)}} \;-\; \overbrace{K\big(\phi_0 - \phi\big)}^{\le K\delta < \underline{\lambda}\eta/2}}{\bar{\lambda}\cdot(\underbrace{\Delta\phi_0(\mathsf{a}(\phi_0, U))}_{\le C/R} + \phi_0 - \phi\,)} \\ &\ge\; \frac{\underline{\lambda}\,U'(\phi_0)\,\Delta\phi_0(\mathsf{a}(\phi_0, U)) - \frac{\underline{\lambda}\eta}{2}}{\bar{\lambda}\,(\frac{C}{R} + \phi_0 - \phi)}. \end{aligned}$$

Thus, to prove inequality (80) and conclude the proof of the lemma, it suffices to show that

$$U'(\phi_0)\Delta\phi_0(\mathsf{a}(\phi_0, U)) \geq \eta.$$

Indeed, by the mean value theorem and the fact that U is constant on $[1-\varepsilon, 1]$, we must have $\Delta U(\phi_0) \leq U'(\bar{\phi})\,\Delta\phi_0(\mathsf{a}(\phi_0, U))$ for some $\bar{\phi} \in [\phi_0, 1-\varepsilon]$. Hence, $\Delta U(\phi_0) \leq U'(\phi_0)\,\Delta\phi_0(\mathsf{a}(\phi_0, U))$ since $U'(\phi_0) = \max\{U'(\phi) : \phi \in [\varepsilon, 1-\varepsilon]\}$, and therefore $U'(\phi_0)\Delta\phi_0(\mathsf{a}(\phi_0, U))) \geq \eta$ follows from (79). ■

We are now ready to prove Proposition F.1.

Proof of Proposition F.1. Our method of proof is to construct a solution $U : (0,1) \to [\underline{g}, \bar{g}]$ as a limit of a sequence of solutions on expanding closed subintervals of $(0,1)$. Using Lemma F.4, for each $n \geq 1$ there exists a continuous, increasing function $U_n : [\frac{1}{n}, 1] \to [\underline{g}, \bar{g}]$ that solves the optimality equation on $[\frac{1}{n}, 1-\frac{1}{n}]$. Since for $m \geq n$ the restriction of U_m to $[\frac{1}{n}, 1]$ solves the optimality equation on $[\frac{1}{n}, 1-\frac{1}{n}]$, by Lemma F.7 the derivative of U_m is uniformly bounded for $m \geq n$, and so the sequence $(U_m)_{m\geq n}$ is uniformly bounded and equicontinuous over the domain $[\frac{1}{n}, 1-\frac{1}{n}]$. By the Arzelà-Ascoli Theorem, for every n there exists a subsequence of $(U_m)_{m\geq n}$ that converges uniformly on $[\frac{1}{n}, 1-\frac{1}{n}]$. Hence, by a standard diagonalization argument, we can find a subsequence $(U_{n_k})_{k\geq 1}$ that converges pointwise to a continuous, increasing function $U : (0,1) \to [\underline{g}, \bar{g}]$ such that the convergence is uniform on every compact subset of $(0,1)$.

It remains to show that U solves the equation on $(0,1)$. If we show that $U'_{n_k}(\phi) \to H(\phi, U)$ uniformly on any closed subinterval $[\phi_0, \phi_1] \subset (0,1)$, it will then follow that U is differentiable and $U'(\phi) = H(\phi, U)$. First, note that from any $\phi \in [\phi_0, \phi_1]$ the posterior cannot jump above $\phi_2 = \phi_1 + \phi_1(1-\phi_1)\frac{\bar{\lambda}-\underline{\lambda}}{\underline{\lambda}}$. Since Proposition 9 and Lemma F.2 imply that $V \mapsto H(\phi, V)$ is Lipschitz continuous on the compact set $\{U|_{[\phi_0,\phi_2]}\} \cup \{U_{n_k}|_{[\phi_0,\phi_2]} : k \geq 1\}$ with a Lipschitz constant that is uniform in $\phi \in [\phi_0, \phi_1]$, and the sequence $(U_{n_k})_{k\geq 1}$ converges to U uniformly on $[\phi_0, \phi_2]$, it follows that $U'_{n_k}(\phi) = H(\phi, U_{n_k})$ converges to $H(\phi, U)$ uniformly on $[\phi_0, \phi_1]$, as required. ■

The following lemma concerns boundary conditions.

Lemma F.8. *Every bounded increasing solution of the optimality equation $U : (0,1) \to \mathbb{R}$ satisfies the following conditions at $p \in \{0,1\}$:*

$$\lim_{\phi\to p} U(\phi) = g(\mathcal{M}(p,0)) \quad \textit{and} \quad \lim_{\phi\to p} \phi(1-\phi)U'(\phi) = 0.$$

Proof. Let us show that $\lim_{\phi\to 0} U(\phi) = g(\mathcal{M}(0,0))$. First, by continuity, as $\phi \to 0$ we have $\Delta U(\phi) = U(\phi + \Delta\phi(\mathsf{a}(\phi, U))) - U(\phi) \to 0$, and hence $(\mathsf{a}(\phi, U), \mathsf{b}(\phi, U)) \to \mathcal{M}(0,0) = (a^N, b^N)$. Therefore, if $\lim_{\phi\to 0} U(\phi) < g(a^N, b^N)$ then the optimality equation implies

$$\lim_{\phi\to 0} \phi U'(\phi) = \frac{r(g(a^N, b^N) - \lim_{\phi\to 0} U(\phi))}{\lambda(a^*) - \lambda(a^N)} > 0.$$

Thus, for $0 < c < r(g(a^N, b^N) - \lim_{\phi\to 0} U(\phi))/(\lambda(a^*) - \lambda(a^N))$ we must have $U'(\phi) > c/\phi$ for all ϕ sufficiently close to 0. But then U cannot be bounded, since the anti-derivative of c/ϕ, which is $c\log\phi$, tends to $-\infty$ as $\phi \to 0$, and this is a contradiction. Thus, we must have $\lim_{\phi\to 0} U(\phi) \geq g(a^N, b^N)$. Moreover, if $\lim_{\phi\to 0} U(\phi) > g(a^N, b^N)$ then a similar argument can be used to show that

$\lim_{\phi\to 0}\ \phi U'(\phi) < 0$, which is impossible since U is increasing. We have thus shown that $\lim_{\phi\to 0}\ U(\phi) = g(\mathcal{M}(0,0))$. The proof for the $\phi \to 1$ case is analogous.

Finally, the condition $\lim_{\phi\to p}\ \phi(1-\phi)U'(\phi) = 0$ follows from the optimality equation and the boundary condition $\lim_{\phi\to p}\ U(\phi) = g(\mathcal{M}(p,0))$. ■

Proposition F.2. *The optimality equation* (36) *has a unique bounded increasing solution on* $(0,1)$.

Proof. By Proposition F.1 the optimality equation has at least one bounded increasing solution. Suppose U and V are two such solutions. Assuming that $U(\phi) > V(\phi)$ for some $\phi \in (0,1)$, let $\phi_0 \in (0,1)$ be the point where the difference $U - V$ is maximized, which is well-defined because $\lim_{\phi\to p} U(\phi) = \lim_{\phi\to p} V(\phi)$ for $p \in \{0,1\}$ by Lemma F.8. Thus, we have $U(\phi_0) - V(\phi_0) > 0$ and $U'(\phi_0) - V'(\phi_0) = 0$.

Let

$$\begin{aligned}\Delta U(\phi_0) &\stackrel{\text{def}}{=} U(\phi_0 + \Delta\phi_0(\mathsf{a}(\phi_0, U))) - U(\phi_0),\\ \Delta V(\phi_0) &\stackrel{\text{def}}{=} V(\phi_0 + \Delta\phi_0(\mathsf{a}(\phi_0, V))) - V(\phi_0).\end{aligned}$$

We claim that $\Delta U(\phi_0) > \Delta V(\phi_0)$. Otherwise, if $\Delta U(\phi_0) \le \Delta V(\phi_0)$, then $\mathsf{a}(\phi_0, U) \le \mathsf{a}(\phi_0, V)$ by Lemma F.1, and hence $\mathsf{b}(\phi_0, U) \le \mathsf{b}(\phi_0, V)$ by Step 1 of the proof of Lemma F.1. Therefore,

$$\begin{aligned} rg(\mathsf{a}(\phi_0,U),\mathsf{b}(\phi_0,U)) + \lambda(\mathsf{a}(\phi_0,U))\Delta U(\phi_0) &\le rg(\mathsf{a}(\phi_0,U),\mathsf{b}(\phi_0,V)) + \lambda(\mathsf{a}(\phi_0,U))\Delta V(\phi_0)\\ &\le rg(\mathsf{a}(\phi_0,V),\mathsf{b}(\phi_0,V)) + \lambda(\mathsf{a}(\phi_0,V))\Delta V(\phi_0),\end{aligned}$$

where the first inequality uses $g_2 \ge 0$ and the second inequality follows from the fact that $(\mathsf{a}(\phi_0,V),\mathsf{b}(\phi_0,V)) \in \mathcal{M}(\phi_0, \Delta V(\phi_0)/r)$. Then, the optimality equation implies

$$\begin{aligned} U'(\phi_0) &= \frac{rg(\mathsf{a}(\phi_0,U),\mathsf{b}(\phi_0,U)) + \lambda(\mathsf{a}(\phi_0,U))\Delta U(\phi_0) - rU(\phi_0)}{\phi_0(1-\phi_0)(\lambda(a^*) - \lambda(\mathsf{a}(\phi_0,U)))}\\ &< \frac{rg(\mathsf{a}(\phi_0,V),\mathsf{b}(\phi_0,V)) + \lambda(\mathsf{a}(\phi_0,V))\Delta V(\phi_0) - rV(\phi_0)}{\phi_0(1-\phi_0)(\lambda(a^*) - \lambda(\mathsf{a}(\phi_0,V)))} = V'(\phi_0),\end{aligned}$$

and this is a contradiction. Thus, $\Delta U(\phi_0) > \Delta V(\phi_0)$, as claimed.

It follows from $\Delta U(\phi_0) > \Delta V(\phi_0)$ that $\mathsf{a}(\phi_0,U) \ge \mathsf{a}(\phi_0,V)$, by Lemma F.1. Hence, $\Delta\phi_0(\mathsf{a}(\phi_0,U)) \le \Delta\phi_0(\mathsf{a}(\phi_0,V))$, and since V is increasing,

$$\begin{aligned} U(\phi_0 + \Delta\phi_0(\mathsf{a}(\phi_0,U))) - V(\phi_0 + \Delta\phi_0(\mathsf{a}(\phi_0,U))) &\ge \\ U(\phi_0 + \Delta\phi_0(\mathsf{a}(\phi_0,U))) - V(\phi_0 + \Delta\phi_0(\mathsf{a}(\phi_0,V))) &> U(\phi_0) - V(\phi_0),\end{aligned}$$

where the last inequality follows from $\Delta U(\phi_0) > \Delta V(\phi_0)$. But this is a contradiction, for we picked ϕ_0 to be the point where the difference $U - V$ is maximized. Thus, the optimality equation has a unique bounded increasing solution. ■

We are now ready to prove Theorem 11.

Proof of Theorem 11. By Proposition F.2, the optimality equation has a unique bounded solution $U : (0,1) \to \mathbb{R}$. By Lemma F.8 such solution satisfies the boundary conditions. The proof proceeds in two steps. In *Step 1* we show that, given any prior $\phi_0 \in (0,1)$ on the behavioral type, there is no sequential equilibrium that yields a payoff different from $U(\phi_0)$ to the normal type. In *Step 2* we show that the Markovian strategy profile $(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}) \stackrel{\text{def}}{=} (\mathsf{a}(\phi_{t-}, U), \mathsf{b}(\phi_{t-}, U))$ is the unique sequential equilibrium yielding the payoff $U(\phi_0)$ to the normal type.

Step 1. Fix a prior $\phi_0 \in (0,1)$ and a sequential equilibrium $(a_t, \bar{b}_t, \phi_t)_{t\geq 0}$. By Theorem 9 the belief process $(\phi_t)_{t\geq 0}$ solves

$$d\phi_t = -\lambda^{\phi_{t-}}(a_t)\Delta\phi_{t-}(a_t)\,dt \;+\; \Delta\phi_{t-}(a_t)\,dN_t,$$

and the process $(W_t)_{t\geq 0}$ of continuation values of the normal type solves

$$dW_t = r(W_{t-} - g(a_t, \bar{b}_t) - \zeta_t \lambda(a_t))\,dt \;+\; r\zeta_t\,dN_t, \tag{82}$$

for some predictable process $(\zeta_t)_{t\geq 0}$ such that

$$(a_t, \bar{b}_t) \in \mathcal{M}(\phi_{t-}, \zeta_t) \quad \text{almost everywhere.} \tag{83}$$

Thus, the process $U_t \stackrel{\text{def}}{=} U(\phi_t)$ satisfies

$$dU_t = -\lambda^{\phi_{t-}}(a_t)\Delta\phi_{t-}(a_t)U'(\phi_{t-})\,dt \;+\; \Delta U_t\,dN_t, \tag{84}$$

where

$$\Delta U_t \stackrel{\text{def}}{=} U(\phi_{t-} + \Delta\phi_{t-}(a_t)) - U(\phi_{t-}).$$

We will now demonstrate that $D_0 \stackrel{\text{def}}{=} W_0 - U(\phi_0) = 0$ by showing that if $D_0 \neq 0$ then the process $D_t \stackrel{\text{def}}{=} W_t - U(\phi_t)$ must eventually grow arbitrarily large with positive probability, which is a contradiction since U and W are both bounded. Without loss of generality, we assume $D_0 > 0$. It follows from (82), (84) and the optimality equation (36) that the process $(D_t)_{t\geq 0}$ jumps by

$$r\zeta_t - \Delta U_t$$

when a Poisson event arrives, and has drift given by

$$rD_{t-} + f(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}, \Delta U_t^{\mathsf{m}}, \phi_{t-}) - f(a_t, \bar{b}_t, \zeta_t, \phi_{t-}),$$

where

$$\Delta U_t^{\mathsf{m}} \stackrel{\text{def}}{=} U(\phi_{t-} + \Delta\phi_{t-}(a_t^{\mathsf{m}})) - U(\phi_{t-}),$$

and for each $(a, b, \zeta, \phi) \in A \times B \times [0, \bar{g} - \underline{g}] \times [0,1]$,

$$f(a, b, \zeta, \phi) \stackrel{\text{def}}{=} rg(a,b) + r\zeta\lambda(a) + \phi(1-\phi)U'(\phi)\lambda(a).$$

Claim. *There exists $\varepsilon > 0$ such that, so long as $D_t \geq D_0/2$, either the drift of D_t is greater than or equal to $rD_0/4$ or D_t jumps up by more than ε upon the arrival of a Poisson event.*

To prove this claim, note that by Lemma F.1, if $r\zeta_t - \Delta U_t^{\mathsf{m}} \leq 0$ then $a_t^{\mathsf{m}} \geq a_t$, and hence $b_t^{\mathsf{m}} \geq b_t$ since the small players' best reply in $\mathcal{M}$ is increasing in the large player's action, as shown in Step 1 of the proof of Lemma F.1. Therefore, $r\zeta_t - \Delta U_t^{\mathsf{m}} \le 0$ implies

$$\begin{aligned} f(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}, \Delta U_t^{\mathsf{m}}, \phi_{t-}) - f(a_t, \bar{b}_t, \zeta_t, \phi_{t-}) &= \overbrace{rg(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}) + \Delta U_t^{\mathsf{m}} \lambda(a_t^{\mathsf{m}})}^{\ge rg(a_t, b_t^{\mathsf{m}}) + \Delta U_t^{\mathsf{m}} \lambda(a_t)} - rg(a_t, \bar{b}_t) - r\zeta_t \lambda(a_t) \\ &\quad + \phi_{t-}(1-\phi_{t-}) U'(\phi_{t-}) (\underbrace{\lambda(a_t^{\mathsf{m}}) - \lambda(a_t)}_{\ge 0 \text{ by } \lambda' > 0}) \\ &\ge \underbrace{rg(a_t, b_t^{\mathsf{m}}) - rg(a_t, \bar{b}_t)}_{\ge 0 \text{ by } g_2 \ge 0} + (\underbrace{\Delta U_t^{\mathsf{m}} - r\zeta_t}_{\ge 0}) \lambda(a_t). \end{aligned}$$

Thus,

$$r\zeta_t - \Delta U_t^{\mathsf{m}} \le 0 \implies f(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}, \Delta U_t^{\mathsf{m}}, \phi_{t-}) \; - \; f(a_t, \bar{b}_t, \zeta_t, \phi_{t-}) \; \ge \; 0.$$

By a continuity/compactness argument similar to the one used in the proof of Lemma C.8, there exists $\varepsilon > 0$ such that for all t and after all public histories,

$$r\zeta_t - \Delta U_t^{\mathsf{m}} \le \varepsilon \implies f(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}, \Delta U_t^{\mathsf{m}}, \phi_{t-}) \; - \; f(a_t, \bar{b}_t, \zeta_t, \phi_{t-}) \; \ge \; -rD_0/4,$$

hence, so long as $D_t \ge D_0/2$,

$$r\zeta_t - \Delta U_t^{\mathsf{m}} \le \varepsilon \implies \text{drift of } D_t \text{ is greater than or equal to } rD_0/4.$$

Thus, so long as $D_t \ge D_0/2$, if the drift of D_t is less than $rD_0/4$, then $r\zeta_t - \Delta U_t^{\mathsf{m}} > \varepsilon$, which implies that $a_t \ge a_t^{\mathsf{m}}$ by Lemma F.1, and hence that $\Delta U_t \le \Delta U_t^{\mathsf{m}}$ since U is increasing; therefore, $r\zeta_t - \Delta U_t$, the jump in D_t upon the arrival of a Poisson event, must be greater than or equal to $r\zeta_t - \Delta U_t^{\mathsf{m}} > \varepsilon$, as claimed.

Since we have assumed that $D_0 > 0$, the claim above readily implies that with positive probability the process $D_t = W_t - U(\phi_t)$ must eventually grow arbitrarily large, which is impossible since $(W_t)_{t\ge0}$ and $(U(\phi_t))_{t\ge0}$ are bounded processes. We have thus shown that no sequential equilibrium can yield a payoff greater than $U(\phi_0)$. A similar argument proves that payoffs below $U(\phi_0)$ also cannot be achieved in sequential equilibria.

Step 2. First, let us show that the Markovian strategy profile $(\mathsf{a}(\phi_t, U), \mathsf{b}(\phi_t, U))_{t\ge0}$ is a sequential equilibrium profile yielding the normal type a payoff of $U(\phi_0)$. Let $(\phi_t)_{t\ge0}$ be the solution of

$$d\phi_t = -\lambda^{\phi_{t-}}(\mathsf{a}(\phi_{t-}, U)) \Delta\phi_{t-}(\mathsf{a}(\phi_{t-}, U))\, dt \; + \; \Delta\phi_{t-}(\mathsf{a}(\phi_{t-}, U))\, dN_t,$$

with initial condition ϕ_0. Thus, using the optimality equation,

$$dU_t^{\mathsf{m}} = (rU_{t-}^{\mathsf{m}} - rg(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}) - \lambda(a_t^{\mathsf{m}}) \Delta U_t^{\mathsf{m}})\, dt + \Delta U_t^{\mathsf{m}}\, dN_t,$$

where $(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}) \stackrel{\text{def}}{=} (\mathsf{a}(\phi_{t-}, U), \mathsf{b}(\phi_{t-}, U))$, $U_t^{\mathsf{m}} \stackrel{\text{def}}{=} U(\phi_t)$ and $\Delta U_t^{\mathsf{m}} \stackrel{\text{def}}{=} U(\phi_{t-} + \Delta\phi_{t-}(a_t^{\mathsf{m}})) - U(\phi_{t-})$. Since $(U_t^{\mathsf{m}})_{t\ge0}$ is bounded and $(a_t^{\mathsf{m}}, b_t^{\mathsf{m}}) \in \mathcal{M}(\phi_{t-}, \Delta U_t^{\mathsf{m}})$, Theorem 9 implies that $(a_t^{\mathsf{m}}, b_t^{\mathsf{m}})_{t\ge0}$ is a sequential equilibrium profile in which the normal type receives a payoff of $U(\phi_0)$.

It remains to show that $(\mathsf{a}(\phi_t, U), \mathsf{b}(\phi_t, U))_{t\ge0}$ is the unique sequential equilibrium. Indeed, if $(a_t, \bar{b}_t)_{t\ge0}$ is an arbitrary sequential equilibrium, then the associated belief/continuation value pair (ϕ_t, W_t) must stay in the graph of U, by Step 1. Then, by (82) and (84), we must have $(a_t, \bar{b}_t) \in \mathcal{M}(\phi_{t-}, \zeta_t)$ a.e., where $r\zeta_t = U(\phi_{t-} + \Delta\phi_{t-}(a_t)) - U(\phi_{t-})$. Therefore, by Proposition 9, $(a_t, \bar{b}_t) = (\mathsf{a}(\phi_{t-}, U), \mathsf{b}(\phi_{t-}, U))$ a.e., as was to be shown. ■

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