

Competitive Experimentation and Private Information

Giuseppe Moscarini and Francesco Squintani

Omitted Analysis not Submitted for Publication

Derivations for the Gamma-Exponential Model

Derivation of expected hazard rates. By Bayes rule, posterior beliefs are

$$\begin{aligned}
 \pi_{t,t'}(\lambda|x,y) &= \pi(\lambda|z_i = x, z_j = y, t_i \geq t, t_j \geq t') \\
 &= \frac{\pi(\lambda) h(x|\lambda) h(y|\lambda) [1 - F_i(t|\lambda)] [1 - F_j(t'|\lambda)]}{\int_{\Lambda} \pi(\lambda') h(x|\lambda') h(y|\lambda') [1 - F_i(t|\lambda')] [1 - F_j(t'|\lambda')] d\lambda'} \\
 &= \frac{e^{-\alpha\lambda} \lambda^{\beta-1} \lambda e^{\lambda x} \lambda e^{\lambda y} e^{-(\zeta_i + \lambda)t} e^{-(\zeta_j + \lambda)t'}}{\int_{\Lambda} e^{-\alpha\lambda'} \lambda'^{\beta-1} \lambda' e^{\lambda' x} \lambda' e^{\lambda' y} e^{-(\zeta_i + \lambda')t} e^{-(\zeta_j + \lambda')t'} d\lambda'} = \frac{e^{-\lambda(\alpha - x - y + t + t')} \lambda^{\beta+1}}{\Gamma(\beta + 2) (\alpha - x - y + t + t')^{-\beta-2}},
 \end{aligned}$$

$$\begin{aligned}
 \pi_{t,t'}(\lambda|x,y+) &= \pi(\lambda|z_i = x, z_j \geq y, t_i \geq t, t_j \geq t') \\
 &= \frac{\pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F_i(t|\lambda)] [1 - F_j(t'|\lambda)]}{\int_{\Lambda} \pi(\lambda') h(x|\lambda') [1 - H(y|\lambda')] [1 - F_i(t|\lambda')] [1 - F_j(t'|\lambda')] d\lambda'} \\
 &= \frac{\lambda^{\beta} [e^{-\lambda(\alpha + t + t' - x)} - e^{-\lambda(\alpha + t + t' - x - y)}]}{\Gamma(\beta + 1) \left[(\alpha + t + t' - x)^{-\beta-1} - (\alpha + t + t' - x - y)^{-\beta-1} \right]}.
 \end{aligned}$$

Taking expectations of the hazard rate λ with respect to these three posteriors yields the following expressions:

$$\begin{aligned}
 \mathbb{E}_{t,t'}^i[\lambda|x,y] &= \frac{\beta + 2}{\alpha - x - y + t + t'}, \\
 \mathbb{E}_{t,t'}^i[\lambda|x,y-] &= \frac{\beta + 1}{\alpha - x - y + t + t'}, \\
 \mathbb{E}_{t,t'}^i[\lambda|x,y+] &= (\beta + 1) \frac{(\alpha + t + t' - x)^{-\beta-2} - (\alpha + t + t' - x - y)^{-\beta-2}}{(\alpha + t + t' - x)^{-\beta-1} - (\alpha + t + t' - x - y)^{-\beta-1}}.
 \end{aligned}$$

Lemma A.3 *In the Gamma-exponential model, $E_{t,t}^i[\zeta_i + \lambda|x,y+]$ satisfies Assumptions 1 and 2.*

Proof. We use the following technical result.

Theorem A.1 Let $q : \Lambda \times \mathfrak{R}^n \rightarrow \mathfrak{R}_+$, integrable in its first argument and differentiable in the remaining n arguments, with $\int_{\Lambda} q(\lambda, \vec{\theta}) d\lambda \in (0, \infty)$. Then the c.d.f. defined by:

$$\varphi(L, \vec{\theta}) \equiv \frac{\int_{\underline{\lambda}}^L q(\lambda, \vec{\theta}) d\lambda}{\int_{\Lambda} q(\lambda', \vec{\theta}) d\lambda'}$$

for every $L \in (\underline{\lambda}, \bar{\lambda})$ is stochastically strictly increasing in every component of $\vec{\theta}$ if $q(\lambda, \vec{\theta})$ is log-supermodular in (λ, θ_i) , i.e. if $\partial \log q(\lambda, \vec{\theta}) / \partial \theta_i$ is strictly increasing in λ .

Proof We want to show that for every $L \in (\underline{\lambda}, \bar{\lambda})$

$$0 > \frac{\partial}{\partial \theta_i} \frac{\int_{\underline{\lambda}}^L q(\lambda, \vec{\theta}) d\lambda}{\int_{\Lambda} q(\lambda, \vec{\theta}) d\lambda} = \frac{\int_{\underline{\lambda}}^L \frac{\partial q(\lambda, \vec{\theta})}{\partial \theta_i} d\lambda}{\int_{\Lambda} q(\lambda, \vec{\theta}) d\lambda} - \frac{\int_{\underline{\lambda}}^L q(\lambda, \vec{\theta}) d\lambda \int_{\Lambda} \frac{\partial q(\lambda, \vec{\theta})}{\partial \theta_i} d\lambda}{\left[\int_{\Lambda} q(\lambda, \vec{\theta}) d\lambda \right]^2}$$

using

$$\int \frac{\partial q(\lambda, \vec{\theta})}{\partial \theta_i} d\lambda = \int \frac{\partial \log q(\lambda, \vec{\theta})}{\partial \theta_i} q(\lambda, \vec{\theta}) d\lambda$$

the claim reads

$$\int_{\Lambda} \frac{\partial \log q(\lambda, \vec{\theta})}{\partial \theta_i} \frac{q(\lambda, \vec{\theta})}{\int_{\Lambda} q(\lambda', \vec{\theta}) d\lambda'} d\lambda > \int_{\underline{\lambda}}^L \frac{\partial \log q(\lambda, \vec{\theta})}{\partial \theta_i} \frac{q(\lambda, \vec{\theta})}{\int_{\underline{\lambda}}^L q(\lambda', \vec{\theta}) d\lambda'} d\lambda$$

A sufficient condition for the latter inequality is that the RHS be strictly increasing in L . Since the RHS is differentiable in L , it suffices that

$$\begin{aligned} 0 &< \frac{\partial}{\partial L} \left[\int_{\underline{\lambda}}^L \frac{\partial \log q(\lambda, \vec{\theta})}{\partial \theta_i} \frac{q(\lambda, \vec{\theta})}{\int_{\underline{\lambda}}^L q(\lambda', \vec{\theta}) d\lambda'} d\lambda \right] \\ &= \frac{\partial \log q(L, \vec{\theta})}{\partial \theta_i} \frac{q(L, \vec{\theta})}{\int_{\underline{\lambda}}^L q(\lambda', \vec{\theta}) d\lambda'} - q(L, \vec{\theta}) \int_{\underline{\lambda}}^L \frac{\partial \log q(\lambda, \vec{\theta})}{\partial \theta_i} \frac{q(\lambda, \vec{\theta})}{\left[\int_{\underline{\lambda}}^L q(\lambda', \vec{\theta}) d\lambda' \right]^2} d\lambda \\ &\text{or } \frac{\partial \log q(L, \vec{\theta})}{\partial \theta_i} > \int_{\underline{\lambda}}^L \frac{\partial \log q(\lambda, \vec{\theta})}{\partial \theta_i} \frac{q(\lambda, \vec{\theta})}{\int_{\underline{\lambda}}^L q(\lambda', \vec{\theta}) d\lambda'} d\lambda. \end{aligned}$$

But, since $\int_{\underline{\lambda}}^L \frac{q(\lambda, \vec{\theta})}{\int_{\underline{\lambda}}^L q(\lambda', \vec{\theta}) d\lambda'} d\lambda = 1$, this follows from the assumption that the LHS is strictly increasing in λ . ■

It now suffices to show that for every x, y, t, t' , the c.d.f. associated with the posterior beliefs $\pi_{t,t'}(\lambda|x, y+)$ are stochastically strictly decreasing in t and in t' and strictly increasing in x and in y . We prove all these results as corollaries of Theorem A1. Let $\vec{\theta} = (x, y, t, t')$,

$$q(\lambda, \vec{\theta}) = \pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)],$$

$$\varphi(L, \vec{\theta}) = \int_{\underline{\lambda}}^L \pi_{t,t'}(\lambda|x, y+) d\lambda.$$

Since the expressions

$$\begin{aligned} \frac{\partial \log q(\lambda, \vec{\theta})}{\partial t} &= -\frac{f(t|\lambda)}{1-F(t|\lambda)} = -\lambda, & \frac{\partial \log q(\lambda, \vec{\theta})}{\partial t'} &= -\frac{f(t'|\lambda)}{1-F(t'|\lambda)} = -\lambda \\ \frac{\partial \log q(\lambda, \vec{\theta})}{\partial x} &= \frac{h'(x|\lambda)}{h(x|\lambda)} = \lambda, & \frac{\partial \log q(\lambda, \vec{\theta})}{\partial y} &= -\frac{h(y|\lambda)}{1-H(y|\lambda)} = \frac{\lambda e^{\lambda x}}{1-e^{\lambda x}} \end{aligned}$$

are strictly monotonic in λ , all monotonicity results follow from Theorem A1. The limit $\lim_{t \rightarrow \infty} \mathbb{E}_{t,t'}[\lambda|x, y+] = 0$ follows from:

$$\lim_{\tau \rightarrow \infty} \pi_{\tau,\tau}(\lambda > \varepsilon|x, y+) = \lim_{\tau \rightarrow \infty} \frac{\int_{\varepsilon}^{\infty} e^{-\alpha\lambda} \lambda^{\beta-1} \lambda e^{\lambda x} (1-e^{\lambda y}) e^{-\lambda 2\tau} d\lambda}{\int_0^{\infty} e^{-\alpha\lambda'} \lambda'^{\beta-1} \lambda' e^{\lambda' x} (1-e^{\lambda' y}) e^{-\lambda' 2\tau} d\lambda'} = 0. \blacksquare$$

Lemma A.4 *In the Gamma-exponential model, $E_{t,t}^i[\zeta_i + \lambda|x, y+]$ satisfies Assumption 3.*

Proof. It is immediate to verify that $\frac{d}{dt} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+] = -2 \frac{d}{dx} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+]$, as it can be appreciated with the change of variable $q = \alpha + 2t - x$, and

$$\begin{aligned} \phi(q, y, \beta) &= (\beta + 1) \frac{q^{-\beta-2} - (q-y)^{-\beta-2}}{q^{-\beta-1} - (q-y)^{-\beta-1}} \\ \frac{d}{dt} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+] &= \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial t} = 2 \frac{\partial \phi}{\partial q}, \\ \frac{d}{dx} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+] &= \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial x} = -\frac{\partial \phi}{\partial q} \end{aligned}$$

Hence, setting \bar{G} just a bit larger than 2 and thus $\bar{G}' < -1/2$ satisfies the inequality in Assumption 3. For the first inequality, we show that it is strict at $\bar{G}' = -1/2$ and therefore, by continuity, for \bar{G}' slightly larger:

$$2 \frac{d}{dy} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+] < -\frac{d}{dt} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+] = 2 \frac{d}{dx} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+].$$

Algebra yields

$$\frac{d}{dx} \mathbb{E}_{t,t'}^i[\zeta_i + \lambda|x, y+] = (\beta + 1) \frac{\left[\begin{aligned} &(\beta + 2) \left[(\alpha + 2t - x)^{-\beta-3} - (\alpha + 2t - x - y)^{-\beta-3} \right] \cdot \\ &\quad \cdot \left[(\alpha + 2t - x)^{-\beta-1} - (\alpha + 2t - x - y)^{-\beta-1} \right] \\ & - (\beta + 1) \left[-(\alpha + 2t - x)^{-\beta-2} + (\alpha + 2t - x - y)^{-\beta-2} \right]^2 \end{aligned} \right]}{\left[(\alpha + 2t - x)^{-\beta-1} - (\alpha + 2t - x - y)^{-\beta-1} \right]^2}$$

$$\begin{aligned} & \frac{d}{dy} \mathbb{E}_{t,t'}^i [\zeta_i + \lambda | x, y+] \\ &= (\beta + 1) \frac{\left[\begin{array}{l} -(\beta + 2)(\alpha + 2t - x - y)^{-\beta-3} \left[(\alpha + 2t - x)^{-\beta-1} - (\alpha + 2t - x - y)^{-\beta-1} \right] \\ + (\beta + 1)(\alpha + 2t - x - y)^{-\beta-2} \left[(\alpha + 2t - x)^{-\beta-2} - (\alpha + 2t - x - y)^{-\beta-2} \right] \end{array} \right]}{\left[(\alpha + 2t - x)^{-\beta-1} - (\alpha + 2t - x - y)^{-\beta-1} \right]^2}. \end{aligned}$$

Hence,

$$\begin{aligned} & -\frac{d}{dx} \mathbb{E}_{t,t'}^i [\zeta_i + \lambda | x, y+] + \frac{d}{dy} \mathbb{E}_{t,t'}^i [\zeta_i + \lambda | x, y+] \\ &= -(\beta + 1) \frac{\left[\begin{array}{l} (\beta + 2) \left[(\alpha + 2t - x)^{-\beta-3} \right] \left[(\alpha + 2t - x)^{-\beta-1} - (\alpha + 2t - x - y)^{-\beta-1} \right] \\ - (\beta + 1) \left[(\alpha + 2t - x)^{-\beta-2} \right] \left[(\alpha + 2t - x)^{-\beta-2} - (\alpha + 2t - x - y)^{-\beta-2} \right] \end{array} \right]}{\left[(\alpha + 2t - x)^{-\beta-1} - (\alpha + 2t - x - y)^{-\beta-1} \right]^2} \\ &\propto (\beta + 1) \frac{(\alpha + t + t' - x)^{-\beta-2}}{(\alpha + t + t' - x)^{-\beta-1} - (\alpha + t + t' - x - y)^{-\beta-1}} \\ &\quad - (\beta + 2) \frac{(\alpha + t + t' - x)^{-\beta-3}}{(\alpha + t + t' - x)^{-\beta-2} - (\alpha + t + t' - x - y)^{-\beta-2}} \\ &= (\beta + 1) \frac{q^{-\beta-2}}{q^{-\beta-1} - (q - y)^{-\beta-1}} - (\beta + 2) \frac{q^{-\beta-3}}{q^{-\beta-2} - (q - y)^{-\beta-2}} \end{aligned}$$

We want this quantity to be strictly negative. This is so if the following derivative is strictly positive for all β, y :

$$\begin{aligned} & \frac{d}{d\beta} \left(\beta \frac{q^{-\beta-1}}{q^{-\beta} - (q - y)^{-\beta}} \right) \\ &= \frac{q^{-\beta-1}}{q^{-\beta} - (q - y)^{-\beta}} + \beta \frac{-q^{-\beta-1} \left[q^{-\beta} - (q - y)^{-\beta} \right] \ln q + \left[q^{-\beta} \ln q - (q - y)^{-\beta} \ln (q - y) \right] q^{-\beta-1}}{\left[q^{-\beta} - (q - y)^{-\beta} \right]^2} \\ &\propto q^{-\beta} - (q - y)^{-\beta} + \beta \left[- \left[q^{-\beta} - (q - y)^{-\beta} \right] \ln q + \left[q^{-\beta} \ln q - (q - y)^{-\beta} \ln (q - y) \right] \right] \\ &= q^{-\beta} - (q - y)^{-\beta} + (q - y)^{-\beta} \beta (\ln q - \ln (q - y)) \\ &: = \chi(\beta, y) \end{aligned}$$

Note that for $y < 0$

$$\frac{d\chi(\beta, y)}{dy} = \beta^2 \frac{\ln q - \ln (q - y)}{(q - y)^{\beta+1}} - \frac{\beta}{(y - q)(q - y)^\beta} - \frac{\beta}{(q - y)^{\beta+1}} = -\frac{(\ln (q - y) - \ln q) \beta^2}{(q - y)^{\beta+1}} < 0,$$

while $\chi(\beta, 0) = q^{-\beta} - q^{-\beta} + q^{-\beta}\beta(\ln q - \ln(q)) = 0$, $\chi(\beta, -\infty) = q^{-\beta} - \frac{1+\beta(\ln(q-y)-\ln q)}{(q-y)^\beta} = q^{-\beta} + \frac{\beta}{(q-y)^\beta} = q^{-\beta} > 0$. So indeed $\chi(\beta, 0) > 0$ for all β, y . ■

Lemma A.5 *In the Gamma-exponential model, there exists a unique monotonic differentiable equilibrium.*

Proof. Proceeding by contradiction, suppose that for some x, y there exist two optimal stopping times as first quitter. We can rephrase this as follows: for some τ_1 , there exist signals $(x, y) \neq (x', y')$ such that $q_A = \mathbb{E}_{\tau_1, \tau_1}[\lambda|x, y+]$, $q_B = \mathbb{E}_{\tau_1, \tau_1}[\lambda|y, x+]$, $q_A = \mathbb{E}_{\tau_1, \tau_1}[\lambda|x', y'+]$, $q_B = \mathbb{E}_{\tau_1, \tau_1}[\lambda|y', x'+]$.

Note that we can take $(x, y) < 0$, and $(x', y') < 0$ by continuity of $dV_{1,t}^i(\tau_1|x)/d\tau_1|_{t=\tau_1}$ and $dV_{1,t}^i(\tau_1|y)/d\tau_1|_{t=\tau_1}$. Simple manipulations of Eq. (6) then yield:

$$\begin{aligned} & q_A^{-1}(\beta+1)(\alpha+2\tau_1-x-y)^{-(\beta+2)} - (\alpha+2\tau_1-x-y)^{-(\beta+1)} = \\ & q_A^{-1}(\beta+1)(\alpha+2\tau_1-x)^{-(\beta+2)} - (\alpha+2\tau_1-x)^{-(\beta+1)} \end{aligned}$$

$$\begin{aligned} & q_B^{-1}(\beta+1)(\alpha+2\tau_1-x-y)^{-(\beta+2)} - (\alpha+2\tau_1-x-y)^{-(\beta+1)} = \\ & q_B^{-1}(\beta+1)(\alpha+2\tau_1-y)^{-(\beta+2)} - (\alpha+2\tau_1-y)^{-(\beta+1)}. \end{aligned}$$

Consider the functions

$$\varphi_i(\xi) = q_i^{-1}(\beta+1)(\alpha+2\tau-\xi)^{-(\beta+2)} - (\alpha+2\tau-\xi)^{-(\beta+1)}$$

for $i = A, B$. Note that

$$\varphi_i'(\xi) \propto q_i^{-1}(\beta+2) + \xi - \alpha - 2\tau,$$

is continuous in ξ , has at most one zero, $\xi_0 = \alpha + 2\tau - q_i(\beta+2)$, and has the same sign as $\xi - \xi_0$. So, suppose that $x' < x$. The conditions $\varphi_A(x) = \varphi_A(x+y)$ and $\varphi_A(x') = \varphi_A(x'+y')$ require $x'+y' > x+y$, and hence $y' > y$. But then, the condition $\varphi_B(x) = \varphi_B(x+y)$ and $\varphi_B(x') = \varphi_B(x'+y')$ require that $x'+y' < x+y$, and we have a contradiction. A similar contradiction is obtained when supposing that $x' > x$. ■

The Binary Prior Model

Suppose that the prize arrives according to an exponential process of intensity λ , i.e. with $F_i(t_i|\lambda) = 1 - e^{-\lambda t_i}$. The prior is concentrated on two points:

$$p_0 = \Pr(\lambda_1) = 1 - \Pr(\lambda_0)$$

where $\lambda_1 > \lambda_0 > 0$. Conditional on λ , the private signal is distributed as $\Pr(z_i \leq z|\lambda) = e^{-\lambda z}$ for $z \geq 0$. Therefore the posterior belief of λ_1 is

$$\begin{aligned} \pi_{t,t'}(x, y) &= \frac{p_0 e^{-\lambda_1 t} e^{-\lambda_1 t'} \lambda_1 e^{\lambda_1 x} \lambda_1 e^{\lambda_1 y}}{(1-p_0) e^{-\lambda_0 t} e^{-\lambda_0 t'} \lambda_0 e^{\lambda_0 x} \lambda_0 e^{\lambda_0 y} + p_0 e^{-\lambda_1 t} e^{-\lambda_1 t'} \lambda_1 e^{\lambda_1 x} \lambda_1 e^{\lambda_1 y}} \\ &= [1 + q(p_0, \lambda_0, \lambda_1, t, t', x, y)]^{-1} \end{aligned}$$

where

$$q(p_0, \lambda_0, \lambda_1, t, t', x, y) = \frac{1-p_0}{p_0} e^{(\lambda_1-\lambda_0)(t+t'-x-y)} \left(\frac{\lambda_0}{\lambda_1}\right)^2$$

So

$$\mathbb{E}_{t,t'}^i [\rho_i(\tau_i|\lambda)|x, y] = \frac{\lambda_0 q(p_0, \lambda_0, \lambda_1, t, t', x, y) + \lambda_1}{q(p_0, \lambda_0, \lambda_1, t, t', x, y) + 1}.$$

This is differentiable in x, y, t, t' . Since q is decreasing in x, y and increasing in t, t' and the expected hazard rate is decreasing in q , then the expected hazard rate is increasing in x, y and decreasing in t, t' .

Next

$$\mathbb{E}_{t,t'}^i [\rho_i(\tau_i|\lambda)|x, y-] = \frac{\lambda_0 q_-(p_0, \lambda_0, \lambda_1, t, t', x, y) + \lambda_1}{q_-(p_0, \lambda_0, \lambda_1, t, t', x, y) + 1}$$

where

$$q_-(p_0, \lambda_0, \lambda_1, t, t', x, y) = \frac{1-p_0}{p_0} e^{(\lambda_1-\lambda_0)(t+t'-x-y)} \left(\frac{\lambda_0}{\lambda_1}\right)$$

This has the same properties as $\mathbb{E}_{t,t'}^i [\rho_i(\tau_i|\lambda)|x, y]$ w.r. to x, y, t, t' .

Finally

$$\mathbb{E}_{t,t'}^i [\rho_i(\tau_i|\lambda)|x, y+] = \frac{\lambda_0 q_+(p_0, \lambda_0, \lambda_1, t, t', x, y) + \lambda_1}{q_+(p_0, \lambda_0, \lambda_1, t, t', x, y) + 1}.$$

where

$$q_+(p_0, \lambda_0, \lambda_1, t, t', x, y) = \frac{1-p_0}{p_0} e^{(\lambda_1-\lambda_0)(t+t'-x)} \frac{1-e^{\lambda_0 y}}{1-e^{\lambda_1 y}} \left(\frac{\lambda_0}{\lambda_1}\right)$$

This has the same properties as $\mathbb{E}_{t,t'}^i [\rho_i(\tau_i|\lambda)|x, y]$ w.r. to x, t, t' . Because

$$\frac{d}{dq} \left(\frac{\lambda_0 q + \lambda_1}{q + 1} \right) = -\frac{\lambda_1 - \lambda_0}{(q + 1)^2} < 0,$$

$e^{(\lambda_1-\lambda_0)(t+t'-x-y)}$ decreases in x and y and increases in t , because

$$\begin{aligned} \frac{d}{dy} \left(\frac{1-e^{\lambda_0 y}}{1-e^{\lambda_1 y}} \right) &= \frac{\lambda_1 e^{y\lambda_1} (1-e^{y\lambda_0}) - \lambda_0 e^{y\lambda_0} (1-e^{y\lambda_1})}{(1-e^{y\lambda_1})^2} \\ &\propto \frac{\lambda_1 e^{y\lambda_1}}{1-e^{y\lambda_1}} - \frac{\lambda_0 e^{y\lambda_0}}{1-e^{y\lambda_0}}, \end{aligned}$$

and $\frac{d}{d\lambda} \left(\frac{\lambda e^{y\lambda}}{1-e^{y\lambda}} \right) = \frac{e^{y\lambda}(1-e^{y\lambda}+y\lambda)}{(1-e^{y\lambda})^2} > 0$, it follows that Assumptions 1 and 2 are satisfied. Because

$$2 \frac{d}{dx} E_{t,t}^i [\rho_i(t|\lambda)|x, y+] = -\frac{d}{dt} E_{t,t}^i [\rho_i(t|\lambda)|x, y+],$$

setting $\bar{G}' = -1/2$, the inequalities in Assumption 3 are satisfied again if:

$$0 < \frac{d}{dy} E_{t,t}^i [\rho_i(t|\lambda)|x, y+] < \frac{d}{dx} E_{t,t}^i [\rho_i(t|\lambda)|x, y+]$$

The first inequality is already verified. The second inequality follows because:

$$\begin{aligned}
& \frac{d}{dx} \left(e^{(\lambda_1 - \lambda_0)(2t-x)} \frac{1 - e^{\lambda_0 y}}{1 - e^{\lambda_1 y}} \right) - \frac{d}{dy} \left(e^{(\lambda_1 - \lambda_0)(2t-x)} \frac{1 - e^{\lambda_0 y}}{1 - e^{\lambda_1 y}} \right) \\
&= -(\lambda_1 - \lambda_0) \frac{1 - e^{\lambda_0 y}}{1 - e^{\lambda_1 y}} e^{(\lambda_1 - \lambda_0)(2t-x)} - \left(-\frac{\lambda_0 e^{y\lambda_0} - \lambda_1 e^{y\lambda_1} - \lambda_0 e^{y\lambda_0} e^{y\lambda_1} + \lambda_1 e^{y\lambda_0} e^{y\lambda_1}}{(1 - e^{\lambda_1 y})^2} e^{(\lambda_1 - \lambda_0)(2t-x)} \right) \\
&= -\frac{\lambda_1 (1 - e^{y\lambda_0}) - \lambda_0 (1 - e^{y\lambda_1})}{(1 - e^{y\lambda_1})^2} < 0
\end{aligned}$$

Finally, the following

$$\lambda_0 < \frac{c_i}{b_i} < p_0 \lambda_1 + (1 - p_0) \lambda_0$$

is sufficient but not even necessary to satisfy Assumption 4.

Specializing Propositions 2 and 3, player i , observing signal x and upon observing that the opponent has left at time τ , will leave the race at time:

$$\sigma_{2,i}^*(x, \tau) = \max \left\{ \tau, x + g_j^*(\tau) - \tau + K + \frac{\log \left(\frac{\lambda_1}{\lambda_0} \right)}{\lambda_1 - \lambda_0} \right\}$$

and

$$\sigma_{2,i}^*(x, 0) = \max \{0, x + \underline{x}_j^* + K\}$$

where

$$K = \frac{1}{\lambda_1 - \lambda_0} \log \left(\frac{\lambda_1 - \frac{c_i}{b_i}}{\frac{c_i}{b_i} - \lambda_0} \frac{p_0}{1 - p_0} \frac{\lambda_1}{\lambda_0} \right)$$

is a measure of the prior bias in favor of λ_1 .

From Proposition 5, we obtain the stopping time of a player i with signal x , conditional on the opponent j still being in the game is:

$$\sigma_{1,i}^*(x) = \max \left\{ 0, \frac{1}{2} \left[x + K + (\lambda_1 - \lambda_0)^{-1} \log \left(\frac{1 - e^{\lambda_1 \sigma_{1,j}^{*-1}(\sigma_{1,i}^*(x))}}{1 - e^{\lambda_0 \sigma_{1,j}^{*-1}(\sigma_{1,i}^*(x))}} \right) \right] \right\}$$

In the symmetric payoff model $g_j^*(\sigma_{1,i}^*(x)) = \sigma_1^{*-1}(\sigma_1^*(x)) = x$, so we obtain a closed-form solution for the first quitting time:

$$\sigma_1^*(x) = \max \left\{ 0, \frac{1}{2} \left[x + K + (\lambda_1 - \lambda_0)^{-1} \log \left(\frac{1 - e^{\lambda_1 x}}{1 - e^{\lambda_0 x}} \right) \right] \right\}.$$

Finally, from Proposition 7, we obtain that in any equilibrium of the public information game, for every pair of signals x, y , the first firm quits the race at time

$$T_1(x, y) = \frac{1}{2} (x + y + Q)$$

and

$$T_2(x, y) = x + y + Q - T_1,$$

where

$$Q = \frac{1}{(\lambda_1 - \lambda_0)} \log \left(\frac{\lambda_1 - c_2/b_2}{c_2/b_2 - \lambda_0} \frac{p_0}{1 - p_0} \left(\frac{\lambda_1}{\lambda_0} \right)^2 \right).$$