Fragility of Reputation and Clustering of Risk-Taking

Guillermo L. Ordoñez*

July 2011

Abstract

Concerns for constructing and maintaining good reputations are known to reduce borrowers’ excessive risk-taking. I find that the self-discipline induced by these concerns is fragile, and can break down without obvious changes in economic fundamentals. Furthermore, at an aggregate level, breakdowns are clustered among borrowers with intermediate and good reputations, which can exacerbate small economic shocks. These results come from an aggregate dynamic model of reputation formation in credit markets. Since reputation introduces multiplicity, I select a unique equilibrium which is robust to small perturbations of information about economic fundamentals.

“A sound banker, alas, is not one who foresees danger and avoids it, but one who, when he is ruined, is ruined in a conventional way along with his fellows, so that no one can really blame him”

John Maynard Keynes, 1931.

*Yale University, Department of Economics, 28 Hillhouse Av., Room 208, New Haven, CT, 06511. E-Mail: guillermo.ordonez@yale.edu. I especially thank Andy Atkeson, Christian Hellwig, and David K. Levine. For useful comments and discussions, I also thank Fernando Alvarez, V.V. Chari, Johannes Horner, Hugo Hopenhayn, Narayana Kocherlakota, David Lagakos, Robert Lucas, Lee Ohanian, Chris Phelan, Larry Samuelson, Rob Shimer, Nancy Stokey, Ivan Werning, Bill Zame, and seminar participants at Arizona State, Brown, Chicago Economics, Chicago Booth, Columbia, CREI-Pompeu Fabra, Federal Reserve Banks of Chicago and Minneapolis, Harvard, Iowa, ITAM, LSE, Minnesota, MIT, NYU Stern, Princeton, UCLA, Washington University in St. Louis, Wharton, Yale, LAMES-LACEA Meetings in Rio de Janeiro and SED Meetings at MIT. I also thank Kathy Rolfé and Joan Gieseke for excellent editorial assistance. The usual waiver of liability applies.
1 Introduction

Reputation concerns are important sources of discipline in financial markets, but this discipline seems to collapse from time to time. In 1963, former Federal Reserve chairman Alan Greenspan wrote "Reputation, in an unregulated economy, is a major competitive tool...Left to their own devices, it is alleged, businessmen would attempt to sell unsafe food and drugs, fraudulent securities, and shoddy buildings...but it is in the self-interest of every businessman to have a reputation for honest dealings and a quality product".\(^1\) This view, which justified a wave of deregulation in financial markets, was buffeted by a major financial crisis during 2007-2008. In relation to this event, Alan Greenspan declared before the House of Representatives: "Those of us who have looked to the self-interest of lending institutions to protect shareholders equity, myself included, are in a state of shocked disbelief".\(^2\) Similarly, current Federal Reserve chairman, Ben Bernanke, highlighted the periodic collapse of financial markets' discipline as a key element of crises: "Market discipline has in some cases broken down, and the incentives to follow prudent lending procedures have, at times, eroded."\(^3\).

Why the discipline in financial markets tends to fail abruptly? Is it possible that firms’ concerns about gaining and maintaining good reputations suddenly disappear? In this paper I show these concerns are, in fact, fragile and their breakdown induce a sudden collapse in market discipline, characterized by more risk-taking and less effort exertion by large and reputable firms in response to small and not obvious changes in aggregate fundamentals.

I begin with a standard model of credit markets in which firms borrow to invest. First, I focus on risk-taking decisions by productive firms to illustrate how reputation concerns that relax moral hazard are fragile and very sensitive to aggregate fundamentals. Then, I discuss how to apply the same basic structure to other forms of moral hazard, such as effort provision, and other forms of economic activity, such as financial intermediation.

In the model, all firms can invest in a "risky" project, while only some of them (strategic firms) can also invest in safer projects, with higher probabilities of success but lower private benefits to managers. Lenders cannot observe which project the firm chooses, but they prefer that strategic firms take "safe" ones, which imply a lower probability of default.

A firm’s reputation is defined as the probability that the firm is strategic. Reputation is updated by lenders after observing both the firm’s continuation – a signal about the firm’s decisions – and the aggregate default rate in the economy – a signal about the average decision of similar firms. Strategic firms want to distinguish themselves from nonstrategic ones so

\(^1\)"The Assault on Integrity". The Objectivist Newsletter, August 1963.
\(^3\)Statement, Board of Governors, December 18, 2007
that they can pay lower interest rates for borrowed funds in the future. The fear of losing reputation, therefore, leads strategic firms to reduce inefficient risk-taking.

I assume firms’ temptation to take risks varies monotonically with a stochastic aggregate fundamentals. However, it turns out that when fundamentals are perfectly observable, the model delivers multiple equilibria based on multiple firms’ beliefs about similar firms’ behavior. There is a range of this fundamental for which two equilibria coexist. At the one extreme, if a strategic firm believes all other strategic firms play it safe, then it does the same. Firms know that in this case the aggregate default rate will be low, lenders will believe no strategic firm takes risks and firms continuation will be attributed at least partially to their good behavior, thereby improving their reputation. At the other extreme, if a strategic firm believes all other strategic firms take risks, then it does the same. In this case, firms know the aggregate default rate will be high, that lenders will believe strategic firms take risks and their continuation will be attributed solely to good luck, not improving their reputation at all. Reputation concerns clearly reduce risk-taking in the first equilibrium but not in the second.

In order to obtain a unique equilibrium, which is robust to small perturbations of information, I use techniques from the global games literature. I assume that after negotiating the loan, but before making a decision about risk-taking, firms do not observe the fundamental but just a private noisy signal about it. The model thus becomes a nonstandard dynamic global game in which strategic complementarities are not just assumed, but are rather obtained endogenously from the concerns behind reputation formation. Uniqueness of equilibrium is characterized for each reputation level by a cutoff in signals about fundamentals, below which strategic firms decide to take risks. Fundamentals, that is, do not only affect the temptation but also become a coordination device for risk-taking.

This equilibrium selection generates the first of two sources of reputation fragility. If signals about fundamentals are precise, small changes of fundamentals around the cutoff of risk-taking produce a clustering of behavior among firms with the same reputation.

The second source of fragility arises at an aggregate level when firms with different reputation are compared. This source of fragility is independent of the equilibrium selection and depends only on primitive learning properties. Safe projects have higher probabilities of continuation, which generate two types of incentives for safe behavior. One type, continuation incentives, increases with reputation; firms with better reputations face lower interest rates in the future, have higher expected future profits, and are more afraid to die. The other type of incentives, reputation formation incentives, is low for extreme reputations and high for intermediate ones; because of learning, priors are harder to change when reputation is either too high or too low. The combination of these two types of incentives lead to the following result:
Poor reputation firms have no incentives to play it safe because their continuation value is low and, if they survive, their reputation cannot improve much.

Intermediate reputation firms do have incentives to play it safe, not because their continuation value is high; rather, if they survive, they can improve their reputation a lot.

Good reputation firms also have incentives to play it safe, but not because they can gain much reputation if surviving; rather, their continuation value is high and they can lose that value if they die.

Hence, intermediate and good reputation firms have similar cutoffs for different reasons. They switch to risk-taking under similar conditions, clustering their behavior. Furthermore, since the distribution of reputation is biased toward intermediate and good reputations, what these firms do strongly affects the aggregate level of risk-taking in the market.

My work shows that reputation concerns are beneficial because they reduce and stabilize aggregate risk-taking and default, but they are also fragile because they generate sudden reactions in risk-taking and default without obvious changes in fundamentals. Sudden clusters of default and risk-taking are not only well documented in the recent financial crisis (Taylor (2009) and Bernanke (2009)) but has been also well documented in previous recessions (Campbell et al. (2001) and Das et al. (2007)).

Relation with the literature: My work here primarily combines two strands of literature: reputation and global games. With regard to the reputation strand, my model is most closely related to the models of Diamond (1989) and Mailath and Samuelson (2001), who analyze the ability of reputation to deter opportunistic behavior in the presence of both adverse selection and moral hazard. Unlike their work, which is focused on reputation incentives for a single agent living in a state-invariant environment, my work here explicitly introduces a cross section of firms in an environment that evolves stochastically, which allows to study the interplay between reputation incentives and economic conditions in determining aggregate behavior. As in their work, my model also exhibit multiple equilibria, but in my case the multiplicity cannot be simply assumed away by using the arguments of Stiglitz and Weiss (1981). While Diamond (1989) focuses on extreme equilibria and Mailath and Samuelson (2001) focus on the most efficient one, I select a unique equilibrium by exploiting fundamentals as a coordination device. Finally, unlike Mailath and Samuelson (2001), here firms’ behavior affects the probability of their continuation, a signal to update reputation, and unlike Diamond’s (1989) model, mine is flexible enough to include the use of additional signals correlated to projects, breaking the perfect correlation between age and reputation.

My work also contributes to the literature of herding behavior generated by reputation concerns. While the work pioneered by Scharfstein and Stein (1990) considers agents that mimic
others and disregard private information, firms in my model cannot observe others’ projects and instead use private information to coordinate behavior. The justification of herding when agents cannot observe others’ projects but instead a correlated signal about an aggregate variable is key in understanding the collapse of the new, more complex, financial system, critically characterized by lack of information about other investors’ positions.

My model contributes more generally to a broad literature on career concerns in financial markets, where institutional investors who care about their reputation generate herding (Zwiebel (1995) and Ottaviani and Sorensen (2006)), prevent information aggregation (Dasgupta and Prat (2006) and (2008)) and amplify price volatility in financial markets (Guerrieri and Kondor (2009)). My work here is less restrictive to experts’ advice and actions, rather focusing on the impact of excessive risk taking by firms in amplifying aggregate defaults in credit markets.

My work also adds to the dynamic global games literature – such as Morris and Shin (2003), Toxvaerd (2008), Giannitsarou and Toxvaerd (2009) and Chassang (2010) – exploiting novel properties coming from the endogenous reputational generation of strategic complementarities. In particular, the range of fundamentals with multiple equilibria depends on the initial reputation of the firm and this is useful in characterizing the schedule of cutoffs for different reputations. Also, since interest rates evolve endogenously with reputation, I endogenously determine the value functions that enter into the solution of the dynamic global game.

Finally, my work also contributes to the scarce literature on learning in global games. While most of that literature studies situations in which players learn about a policymaker or a status quo (as in, for example, the 2006 and 2007 work of Angeletos, Hellwig and Pavan), my model deals with the opposite situation, in which the market learns about players’ types and so generates coordination problems. My work here is the first to exploit fundamental-driven incentives to create a reputation global game and select a unique equilibrium.

In the next section, I show how, with incomplete information about economic fundamentals, a dynamic global game analysis of reputation formation in credit markets delivers a unique equilibrium. In Section 3, I show that firms’ concerns about reputation impose a discipline that is fragile. To conclude, in Section 4, I review empirical evidence of the model’s testable predictions and discuss potential extensions.

2 Selecting a Unique Reputation Equilibrium in Credit Markets

In this section I show that reputation concerns create strategic complementarities across firms, generating equilibrium multiplicity, and discuss the role of imperfect information in selecting a unique equilibrium. First, I describe the general model. Second, I illustrate the main
results using a single period version, with specific distribution and payoff functions and ex-
ogenous interest rates and continuation values. Finally, I extend the results to a finite horizon
game, generalizing distribution and payoff functions and endogeneizing interest rates and
continuation values.

2.1 The Model

Here I describe my model of reputation concerns in credit markets and the timing of its events.

2.1.1 Description

Credit markets are composed of a continuum of long-lived, risk-neutral firms (with mass 1)
and an infinite number of short-lived risk-neutral lenders that provide funds to those firms.

Each firm runs a unique project per period. The project can be safe (s) or risky (r). As in
Fama and Miller (1972) and Jensen and Meckling (1976), a safer project is one with a higher
probability of success and firms’ continuation (c).

**Assumption 1** Safe projects make firms’ continuation more likely \( \Pr(c|s) = p_s > \Pr(c|r) = p_r \).

If the firm does not continue, or dies, then current and future cash flows are zero. If the
firm plays it safe and continues, the project just delivers pledgable cash flows \( \Pi_s = K \) in
the period. If the firm plays it risky and continues, the project not only delivers pledgable
cash flows \( K \), but also non-pledgable cash flows (or non-pecuniary benefits to managers) \( \theta \),
such that \( \Pi_r = K - \theta \) in the period. The single-dimensional variable \( \theta \in \mathbb{R} \) is common to all
firms (we call it a fundamental), is independently and identically distributed over time and is
normally distributed \( \theta \sim \mathcal{N}(\mu, \frac{1}{\alpha}) \), with mean \( \mu \) and precision \( \alpha \) in each period.\(^4\)

There are two types of firms, defined by their access to projects. **Strategic** firms \( S \) can choose
between safe and risky projects. **Risky** firms \( R \) can only endeavor in risky projects, with a
low probability of success. Firms know their own type but lenders do not. Firm’s reputation
is defined by \( \phi = \Pr(S) \), the probability the firm is strategic.\(^5\)

To run a project, each firm needs external funds (normalized to one per period), which can
be provided by lenders, whose outside option is an alternative investment in a risk-free bond

\(^4\)We relax the assumption of linearity in cash flows and normality in the Appendix.
\(^5\)The introduction of these two types is based on my (maybe pessimistic) belief that all firms can take risks, but
not all of them can play it safe. While all firms can perform trial-error procedures, not all of them have access to
well-designed procedures. An alternative assumption is that strategic firms have a positive discount factor, while
others (risky ones) have a zero discount factor. Still, if nonstrategic firms only have access to safe technologies,
the main result of reputation fragility remains unchanged.
that pays $\bar{R} > 1$.\footnote{Since lenders are the long side of the market, there is no competition for funds. The introduction of such competition makes reputation more important and magnifies the results. Alternative assumptions are that fundamentals do not only affect cash flows ($\Pi$), but also the probability of continuation ($p_0$) and/or the risk-free interest rate ($\bar{R}$). These alternatives do not modify my main results either. See Ordonez (2008).} Failure to repay loans (default) triggers a bankruptcy procedure that destroys the firm’s pledgable output. This is a straightforward way to introduce truth-telling by firms: when pledgable cash flows are greater than debts, firms always find optimal to repay the loan and get the positive differential than to default and get nothing. I assume that, conditional on continuation, firms are always solvent and pay their loans (specifically $K > \frac{1}{p_r}$, where $\frac{1}{p_r}$ will be the largest possible face value of debt). Contrarily, if a firm dies, cash flows are zero and the firm has to default.\footnote{The goal is to understand how reputation concerns and aggregate conditions interact in a market with short-term debt contracts, not the optimality of such contracts. Then, for simplicity I rule out equity, asset accumulation and other alternative forms of financing, which are interesting extensions but outside the scope of the paper.}

2.1.2 Timing

These decisions are repeated during a finite number of periods. The order of events in each period $t$ is the same in all periods $t = \{0, 1, ..., T\}$ and is given by:

- Firms and lenders meet. Lenders do not observe the firm’s type, just its continuation, its last period reputation ($\phi_{t-1}$) and the last period aggregate default among all firms with the same reputation. Based on this information each lender revises beliefs about the firm to $\phi_t$. New firms start with exogenous $\phi_0$.
- Each firm acquires a loan of 1 at a rate that depends on its new reputation, $R(\phi_t)$.
- Firms observe the fundamental $\theta_t$ (potential non-pledgable cash flows).
- Strategic firms decide between using safe ($s$) or risky ($r$) projects. Risky firms only endeavor on risky projects ($r$).
- Production occurs, and the firms either continue or die.
- If a firm dies, it defaults on its loan. If a firm continues, it pays to lenders $R(\phi_t)$ and consumes the remaining cash flows.

2.1.3 Reputation Updating

When updating a continuing firm’s reputation from $\phi_t$ to $\phi_{t+1}$, lenders should have a belief about strategic firms’ behavior, which they can infer from the realized default of similar firms.
Denote $D_t(\phi_t, \hat{x}_t)$ the aggregate default rate of all firms $\phi_t$ at period $t$, where $\hat{x}_t(\phi_t, \theta_t)$ is the fraction of strategic firms $\phi_t$ taking risks at period $t$,

$$D_t(\phi_t, \hat{x}_t) = [(1 - p_r)\hat{x}_t + (1 - p_s)(1 - \hat{x}_t)]\phi_t + (1 - p_r)(1 - \phi_t)$$

From this equation, lenders can infer $\hat{x}_t(\phi_t, \theta_t)$ and update the reputation of a single continuing firm using Bayes’ rule,

$$Pr(S|c) = \phi_{t+1}(\phi_t, \hat{x}_t) = \frac{[p_r\hat{x}_t + p_s(1 - \hat{x}_t)] \phi_t}{1 - D_t(\phi_t, \hat{x}_t)}$$

(1)

Note that reputation is non-decreasing as firms continue and reputation increases less when many similar firms die. This seemingly counterintuitive result arises because a “good” firm is one that has the chance to behave well, but not necessarily does it. This is why a high aggregate default is not good news for surviving firms, since continuation is not assigned to quality but to luck. Note that, for $\phi_t \in (0, 1)$, $\phi_{t+1} = \phi_t$ when $\hat{x}_t = 1$ and $\phi_{t+1} > \phi_t$ when $\hat{x}_t < 1$, with the gap $(\phi_{t+1} - \phi_t)$ increasing as $\hat{x}_t$ goes to 0.

Graphically, firms reputation evolves as in Figure 1. Reputation priors $\phi_t$ are represented on the horizontal axis; reputation posteriors $\phi_{t+1}$ on the vertical axis. For any prior $\phi_t$,

- If lenders observe all strategic firms playing it safe (that is, if $\hat{x}_t = 0$), then the gap $\phi_{t+1} - \phi_t$ represents the gains to the firm from continuation, in terms of reputation.
- If lenders observe all strategic firms taking risks (that is, if $\hat{x}_t = 1$), then $\phi_{t+1} = \phi_t$ and firms get no gains in terms of reputation.
- Regardless of $\hat{x}_t$, the updating is weaker when priors are stronger (that is, close to $\phi_t = 0$ or $\phi_t = 1$). In particular, regardless of $\hat{x}_t$, $\phi_{t+1} = \phi_t$ for $\phi_t = 0$ and $\phi_t = 1$. The maximum gap, $(\phi_{t+1} - \phi_t)$ is obtained at some intermediate reputation level $\phi_M$.

2.2 A Single Period Version

In this section I introduce many simplifying assumptions that are useful to highlight the essence of reputational multiplicity and the role of imperfect information in selecting a unique equilibrium. We assume a single period. Firms start with a given reputation $\phi$ and lenders charge an exogenous interest rate $R(\phi)$ (decreasing in $\phi$) for the loan. The timing described above proceeds. At the end of the period, lenders observe aggregate default and update the reputation of continuing firms up to $\phi'$. Finally an exogenous continuation value $V(\phi')$
(increasing in $\phi'$) is transferred to each continuing firm. In next sections I relax these simplifications, endogeneizing interest rates and repeating the game an arbitrarily large number of periods to endogeneize continuation values, showing that the properties we assume here hold in equilibrium.

To eliminate many equilibria, standard in models with public signals, that require an implausible degree of coordination between the firm’s behavior and its beliefs about other firms’ behavior, I restrict attention to Markovian strategies, such that $x(\phi, \theta)$ is the probability that a firm with reputation $\phi$ that observes fundamentals $\theta$ takes risks.\(^8\)

Given the monotonicity of payoffs on $\theta$, I also focus on equilibria in cutoff strategies,\(^9\) in which a firm with reputation $\phi$ decides to take risks if it observes fundamentals below a certain cutoff point, $k(\phi)$ and play it safe if it observes fundamentals above that cutoff

$$x(\phi, \theta) = \begin{cases} 0 & \text{if } \theta > k(\phi) \\ 1 & \text{if } \theta < k(\phi) \end{cases}.$$ \hspace{1cm} (2)

**Definition 1** A Markov perfect equilibrium in cutoff strategies consists of a (symmetric) strategy for the firms $k(\phi) = \theta^*(\phi) : [0, 1] \rightarrow \mathbb{R}$ and posteriors $\phi'(\phi, \bar{x}) : [0, 1] \times [0, 1] \rightarrow [0, 1]$, for all $\phi \in [0, 1]$, such that

\(^8\)See the discussion in Mailath and Samuelson (2006).

\(^9\)We show later these are the only Markovian strategies that survive the global game refinement.
• $k(\phi) = \theta^*(\phi)$ defines the $x^*(\phi, \theta) \in \arg\max_{x \in [0, 1]} U(\phi, \theta, x|\hat{x})$ for all $\theta$, where,

$$U(\phi, \theta, x|\hat{x}) = x p_r [K - \theta - R(\phi) + \beta V(\phi'(\phi, \hat{x}))] + (1 - x) p_s [K - R(\phi) + \beta V(\phi'(\phi, \hat{x}))]$$

and $\beta$ is the discount factor.

• $\phi'(\phi, \hat{x})$ is obtained using Bayes’ rule (equation (1)), where $\hat{x}(\phi, \theta) = x^*(\phi, \theta)$ for all $\theta$, is the updating rule that lenders must use if their beliefs are to be correct (this is, consistent with equilibrium strategies of a continuum of firms $\phi$, which determine aggregate default $D(\phi, x^*)$).

### 2.2.1 Multiple Equilibria with Complete Information

Now I show that in this baseline model, when firms perfectly observe the fundamental, a multiplicity of Markovian perfect equilibria exists in monotone cutoff strategies. First I discuss properties of the firms’ differential gains from taking safe projects relative to risky projects, which characterize each firm’s decisions. Then I show how these properties interact with firms’ beliefs about other firms’ actions to create multiple equilibria.

Define by $\Delta(\phi, \theta|\hat{x}) = U(\phi, \theta, x = 0|\hat{x}) - U(\phi, \theta, x = 1|\hat{x})$ the differential gains to firms from taking safe projects relative to risky projects when a firm with reputation $\phi$ observes a fundamental $\theta$, conditional on beliefs $\hat{x}(\phi, \theta)$ (expected fraction of strategic risk-taking that lenders will recover from the end of period’s aggregate default). A firm decides to play it safe if $\Delta(\phi, \theta|\hat{x}) > 0$ and to take risks if $\Delta(\phi, \theta|\hat{x}) < 0$.

$$\Delta(\phi, \theta|\hat{x}) = (p_s - p_r) \left[ \frac{K - R(\phi) + p_r}{p_s - p_r} \theta + \beta V(\phi) + \beta [V(\phi'(\phi, \hat{x})) - V(\phi)] \right]$$

(3)

Equation (3) displays the four essential components of these differential gains:

• **Short-Term** captures the differential gains (in expected net pledgable cash flows from a higher probability of continuation) from choosing safe projects.

• **MH** refers to Moral Hazard. It captures the relative temptation to take risks. This is the only part of the differential gains that depends on $\theta$.

• **Cont** captures that taking safe projects increases the probability of the firm’s continuation, which has a value that depends on reputation $\phi$. 
• Reputation Formation captures that taking safe projects also increases the probability of reputation improvement from $\phi$ to $\phi'$.

It is clear that risk-taking is less tempting as fundamentals increase ($\frac{\partial \Delta(\phi, \theta|\hat{x})}{\partial \theta} = p_r > 0$) and strategic risk-taking declines ($\frac{\partial \Delta(\phi, \theta|\hat{x})}{\partial \hat{x}} \leq 0$, since $\frac{\partial (\phi'-\phi)}{\partial \hat{x}} \leq 0$ and $\frac{\partial V(\phi')}{\partial \theta'} > 0$).

Before discussing multiplicity, I assume uniform limit dominance, which defines ranges of fundamentals for which, regardless of other firms’ actions, a firm decides to take risks (fundamentals below a lower bound $\theta$) or play it safe (fundamentals above an upper bound $\bar{\theta}$).

**Assumption 2** (Uniform Limit Dominance in a Single Period Model)

• For each $\phi$, there is a lower bound $\theta(\phi)$ such that $\Delta(\phi, \theta|\hat{x} = 0) = 0$

• For each $\phi$, there is an upper bound $\bar{\theta}(\phi)$ such that $\Delta(\phi, \bar{\theta}|\hat{x} = 1) = 0$

In this simple single period version of the model,

$$\theta(\phi) = -\frac{p_s - p_r}{p_r} [K - R(\phi) + \beta V(\phi', \hat{x} = 0)]$$

$$\bar{\theta}(\phi) = -\frac{p_s - p_r}{p_r} [K - R(\phi) + \beta V(\phi)].$$

The gap $\bar{\theta}(\phi) - \theta(\phi) = \beta \frac{p_s - p_r}{p_r} [V(\phi', \hat{x} = 0) - V(\phi)] \geq 0 (=0$ for $\phi = 0$ and $\phi = 1$) achieves a maximum at some intermediate reputation level $\phi_M$. This is simply a mapping from the updating gap $\phi'(\phi, \hat{x} = 0) - \phi$ (illustrated in Figure 1) into value functions increasing in $\phi$.

The next Proposition formalizes the multiplicity of equilibria in this model.

**Proposition 1** (Multiplicity in a Single Period Model)

For all reputation levels $\phi \in (0, 1)$, all $\theta \in [\theta(\phi), \bar{\theta}(\phi)]$ are equilibrium strategy cutoffs $\theta^*(\phi)$. Only for reputation levels $\phi = 0$ and $\phi = 1$, is there a unique equilibrium cutoff, $\theta^*(0)$ and $\theta^*(1)$, respectively.

The proof is in the Appendix but Figure 2 provides a graphical intuition of multiplicity. Consider a particular risk-taking cutoff $\theta^*(\phi)$ for some firm with reputation $\phi \in (0, 1)$, such that $\theta^*(\phi) \in [\theta(\phi), \bar{\theta}(\phi)]$. Then, the symmetric equilibrium differential gain $\Delta(\phi, \theta|x^*)$ for different levels of fundamentals is the bold function with a discrete jump at $\theta^*(\phi)$. This is an equilibrium because it is a best response for any realization of the fundamental $\theta$ such that firms’ beliefs about other firms’ actions are correct. Playing it safe is optimal for all $\theta \geq \theta^*(\phi)$ (since $\Delta(\phi, \theta|x^* = 0) \geq 0$ for all $\theta \geq \theta^*(\phi)$), and taking risks is optimal for all $\theta \leq \theta^*(\phi)$ (since $\Delta(\phi, \theta|x^* = 1) \leq 0$ for all $\theta \leq \theta^*(\phi)$).
In words, for a fundamental to be a cutoff in equilibrium, it should be the case that two equilibria coexist at exactly that cutoff. At the one extreme, if firms believe all other strategic firms will play it safe and aggregate default will be low, it is in firms’ best interest to play it safe. Firms know that in this case their continuation and loan repayment will be attributed at least partly to their good behavior, thereby improving their reputation. At the other extreme, if firms believe all other strategic firms will take risks and aggregate default will be high, it is in firms’ best interest to take risks. Under these beliefs, firms know that their continuation and repayment will be attributed solely to good luck and won’t improve their reputation at all. Since the difference of payoffs between these two extremes is strictly positive, a continuum of fundamentals fulfill this condition.

The multiplicity I have described here thwarts attempts to draw conclusions about the effectiveness of reputation to reduce risk-taking, leaving a big role to self-fulfilling beliefs and payoff irrelevant sunspots. However, what really creates the multiplicity is the assumption of complete information about fundamentals, which at the same time requires an implausible degree of coordination and prediction of other firms’ behavior in equilibrium. This orient us what to do next to move toward the selection of a unique equilibrium.

2.2.2 A Unique Equilibrium with Incomplete Information

What I do is modify the assumption that information about fundamentals is complete. I assume instead that firms observe a private noisy signal about the fundamental before deciding whether to take safe or risky projects.\footnote{The assumption about the timing is important. If interest rates reveal, through the market’s aggregation of information, the true fundamental before production occurs, we go back to complete information and a unique equilibrium cannot be pinned down by introducing heterogeneity through signals. See Atkeson (2001).} This noise, when small, leads to the selection of a
unique equilibrium. What creates the multiplicity is the strategic complementarity across firms, which works through lenders beliefs. With complete information, each equilibrium is sustained by different fulfilling expectations about what other firms do, hence in equilibrium firms can perfectly forecast each other actions and coordinate on multiple courses of action. With incomplete information, private signals serve as an anchor for firm’s actions that avoid the indeterminacy of expectations about other firms’ actions and hence avoid the indeterminacy of beliefs lenders will use to update reputation.

Formally, the new assumption about the information structure is,

**Assumption 3** Each firm $i$ observes a signal about economic fundamentals $z_i = \theta + \epsilon_i$, which is identically and independently distributed across $i$. The noise $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ is unbiased and has a variance $\sigma^2 \equiv \frac{1}{\gamma}$ (precision $\gamma$).

I extend the proof to more general distributions in the Appendix. Signals are useful not only to infer $\theta$ but also to infer other firms’ actions, and the aggregate default lenders will use to update reputation. Given this revised, incomplete information structure, the firm uses a cutoff strategy defined over the set of signals rather than over the set of fundamentals. For a current signal $z_i$, a strategy of a firm $\phi$ is a real number $k_z(\phi)$ such that the firm uses safe technologies ($x(\phi, z_i) = 0$) for $z_i > k_z(\phi)$ and risky ones ($x(\phi, z_i) = 1$) for $z_i < k_z(\phi)$. The strategic risk-taking that lenders infer from aggregate default ($\hat{x}(\phi, \theta)$) still depends on the fundamental. Firms use their signal $z_i$ to take expectations about $\hat{x}(\phi, \theta)$.

We can now define $\Delta(\phi, z|k_z)$ as firm $\phi$’s expected differential gains from playing it safe if receiving a signal $z$ and other strategic firms $\phi$ use a cutoff $k_z(\phi)$. Formally,

$$
\Delta(\phi, z|k_z) = E_{\theta|z} (\Delta(\phi, \theta|\hat{x})|k_z)
$$

The next proposition states that under this incomplete information structure, when signals are precise enough ($\sigma \to 0$), there exists a unique Markovian perfect Bayesian equilibrium in monotone cutoff strategies for each reputation level $\phi$.

**Proposition 2** (Uniqueness in a Single Period Model)

For a given $\phi$, as $\sigma \to 0$, there exists a unique cutoff signal $k_z(\phi) = z^*(\phi)$ in equilibrium such that $\Delta(\phi, z|z^*) = 0$ for $z = z^*(\phi)$, $\Delta(\phi, z|z^*) > 0$ for $z > z^*(\phi)$, and $\Delta(\phi, z|z^*) < 0$ for $z < z^*(\phi)$, where $z^*(\phi)$ is given by

$$
z^*(\phi) = - \frac{p_s - p_r}{p_r} [K - R(\phi) + \beta V(\phi', \hat{x} = 0.5)]
$$
Proof Conditional on observing a signal \( z_i \), firm \( i \)'s expected \( \theta \) is

\[
\hat{\theta}_i = E(\theta|z_i) = \frac{\alpha \mu + \gamma z_i}{\alpha + \gamma}
\]

Given this update on \( \theta \), the conditional distribution of signals of another firm \( j \) is

\[
z_j|\hat{\theta}_i \sim N\left(\hat{\theta}_i, \frac{1}{\alpha + \gamma} + \frac{1}{\gamma}\right)
\]

The expected fraction of other firms having a signal smaller than \( z_i \) (and hence taking risks if firm \( i \) is indifferent at \( z_i \)) is

\[
E(\hat{x}(\phi, \theta)|z_i) = \Pr\left(z_j < z_i|\hat{\theta}_i\right) = \Phi\left[\sqrt{\zeta(\hat{\theta}_i - \mu)}\right]
\]

where \( \Phi \) is the standard normal cdf and

\[
\zeta = \frac{\alpha^2(\alpha + \gamma)}{\gamma(\alpha + 2\gamma)}
\]

The equilibrium cutoff \( z^*(\phi) \) is given by the signal at which firms will be in expectation indifferent between taking safe or risky projects, when other firms also follow \( z^*(\phi) \), such that

\[
\Delta(\phi, z^*|z^*) = 0
\]

\[
E_{\theta|z^*}(\Delta(\phi, \theta|\hat{x})|z^*) = (p_s - p_r) \left[K + \frac{p_r}{p_s - p_r} E(\theta|z^*) - R(\phi) + \beta E[V(\phi'(\phi, \hat{x}|z^*)] = 0
\]

As \( \sigma \to 0 \) (or \( \gamma \to \infty \)), \( E(\theta|z^*) \to z^* \) and \( E[V(\phi'(\phi, \hat{x}|\theta)|z^*)] \to V(\phi'(\phi, \hat{x} = 0.5)) \) (since \( \zeta \to 0 \)). Then, we have equation 5.

Intuitively, relaxing the assumption of complete information about fundamentals and making signals very precise, we can use the approach provided by global games to select a unique equilibrium by iterated deletion of dominated strategies. Assume, for example, that a strategic firm \( \phi \) uses a cutoff strategy \( k_z(\phi) = \theta(\phi) \), which we know is an equilibrium sustained by \( \hat{x} = 0 \) under complete information. If signals are very precise, it means that a firm that observes \( z_i = \theta(\phi) \) believes that around 50% of other strategic firms \( \phi \) that use the same cutoff, observe a signal below \( \theta(\phi) \) and will decide to take risky projects. Since there is a continuum of firms with reputation \( \phi \), lenders will observe 50% of the firms taking risks and will update reputation using \( \hat{x} = 0.5 \). However, with updating based on \( \hat{x} = 0.5 \) the firm would not be indifferent between risky and safe projects at \( \theta(\phi) \), strictly preferring to take risky projects.

The only cutoff in equilibrium is the signal at which a firm is indifferent between taking safe and risky projects when the expected fraction of strategic risk-taking that lenders use to up-
date beliefs is \( \hat{x} = 0.5 \), as in Proposition 2. Morris and Shin (2003) showed these laplacian beliefs “following Laplace’s (1824) suggestion that one should apply a uniform prior to unknown events from the principle of insufficient reason.” arise endogenously in global games for a player observing the threshold signal.

Here, fundamentals do not only affect incentives but also become a coordination device. If a firm observes a low signal, it believes the fundamental is low with high probability, which directly induces the firm to take risks. On top of that, the firm also believes that other similar firms have observed a low signal and will take risks as well, which indirectly induces the firm to take risks. This is why, fundamentals, through the generation of signals, pin down firms’ expectations about other firms’ strategic risk-taking and hence coordinate firms’ actions.

Interestingly, under the alternative assumption that only one firm exists and the lender observes fundamentals instead of aggregate default, multiplicity still arises from reputation formation because lenders’ beliefs about the firm’s actions should be correct in equilibrium. Even when this multiplicity does not rely on complementarities with other firms, still a unique equilibrium can be selected by using global games, under the assumption that both the firm and the lender observes the fundamental with noise.

### 2.3 The General Model

In this section I generalize the previous results by making interest rates and continuation values endogenous.\(^{11}\) In each case I discuss how multiplicity and the selection of a unique equilibrium apply. This generalization is relevant because we are ultimately interested in comparing the cutoffs that firms with different reputation follow. In order to make such a comparison it is key to obtain continuation values endogenously, and this can be done only by endogeneizing interest rates first and then solving the full game as \( T \to \infty \).

#### 2.3.1 Endogenous Interest Rates

In this section, I relax the assumption of exogenous interest rates in a single period. This is a first step to endogeneize value functions later by repeating the game many periods.

Since lenders are competitive and have a fixed outside option \( R \), at the time the loan is negotiated they charge an interest rate that pays in expectation the same return as the outside option. This is an additional condition in the definition of equilibrium. Specifically, in the

\(^{11}\) In the Appendix we also extend the results to more general payoff functions, distributions of fundamentals and noise structures.
case of complete information about fundamentals, interest rates are defined by the risk-free interest rate $R$ divided by the expected probability of firm’s continuation,

$$R(\phi|k) = \frac{R}{Pr(c|\phi, k)},$$

where

$$Pr(c|\phi, k) = (1 - \phi)p_r + \phi [p_r N(k) + p_s (1 - N(k))],$$

$k(\phi)$ are lenders’ beliefs about the cutoff that strategic firms $\phi$ follow and $N(k)$ is the expected strategic risk-taking by those firms (normal cumulative density up to $k(\phi)$). Since interest rates depend on $k(\phi)$, expected gains of playing it safe (equation 3) under complete information can be rewritten as,

$$\Delta(\phi, \theta|k, \hat{x}) = (p_s - p_r) \left[ K + \frac{p_r}{p_s - p_r} \theta - R(\phi|k) + \beta V(\phi'(\phi, \hat{x})) \right]$$

To discuss the complications endogenous interest rates introduce, assume for a moment reputation cannot change (this is, $\hat{x} = 1$ always and $\phi$ is never updated). In the previous section this would eliminate the reputation source of multiplicity. However, endogenous interest rates still may generate a finite number of equilibrium cutoffs $\theta^*(\phi)$. For example, there may be three equilibria, with high, intermediate and low $\theta^*(\phi)$ and $R(\phi|\theta^*)$.

To be more explicit, without reputation formation, the equilibrium with complete information is characterized by,

$$\Delta(\phi, \theta|k = \theta^*, \hat{x} = 1) = (p_s - p_r) \left[ K + \frac{p_r}{p_s - p_r} \theta^* - R(\phi|k) + \beta V(\phi') \right] = 0$$

To guarantee there is just one equilibrium we need a unique best response to $k(\phi)$ (this is $\frac{\partial \theta^*}{\partial k} < 1$). Taking derivatives with respect to $k$, $p_r \frac{\partial \theta^*}{\partial k} = (p_s - p_r) \frac{\partial R(\phi)}{\partial k}$ and considering $\frac{\partial R(\phi)}{\partial k} = \frac{R(p_s - p_r)^2 \phi n(k)}{Pr(c|\phi, k)^2}$ (where $n(k)$ is the normal density evaluated at $k(\phi)$), the condition for uniqueness is then,

$$\frac{R(p_s - p_r)^2 \phi n(k)}{p_r Pr(c|\phi, k)^2} < 1.$$ 

This condition is the most stringent at $\phi = 1$ and $Pr(c|\phi, k) = p_r$. Since $n(\theta) \leq \sqrt{\frac{1}{2\pi}}$ for all $\theta$, a sufficient condition for uniqueness is

$$\alpha < 2\pi \left[ \frac{p_r^3}{R(p_s - p_r)^2} \right]^2.$$ 

16
This condition basically requires a precision of fundamentals low enough such that interest rates do not change too quickly with changes in beliefs about the cutoffs $k(\phi)$ that firms follow. If condition 8 is not fulfilled, then we may have a finite multiplicity of equilibria just created by endogenous interest rates.

As discussed by Stiglitz and Weiss (1981), it is possible to select a unique equilibrium in this case just by assuming Bertrand competition, in which lenders first offer a rate and then firms choose the best offer. Assume there are two equilibria and all lenders charge the highest rate. In this case there are incentives for a single lender to deviate, offering the lower rate from the other equilibrium, attracting firms and still breaking even. Then, the lenders that effectively provide loans are the optimistic ones. This refinement rationalizes as the unique equilibrium the one with the lowest possible rate.

Now, consider again the environment with reputation formation. Since complementarity is introduced, the problem of multiplicity grows from a possible finite multiplicity to a continuum of multiple equilibria. Can we still apply the selection mechanism proposed by Stiglitz and Weiss (1981)? Yes, but only if the lenders who update reputation next period are the same than those who provide loans in the current period. Only in this uninteresting case in which a firm only obtains financing from a single lender all its life, there is not a meaningful complementarity problem.

However, if the lenders that set interest rates in the current period are different than the lenders who provide funds in the following period (or at least there is some chance lenders are not the same), then interest rates cannot be used to select an equilibrium. Assume again all lenders charge a high rate and then firms take risks with high probability. A single lender do not have incentives to deviate and charge a lower interest rate (contrarily to the Bertrand intuition) because the firm taking his loan still would be induced to take risks knowing that future lenders will likely observe a high aggregate default and will not update its reputation. Hence, even when Bertrand competition can solve multiplicity generated by pure moral hazard, it cannot solve the multiplicity created by complementarity in reputation formation.

In what follows, I assume the finite static multiplicity generated by interest rates is not an issue (condition 8 holds), so I can focus on the more interesting multiplicity created by reputation formation. Since for each pair $\phi$ and $k(\phi)$ there is a unique $R(\phi|k(\phi))$, the uniform limit dominance assumption can be restated in the following terms,

**Assumption 4 (Uniform Limit Dominance)**

- For each $\phi$ and $k(\phi)$, there is a lower bound $\underline{\theta}(\phi|k)$ such that $\Delta(\phi, \underline{\theta}|k, \hat{x} = 0) = 0$

- For each $\phi$ and $k(\phi)$, there is an upper bound $\overline{\theta}(\phi|k)$ such that $\Delta(\phi, \overline{\theta}|k, \hat{x} = 1) = 0$
This is Assumption 2, but with endogenous interest rates, such that \( \theta(\phi) \) and \( \bar{\theta}(\phi) \) depend on \( k(\phi) \). Like in that case, \( \theta(\phi|k) < \bar{\theta}(\phi|k) \) for all \( \phi \) and \( k(\phi) \), but the range \([\theta(\phi|k = \theta), \bar{\theta}(\phi|k = \bar{\theta})]\) is wider with endogenous interest rates than with exogenous interest rates.\(^\text{12}\)

Given this range of multiplicity, it is straightforward to show that we can still obtain uniqueness assuming imperfect information about fundamentals with small noise and applying global games techniques as before.

2.3.2 Endogenous Value Functions

To endogeneize value functions I develop now the full fledged finite horizon model as the terminal period \( T \) goes to infinity (\( T \to \infty \)) with endogeneous interest rates. First we discuss multiplicity under complete information and then uniqueness under incomplete information with noise approaching to the limit, \( \sigma \to 0 \).

With complete information, total discounted profits for a firm with reputation \( \phi \), that observes a fundamental \( \theta \), takes risks with probability \( x(\phi, \theta) \) and believes strategic firms follow a cutoff \( k(\phi) \) (which generates \( \hat{x}(\phi, \theta) \)), for all \( \theta \), are,

\[
\tilde{V}(\phi, \theta, x|k, \hat{x}) = x \left[ p_r[K - \theta - R(\phi|k)] + \beta p_r V(\phi'(\phi, \hat{x})) \right] \\
+ (1 - x)[p_s[K - R(\phi|k)] + \beta p_s V(\phi'(\phi, \hat{x}))]
\] (10)

Define,

\[ V(\phi, \theta|k, \hat{x}) = \max_{x \in [0, 1]} \tilde{V}(\phi, \theta, x|k, \hat{x}) \]

and \( V(\phi'(\phi, \hat{x})) = \int_0^\infty V(\phi', \theta'|k', \hat{x'})dN(\theta') \) is the expected continuation value for \( \phi' \), where the expectation is over all possible \( \theta' \) next period. \( V(\phi'(\phi, \hat{x})) \) is an element of a given set of expected continuation values \( \Upsilon' = \{ V(\phi') \}_{\phi = 0}^1 \) for all \( \phi \).

Now, I formally define the equilibrium in a given period, with endogenous interest rates and for any arbitrary set of expected continuation values. Since now we are analyzing the repeated game I explicitly denote variables for each period with a subscript \( t \). To save notation I just refer to generic \( \phi \) and \( \phi' \), since the proposition applies to each \( \phi \in [0, 1] \) at each period \( t \).

**Definition 2** A single period Markov perfect equilibrium in cutoff strategies consists of a (symmetric) strategy for the firms \( k_t(\phi) = \theta^*_t(\phi) : [0, 1] \to \mathbb{R} \), interest rates \( R_t(\phi|k_t) \) and posteriors

\(^{12}\)It is straightforward to show that 

\[
\bar{\theta}(\phi|k = \bar{\theta}) - \theta(\phi|k = \bar{\theta}) = \bar{\theta}(\phi) - \theta(\phi) + \frac{p_s - p_r}{p_r} \beta (R(\phi|k = \bar{\theta}) - R(\phi|k = \theta)) > \bar{\theta}(\phi) - \theta(\phi)
\]
\( \phi'(\phi, \hat{x}_t) : [0, 1] \times [0, 1] \rightarrow [0, 1] \), for each \( \phi \in [0, 1] \), such that, for a given set of expected continuation values \( \Upsilon' = \{ V(\phi') \}_{\phi'=0}^1 \).

- \( k_t(\phi) = \theta_t^*(\phi) \) defines the \( x_t^*(\phi, \theta_t) \in \arg \max_{x_t \in [0,1]} V(\phi, \theta_t, x_t|k_t, \hat{x}_t) \) for all \( \theta_t \).
- Lenders charge \( R_t(\phi|k_t) \) to obtain the risk-free rate \( R \) in expectation, where \( k_t(\phi) = \theta_t^*(\phi) \).
- \( \phi'(\phi, \hat{x}_t) \) is obtained using Bayes’ rule (equation (1)), where \( \hat{x}_t(\phi, \theta_t) = x_t^*(\phi, \theta_t) \) for all \( \theta_t \) is the updating rule that lenders must use if their beliefs are to be correct (this is, consistent with equilibrium strategies of a continuum of firms \( \phi_t \), which determine aggregate default \( D_t(\phi, x_t^*) \)).

The equilibrium in a game in which firms live for a finite time \( T \) and \( V_{T+1}(\phi) = 0 \), for all \( \phi \), is

**Definition 3** A **finite-horizon** Markov perfect equilibrium in cutoff strategies consists of a (symmetric) strategy for the firms \( \{ k_t(\phi) = \theta_t^*(\phi) \}_{t=0}^T \), interest rates \( \{ R_t(\phi|k_t) \}_{t=0}^T \), posteriors \( \phi'(\phi, \hat{x}_t) : [0, 1] \times [0, 1] \rightarrow [0, 1] \), and firms’ expected continuation values \( \{ V_t(\phi) \}_{t=0}^T \) for each \( \phi \), such that

- A single period Markov perfect equilibrium exists in each \( t \in \{0, 1, ..., T\} \).
- \( V_t(\phi) = \int_{-\infty}^{\theta_t^*(\phi)} \hat{V}_t(\phi, \theta_t, x_t = 1|\theta_t^*, \hat{x}_t = 1) \, d\mathcal{N}(\theta_t) + \int_{\theta_t^*(\phi)}^{\infty} \hat{V}_t(\phi, \theta_t, x_t = 0|\theta_t^*, \hat{x}_t = 0) \, d\mathcal{N}(\theta_t). \)
- At each period \( t \), \( \theta_t^*(\phi) \) is determined by the corresponding \( V_{t+1}(\phi') \).

In the finite-horizon game with endogenous continuation values the range of multiple equilibria cutoffs in each period widens. Since multiplicity exists in every single period, multiple streams of future expected continuation values (consistent with multiple equilibria in future periods) can be used to construct the differential of taking safe projects. By introducing extreme streams of continuation values determined by the highest (\( \bar{\Upsilon}' \)) and the lowest (\( \underline{\Upsilon}' \)) probability of risk-taking in all future periods for all reputation levels, we can construct extreme bounds \( \underline{\theta}(\phi|\bar{\Upsilon}') < \theta(\phi|\bar{\Upsilon}') \) and \( \bar{\theta}(\phi|\underline{\Upsilon}') > \theta(\phi|\underline{\Upsilon}') \) such that the region of multiplicity in a given period is wider when considering the multiplicity of equilibria in future periods.

Again, by assuming incomplete information about fundamentals, I show uniqueness in this full model, which is characterized by a unique sequence of equilibrium cutoffs as signals become very precise. Also, as the last period goes to infinity, there is a unique limit to the sequence of perfect Markovian equilibrium for the finite game.

The following Proposition states that, based on the boundary condition \( V_{T+1}(\phi) = 0 \) for all \( \phi \), expected continuation values \( V_t(\phi) \) are well-defined in each period \( t \) for all reputation levels \( \phi \) and then, a unique equilibrium exists in the finite horizon game as \( \sigma \rightarrow 0 \). In order to solve this finite dynamic global game, I follow the literature from Morris and Shin (2003), Toxvaerd (2008), Steiner (2008) and Giannitsarou and Toxvaerd (2009).
Proposition 3 (Uniqueness in a Finite Horizon Game)

For each reputation $\phi$, in each period $t$, as $\sigma \to 0$, a unique cutoff signal $z_t^*(\phi)$ exists such that the expected differential gains from playing it safe $\Delta_t(\phi, z_t|z_t^*) = 0$ for $z_t = z_t^*(\phi)$, $\Delta_t(\phi, z_t|z_t^*) > 0$ for $z_t > z_t^*(\phi)$, and $\Delta_t(\phi, z_t|z_t^*) < 0$ for $z_t < z_t^*(\phi)$, where $\Delta_t(\phi, z_t|z_t^*)$ and $z_t^*(\phi)$ (as defined in Proposition 2) depend on $V_{t+1}(\phi)$ and $V_{t+1}(\phi', x_t^* = 0)$). Continuation values $V_t(\phi)$ are well-defined and, given the boundary condition $V_{T+1}(\phi) = 0$, are recursively determined by

$$V_t(\phi) = \int_{-\infty}^{z_t^*(\phi)} p_t[K - \theta_t - R_t(\phi|z_t^*) + \beta V_{t+1}(\phi)]v(\theta_t)dN(\theta_t)$$

$$+ \int_{z_t^*(\phi)}^{\infty} p_t[K - R_t(\phi|z_t^*) + \beta V_{t+1}(\phi'(\phi, x_t^* = 0))]v(\theta_t)dN(\theta_t)$$

(11)

The Appendix contains a proof for any prior distributions of fundamentals with a c.d.f. $V(\theta)$ strictly increasing over a connected set that includes both dominance regions from Assumption 4 and any distribution of signals with a c.d.f. strictly increasing over a connected set.

The next proposition establishes that under certain conditions, in particular when the variance of the fundamentals distribution is large enough, it is possible to treat the infinite horizon game as a limit of the finite-horizon game.

Proposition 4 If $V_T(\phi) \to \overline{V}(\phi)$ as $T \to \infty$ and $\sigma \to 0$, then a cutoff $z^*(\phi)$ exists for each $\phi$ that is a unique limit to the sequence of cutoffs $\{z_t^*(\phi)\}_{t=0}^T$ of the finite-horizon Markov perfect equilibria described in Proposition 3.

Proof In the Appendix, I show the sufficient condition for $V_T(\phi) \to \overline{V}(\phi)$ for all $\phi$ as $T \to \infty$ is $\alpha < 2\pi \left[\frac{\beta p_{1z}}{R(p_{2z} - p_{1z})}\right]^2 \left[1 - \frac{\beta p_{1z}}{\beta p_{2z}}\right]^2$ for all $\theta \in \mathbb{R}$. Having shown uniqueness for an arbitrary finite-horizon $T$, I must show that the same reasoning is extended as $T \to \infty$. First, note that the values of taking safe and risky projects are bounded and well-behaved monotone functions of $T$, as they converge to a fixed point $\overline{V}(\phi)$ when $T \to \infty$. Second, as defined in equation (3), $\Delta_t(\phi, z_t|z_t^*)$ also converges to a unique limit as $T \to \infty$. Then $\lim_{T \to \infty} z_t^*(\phi|T) = z^*(\phi)$ for all $t$, where the dependence on $T$ indicates the length of the game.

Intuitively, if we solve backwards from some $T$ by using as a boundary condition the fixed point $V_{T+1}(\phi) = \overline{V}(\phi)$, rather than $V_{T+1}(\phi) = 0$, then we obtain a unique $z^*(\phi)$ for each $\phi$ in all periods $t < T$. This matters because, as $\sigma \to 0$, in a period $t$ far enough from the last period $T \to \infty$, unique cutoffs and ex ante probabilities of risk-taking are constant over time for each reputation level $\phi$.

---

13Note the sufficient condition for convergence is more stringent than the sufficient condition for uniqueness (equation 9) when $\beta p_{2z} > 0.5$. 

20
3 Establishing the Fragility of Reputation Concerns

Now I use the unique equilibrium from the last proposition to show how concerns about reputation impose discipline and reduce risk-taking, and how this discipline is fragile and can suddenly break down with small and not obvious changes in economic fundamentals. Furthermore, when discipline collapses, it collapses for a bunch of different firms with intermediate and good reputation, generating a clustering of risk-taking that can have far-reaching aggregate negative consequences. I demonstrate these results first in a formal abstract analysis and then in an illustrative numerical simulation of the model.

3.1 The Formal Analysis

Reputation concerns imposes discipline to strategic firms, discouraging them to take inefficient risky projects. This discipline is however fragile and may suddenly collapse, creating clustering of risk-taking in the aggregate.

3.1.1 Reputation Imposes Discipline

The next proposition shows why firms are concerned about constructing and maintaining good reputations. A better reputation for a firm implies a lower ex-ante probability of risk-taking, hence the firm has to pay lower interest rates and enjoy higher continuation values. I focus on the case $T \to \infty$ and $\sigma \to 0$ (as in Proposition 4), in which cutoffs, interest rates and value functions are time independent functions of $\phi$. The proof is in the Appendix.

**Proposition 5** (Reputation, Risk-Taking, Lending Rates and Continuation Values)

As a firm’s reputation $\phi$ improves

i) Its ex ante probability of risk-taking monotonically decreases (that is, $\frac{d\mathcal{N}(z^*(\phi))}{d\phi} < 0$ for all $\phi \in [0, 1]$).

ii) The interest rate it faces monotonically decreases (that is, $\frac{dR(\phi)}{d\phi} < 0$ for all $\phi \in [0, 1]$).

iii) Its continuation value monotonically increases (that is, $\frac{dV(\phi)}{d\phi} > 0$ for all $\phi \in [0, 1]$).

Even when lenders always prefer firms to play it safe, since $p_s > p_r$ independently of $\theta$, sometimes it is efficient for firms to take risks.

**Lemma 1** (Inefficient Risk Taking)

There is always a region of inefficient risk-taking $\theta \in [\theta_E, z^*(\phi)]$ where $\theta_E \equiv -\frac{p_s - p_r}{p_r} [K + \beta V(1)]$ is the cutoff above which it is efficient that strategic firms take safe projects.
If actions and types are observable, risk taking is efficient only if 
\[ p_s(K - \frac{1}{p_s} + \beta V(1)) < p_r(K - \theta - \frac{1}{p_r} + \beta V(1)) \].
Then, there is a cutoff \( \theta_E \equiv \frac{p_r - p_s}{p_r} [K + \beta V(1)] \) above which it is inefficient that strategic firms take risky projects. Information frictions add a moral hazard component that induces excessive risk-taking. It is straightforward to see, from equation 5, that \( z^*(\phi) \geq \theta_E \) for all \( \phi \). This implies there is always a region of inefficient risk-taking \( \theta \in [\theta_E, z^*(\phi)] \).

We show next that reputation concerns relax this inefficiency by reducing \( z^*(\phi) \). In the next Proposition I prove reputation concerns shorten the region \( \theta \in [\theta_E, z^*(\phi)] \) by comparing an environment with reputation to an artificial environment in which firms are not concerned at all about their reputation, simply because the reputation cannot change.\(^{14}\)

**Proposition 6 (Reputation Concerns Reduce Inefficient Risk-Taking)**

Define \( \tilde{z}^*(\phi) \) the risk-taking cutoffs when reputation is not a concern (this is, when reputation cannot change). Reputation concerns reduce the ex-ante probability of risk-taking (this is, \( z^*(\phi) < \tilde{z}^*(\phi) \)) for all \( \phi \in (0, 1) \) and does not change it (this is \( z^*(\phi) = \tilde{z}^*(\phi) \)) only if \( \phi \in \{0, 1\} \).

**Proof** With reputation concerns, \( z^*(\phi) \) is determined by equation 5 in the following way:

\[
z^*(\phi) = -\frac{p_s - p_r}{p_r} [K - R(\phi|z^*) + \beta V(\phi', \hat{x} = 0.5)]
\]

Without reputation concerns, \( \tilde{z}^*(\phi) \) is determined by

\[
\tilde{z}^*(\phi) = -\frac{p_s - p_r}{p_r} [K - R(\phi|\tilde{z}^*) + \beta V(\phi)]
\]

since the restriction that reputation cannot change is exactly the same as assuming \( \hat{x} = 1 \).

Take \( \phi \in (0, 1) \). Since \( V(\phi', \hat{x} = 0.5) > V(\phi) \) from Proposition 5, then \( z^*(\phi) < \tilde{z}^*(\phi) \). This implies \( R(\phi|z^*) < R(\phi|\tilde{z}^*) \), which reinforces that \( z^*(\phi) < \tilde{z}^*(\phi) \). For \( \phi \in \{0, 1\} \), \( V(\phi', \hat{x} = 0.5) = V(\phi) \) and \( z^*(\phi) = \tilde{z}^*(\phi) \).

Q.E.D.

### 3.1.2 The Reputational Discipline is Fragile

Now I can go further and establish that the existence of reputation concerns may suddenly collapse by creating sudden changes in aggregate risk-taking behavior without obvious changes in fundamentals. First, we can characterize firms’ equilibrium cutoffs \( z^*(\phi) \) for different reputation levels \( \phi \). The next lemma shows that the concerns for reputation formation convexifies the schedule of those cutoffs.

\(^{14}\)For example, credit histories are erased or lenders cannot observe the age of the firm.
Lemma 2  Reputation concerns convexify the schedule of cutoffs (that is, \( \frac{d^2z^\ast(\phi)}{d\phi^2} > \frac{d^2\tilde{z}^\ast(\phi)}{d\phi^2} \) for all \( \phi \in [0, 1] \), where \( \tilde{z}^\ast(\phi) \) are the cutoffs without reputation concerns). Furthermore, there are always signals of the firm’s type precise enough (\( \frac{p^s}{p^c} \) high enough) such that the schedule of cutoffs is strictly convex (this is, \( \frac{d^2z^\ast(\phi)}{d\phi^2} > 0 \) for all \( \phi \)).

The proof is in the Appendix but I use Figure 3 to convey the intuition. At each period, each firm is a point in the graph, a combination of reputation \( \phi \) and a signal \( z \), and follows a cutoff \( z^\ast(\phi) \). Assume, for example, that without reputation concerns, the schedule of cutoffs (\( \tilde{z}^\ast(\phi) \)) is linear in \( \phi \) (as in the figure). As we know from Proposition 6, reputation concerns reduce risk-taking (that is, reduce cutoffs from \( \tilde{z}^\ast(\phi) \) to \( z^\ast(\phi) \)) for all \( \phi \). However, the strength of this force is not the same across reputation levels. I start discussing \( \phi = 0 \) and gradually follow for higher levels of \( \phi \).

Firms with reputation \( \phi = 0 \) cannot change their reputation, which means that the cutoff for risky behavior is the same, with and without reputation concerns (\( z^\ast(0) = \tilde{z}^\ast(0) \)). As we consider higher levels of \( \phi \), firms have higher reputation concerns, which rapidly reduce cutoffs. This effect achieves the maximum at \( \phi_M \), where reputation changes the most. After this point, further increments in \( \phi \) reduce the role of reputation concerns. At the extreme \( \phi = 1 \), reputation cannot improve further, so the cutoff is again the same with and without reputation concerns (\( z^\ast(1) = \tilde{z}^\ast(1) \)). For firms with poor reputation two types of incentives – continuation and reputation formation – reinforce each other in reducing risk-taking. For firms with good reputation, while continuation effects increase with \( \phi \), reputation formation effects become less important.

Figure 3: Reputation and Cutoffs for Risk-Taking Behavior
To be more concise, classify firms in three bins: Firms with poor, intermediate and good reputation. Poor reputation firms are prone to take risks because their gains from surviving are low (they will have to pay high interest rates in the future) and they cannot change their reputation much. Intermediate reputation firms want to take safe projects not because they can lose a lot if they die, but because they can improve their reputation a lot if they survive. Good reputation firms want to take safe projects for the reverse reason, not because they can improve their reputation a lot if they survive but because they can lose a lot if they die.  

This leads us to a final, crucial proposition:

**Proposition 7** (Fragility of reputation and clustering of risk-taking)

i) Reputation is fragile at a firm level. For highly precise signals about fundamentals \((\sigma \to 0)\), small changes in fundamentals \(\theta\) around the optimal cutoff \(z^*(\phi)\) induce a change in risk-taking, clustered among firms with reputation level \(\phi\).

ii) Reputation is fragile at an aggregate level. For strong reputation concerns \((p_s \to p_r \text{ high enough})\), small changes in fundamentals \(\theta\) around the optimal cutoff \(z^*(\phi_M)\) for intermediate reputation firms induce a change in risk-taking, clustered among firms with intermediate and good reputation levels.

The first part of this proposition is just a result from global games. If \(\theta < z^*(\phi)\), then when the signal noise goes to zero, almost all firms with reputation level \(\phi\) receive a signal \(z < z^*(\phi)\) and decide to take risks. Hence, reputation concerns are fragile in the sense that small changes in fundamentals around \(z^*(\phi)\) induce sudden changes in risk-taking for firms with reputation \(\phi\). Our equilibrium selection creates a clustering of risk-taking among firms with the same reputation level.

The second part of proposition 7 is a corollary of Lemma 2 for a given distribution of reputation levels. When fundamentals are strong enough (high \(\theta\)), small variations do not induce firms of different reputation to modify their risk-taking behavior. Contrarily, when fundamentals are weak enough (low \(\theta\)), small variations may induce firms of different reputation to modify their risk-taking behavior under the same shock. Changes around weak fundamentals generate clustering of risk-taking among firms with different reputation levels. This fragility is generated at an aggregate level by learning primitives. Figure 4 illustrates fragility both at a firm and at an aggregate level.

While reputation concerns reduce excessive risk-taking, these effects are fragile and their breakdown can lead to sudden and isolated clusters of risk-taking, large spikes in aggregate default, and large losses for lenders.

\(^{15}\text{This fragility is robust to continuous types and continuous actions, as long as supports are bounded.}\)
Before illustrating reputation fragility with a simulation exercise (which also shows how to solve the model numerically), it is important to highlight that clustering depends not only on the convex schedule of cutoffs, but also on the distribution of reputation in the market. In particular, a distribution with a large mass of intermediate and good reputation firms strengthens the results. In the numerical exercise below, I derive the endogenous stationary distribution of reputation and show it is indeed skewed toward intermediate and good reputation levels, where cutoffs are similar, reinforcing the aggregate effects of clustering.

3.2 A Numerical Exercise

Here I use a numerical example to illustrate the results just established in the formal analysis: reputation concerns reduce inefficient risk-taking, but may also generate large changes in aggregate risk-taking in response to small changes in aggregate fundamentals.

3.2.1 The Exercise

As in the model, payoffs are linear and risky projects are more tempting for low values of $\theta$. Short-term cash flows are $\Pi_s = K$ and $\Pi_r = K - \theta$, where $\theta \sim \mathcal{N}(\mu, 1/\alpha)$. I assume $K = 1.5$, $\mu = -0.4$ and $\alpha = 25$, which means the return of the safe project is 50%, the average return of the risky project (both from pledgable and non-pledgable sources) is just 10% and

---

16To derive theoretically the endogenous distribution of reputation is beyond the scope of the paper, however the extension is feasible and interesting. The distribution (independent of cutoffs $z^*(\phi)$) depends purely on entry assumptions, as in Atkeson, Hellwig, and Ordonez (2010).
the return of the risky project is volatile – the range of returns at two standard deviations is $[-80\%, 100\%]$. Additionally, I assume the probability of continuation is $p_s = 0.9$ when playing it safe, $p_r = 0.7$ when taking risks, a discount factor $\beta = 0.95$ and a risk-free interest rate of $R = 1$. These parameters have been chosen to fulfill four conditions.\footnote{Parameters are chosen purely for illustrative purposes. Still, results are highly robust to alternative choices.}

- Risk-taking is almost never efficient. It happens for fundamentals below $\theta_E = -4$, which occurs ex-ante with a probability of only $0.001\%$.

- Pledgable short-term cash flows are higher than equilibrium interest rates ($K > \frac{1}{p_r}$). Hence, firms can always pay back debts if they continue.

- Without reputation concerns, interest rates are not convex in $\phi$, so I can show the forces of reputation concerns in convexifying them.

- Conditions for uniqueness from interest rates (equation 9) and convergence of continuation values to a fixed point (from Proposition 4) are fulfilled.

Finally, I introduce an additional set of signals, that are correlated to projects and observable if the firm continues. Assume, for example, a firm “grows” or “produces a new idea” (generates a “good signal”) more likely if the project was safe ($\alpha_s = 0.8 > \alpha_r = 0.4$). I make this extension for two reasons. First, it shows the flexibility of the model. Second, it breaks down the perfect correlation between reputation and age that arises when the only positive signal about the firm’s type is continuation.

Using these parameters I compare the results of the basic model (with reputation concerns) with an artificial situation in which the firm’s reputation cannot change (without reputation concerns), and identify how the discipline that reputation imposes – reducing inefficient risk-taking – is fragile.

I proceed in two steps. First I compute the limit of the schedule of cutoffs, with and without reputation concerns, for different reputation levels, as $T \to \infty$ and $\sigma \to 0$. Then I simulate realizations of fundamentals for 100 periods and I aggregate the strategic risk-taking behavior of firms that follow those cutoffs. The computational procedure is described in the Appendix.

### 3.2.2 Discipline from Reputation

Figure 5 shows, for each reputation level $\phi$, cutoffs with and without reputation concerns ($z^*(\phi)$ and $\tilde{z}^*(\phi)$, respectively) and ex ante probabilities of risk-taking ($N(z^*(\phi))$ and $N(\tilde{z}^*(\phi))$).
that apply to any period when $T \to \infty$. As stated in Proposition 5, the probability of risk-taking decreases with reputation. Furthermore, as stated in Proposition 6, for all reputation levels, the probability that firms take risks is lower with reputation concerns. For example, the ex ante probability that a firm with a reputation level $\phi = 0.4$ takes risks is only 4% with reputation concerns but 50% without reputation concerns. By construction, risk-taking is almost never efficient, hence the gap between the two curves in the second plot of Figure 5 shows the reduction in ex ante probabilities of inefficient risk-taking that reputation generates.

Figure 5: Reputation Concerns Reduce the Ex Ante Probability of Risk-Taking

Figure 6 shows the expected continuation values and lending rates for different reputation levels $\phi$. Also as stated in Proposition 5, continuation values increase and lending rates decrease with reputation. Furthermore, firms have higher expected continuation values and pay lower interest rates when reputation is a concern.

Figure 6: Reputation Concerns Increase Continuation Values and Reduce Lending Rates
3.2.3 Fragility of Reputation

Here I simulate fundamental realizations during 100 periods and aggregate strategic risk-taking in the economy. The fragility of reputation concerns can create sudden and isolated events of clustering of risk-taking, with spikes in aggregate defaults and lenders’ losses, without obvious changes in aggregate fundamentals.

To perform the simulations I make three additional assumptions. First, I’m aggregating only strategic firms, whose behavior changes with fundamentals. If all of them play safe, the aggregate probability of default is 10% (since \( p_s = 0.9 \)). If all of them take risks, the aggregate probability of default is 30% (since \( p_r = 0.7 \)).\(^{18}\) Second, I aggregate over firms, first assuming an invariant uniform reputation distribution in all periods and then an evolving reputation distribution. Finally, I choose 100 periods where risk-taking is never efficient (this is, no fundamental realization is below \(-4\) in any period).

The first plot of Figure 7 shows aggregate default probabilities in the 100 periods for a fixed uniform reputation distribution. Without reputation concerns, aggregate default closely follows changes in fundamentals. With reputation concerns, aggregate default is less sensitive to changes in fundamentals. It is low in general and spikes when conditions weaken enough. This comparison reveals that reputation concerns in general reduce inefficient risk-taking (discipline from reputation) but also creates large spikes in default rates without obvious changes in fundamentals (fragility of reputation). It is clear from Figure 5 that reputation effects convexify the schedule of cutoffs and generate clustering of risk-taking among firms with intermediate and good reputations when fundamentals go below values around \(-2\).

In the second plot of Figure 7, I allow the reputation distribution to evolve over time, starting from a uniform distribution in the initial period. I assume that entrants replace failing firms with an exogenous prior reputation \( \phi_0 = 0.5 \). This distribution evolves biasing towards good reputations, which reinforces not only the discipline of reputation concerns in the aggregate (less inefficient aggregate default at most periods), but also the fragility of reputation concerns in the aggregate (spikes in aggregate default rates are even more isolated and sensible to small changes in fundamentals). In essence, the market becomes mostly populated by firms with intermediate and good reputations, whose behavior react similarly to fundamentals.

Finally, Figure 8 shows aggregate net returns to lenders.\(^{19}\) When fundamentals weaken enough, we see that returns decline catastrophically, since most firms, regardless of their reputations, take risks. Since lenders charge low rates to good reputation firms, sudden losses

---

\(^{18}\)Risky firms always have a default probability of 30%. The fraction of these firms in the aggregate is lower than \( \phi_0 \), since in expectation they die more frequently than strategic firms.

\(^{19}\)First I obtained individual net returns for each reputation level (computed by the lending rate charged to \( \phi \) multiplied by the true probability of no default minus the risk-free rate). Then I calculated the weighted sum of individual returns to obtain aggregate net returns.
Figure 7: Simulated Default Probability with Fixed and Evolving Reputation Distribution

Figure 8: Simulated Aggregate Net Returns to Lenders

are large. With reputation concerns, lending rates are lower and more stable, while lenders losses are greater when they rarely occur. Reputation concerns reduce the frequency of crises, but they magnify lenders’ losses when crises do occur.

This exercise has illustrated that reputation concerns are effective in imposing a discipline that is fragile. Reputation concerns deter excessive risk-taking in general, but also generate sudden clusters of risk-taking behavior, characterized by larger defaults and losses to lenders, even without noticeable changes in fundamentals. Even when the exercise is based on arbitrary parameters, results are highly robust, as long as reputation concerns remain important and conditions for uniqueness and convergence are guaranteed.
4 Empirical Implications and Potential Extensions

In this concluding Section I summarize the empirical implications and testable predictions of the model, highlighting how they relate to the existing evidence. I also discuss how to extend the model to accommodate other sources of moral hazard and industries, with especial reference to the recent financial crisis.

4.1 Empirical Implications

Implication 1: (More reputable firms take less risks and pay less rates than less reputable firms) This is the set of predictions from Proposition 6. Evidence has been discussed by Kose and Nachman (1984) for investment behavior and more recently by Chatterjee and Rios-Rull (2011) for households’ credit. More generally, there is evidence about a positive correlation between reputation, performance and compensations among investment banks (Miles and Miller (2000), Jackson (2005), Fang (2005)), IPOs underwriting (Carter, Dark, and Singh (1998), An and Chan (2008)) and syndicated loans (Tykvova (2007), Gopalan, Nanda, and Yerramilli (2010)). This last paper, for example, finds that a large scale bankruptcy filing by a lead-arranger’s borrower is indicative of poor performance by the lead arranger, damaging its reputation. In the years after a loss of reputation the lead manager keeps larger fractions of the loans it arranges, it is less likely to syndicate a new loan and it is less likely to attract lenders to participate in its syndicates.

Implication 2: (Risk-Taking clustering) This is the prediction from Proposition 7. Risk-taking and defaults cluster around recessions, to an extent that cannot be explained just from a weakening in fundamentals, as shown by Campbell et al. (2001) and Das et al. (2007). Furthermore, Menkhoff, Schmidt, and Brozynski (2006) documented that high reputation fund managers cluster more in their projects, while decisions of less reputable managers are less correlated.

Implication 3: (Reputation changes less for highly reputable and non reputable firms) This is the relation between $\phi_{t+1}$ and $\phi_t$ highlighted in Figure 1, which generates the convexification in Lemma 2. Among syndicated loans, Gopalan, Nanda, and Yerramilli (2010) find that borrower bankruptcies seem to have little impact on lead arrangers with very dominant or very poor market positions, where market positions are positively correlated with reputation.

Implication 4: (Reputation changes less during periods of clustering) This is the relation between $\phi_{t+1}$ and $\hat{x}$ highlighted in Figure 1, which generates the multiplicity in Proposition 1. This is supported empirically by Nickell, Perraudin, and Varotto (2000), Bangia et al. (2002), Altman and Rijken A. (2006), and Ordonez (2008) using data on corporate credit-rating transitions over the business cycle. This pattern has been also observed during the latest financial
crisis: the July 2008 Moody’s credit policy report shows the rating volatility decreased significantly, almost 50% with respect to historical averages. Furthermore, among syndicated loans, Gopalan, Nanda, and Yerramilli (2010) find that beliefs do not change dramatically in years in which several other lead arrangers also experience borrower bankruptcies.

4.2 Potential Extensions

Reputation relaxes moral hazard, but its effects are fragile. We focused above on a particular manifestation of moral hazard, which is risk-taking by debt issuers operating in the productive sector. However the conclusions also extend naturally to other forms of moral hazard, such as costly and unobservable effort provision and to other industries, in particular the financial market.

One of the main vulnerabilities that spread and magnify the recent global financial crisis seems to have been a poor risk management by large and reputable security issuers and investors.\(^\text{20}\) We can modify the basic setting to accommodate risk management practices in the financial sector. Consider, for example, an environment where moral hazard is determined by monitoring efforts of mortgage originators. There are two types of originators: Bad types cannot monitor the riskiness of a new mortgage and Good types have the choice to monitor at a given cost \(C\). Monitoring increases the probability of the originator’s survival (say from \(p_r\) to \(p_s\)). Monitoring here is akin to safe projects in the basic setting: a non-observable costly action that increases the probability of survival, a good signal about the firm’s type.

Denote \(P(\phi, \theta)\) the originators’ differential profits from monitoring. Assume these profits increase with reputation \(\phi\) (investors are willing to charge lower rates to originators who, in expectation, are more careful in issuing mortgages) and increase with an aggregate fundamental \(\theta\) (for example, expected depreciation of housing prices increase the liquidity costs from repossession in case of mortgage default, and render monitoring more beneficial). As in our main model, assume \(\theta\) (or a signal about it) is observable before monitoring decisions.

We can define the differential gains from monitoring in the same way that we defined the differential gains from taking safe projects in equation (3),

\[
\Delta(\phi, \theta|\bar{x}) = (p_s - p_r) \left[ X(\phi, \theta) + \beta V(\phi) + \beta[V(\phi'(\phi, \bar{x})) - V(\phi)] \right]
\]

where \(X(\phi, \theta) = P(\phi, \theta) - \frac{C}{p_s - p_r}\), increasing both on \(\phi\) and \(\theta\). Note equation (3) has exactly the same structure, with \(X(\phi, \theta) = K - R(\phi) + \frac{p_r}{p_s - p_r} \theta\). Naturally, value functions \(V(\phi)\) are

endogeneous, but in both settings they have the same basic properties that are just inherited from learning (i.e., they are monotonically increasing in $\phi$ and convexified by the formation of reputation). Since the solution ultimately hinges on the properties of $\Delta(\phi, \theta)$, these problems are effectively isomorphic.

This is just one of many possible applications. The model can potentially accommodate other situations where an unobservable effort is costly but increases the probability of continuation, the market values this higher probability and there is a $\theta$ that affect $\Delta(\phi, \theta)$ monotonically. In particular, the model can accommodate some other key forces of the recent crisis.

First, there was excessive leverage previous the crisis. If we interpret unobservable leverage as a risky alternative for investors and expected housing price as $\theta$ – which increases the temptation of taking levered positions – as in Geanakoplos (2009), our model predicts a sudden movement towards unobservable leverage and excessive use of derivatives by large and reputable firms, as expected housing prices increase.

Second, there was a collapse in loan syndication. A loan syndicate comprises a lead arranger that originates the loan, and participant lenders that fund parts of the loan, and delegate screening and monitoring of the borrower to the lead arranger. These activities are costly to the lead arranger and non-observable to participant lenders. This framework creates a moral hazard problem isomorphic to our setting.

Third, there is a widespread view that deregulation of financial markets during the years leading to the financial crisis was responsible for excessive risk-taking. As documented by Keeley (1990), the main changes in regulation since the 1980s led to a decline in bank charter values, through reductions in entry restrictions, easier conditions for competition and interest rate ceilings. This is also consistent with my model when interpreting $\theta$ as the level of competition faced by financial institutions, which reduces profits, the long term gains from maintaining reputation, and then discipline.

Finally, Plantin (2009) shows that liquidity spreads respond to default risk highly non-linearly in the presence of learning by holding securities, providing a new interpretation for the magnitude and momentum of the recent subprime crisis. This paper also highlights the importance of coordination failures in explaining sudden financial breaks down. Unlike his, my work focuses on the role of reputation in generating such coordination failures and can

---

21Fostel and Geanakoplos (2010) show leverage of large financial institutions (measured as the average down payment for 50% lowest down payment subprime new borrowers) suddenly increases after housing prices (measured by Case Shiller National Home Price Index) reach a certain level and declines when expected housing prices decline. Bernanke specially highlighted the unobservable part of the excessive exposure to derivatives, most notably by the insurance company American International Group (AIG).

then be useful to study avenues and conditions under which aggregate confidence based on reputation can flourish and collapse over time.

Even when these extensions are purely suggestive, further research on the interactions of aggregate fundamentals and short-term payoffs can shed light on the specific role of reputation in magnifying the recent financial crisis. For example, versions of this model can be used to understand "shadow banking", a system sustained by confidence and reputation incentives (Gorton and Metrick (2009) and Gorton (2010)). Recent applications of this model to special purpose vehicles and secondary loan markets are Ordonez (2010) and Chari, Shourideh, and Zetlin-Jones (2009) respectively.

References


A Appendix

A.1 Proof of Proposition 1

First we will prove two lemmas that describe single crossing properties to identify a unique cutoff in the set of fundamentals ($θ$) and in the set of beliefs ($x$).

**Lemma 3** (Fundamental single crossing) For every reputation level $φ ∈ (0, 1)$, fix a $x ∈ [0, 1]$ for all $θ$. There exists a unique $θ^* ∈ [θ(φ), β(φ)]$ such that $Δ(φ, θ| x) < 0$ for $θ < θ^*$, $Δ(φ, θ|x) = 0$ for $θ = θ^*$, and $Δ(φ, θ|x) > 0$ for $θ > θ^*$. Furthermore, $θ^*$ is increasing in $x$. For $φ = \{0, 1\}$, $θ^* = θ(φ) = β(φ)$ for any $x$.

**Proof** For $φ ∈ (0, 1)$, by Assumption 4 and since $\frac{∂Δ(φ, θ|x)}{∂θ} > 0$, there is a unique $θ^*$ such that $Δ(φ, θ^*| x) = 0$. Since $x ∈ [0, 1]$, $θ^* ∈ [θ(φ), β(φ)]$. Since $\frac{∂Δ(φ, θ|x)}{∂θ} < 0$, $θ^*$ is increasing in $x$ (this is, if $x' > x$, $Δ(φ, θ^*| x') = 0$ at $θ^* > θ^*$). Finally, for $φ ∈ \{0, 1\}$, $x$ is irrelevant for updating, then $θ^* = θ(φ) = β(φ)$ for any $x$.

Q.E.D.

**Lemma 4** (Belief single crossing) For every reputation level $φ ∈ (0, 1)$, fix a $θ ∈ [θ(φ), β(φ)]$. There exists a unique $x^* ∈ [0, 1]$ such that $Δ(φ, θ|x) > 0$ for $x < x^*$, $Δ(φ, θ|x) = 0$ for $x = x^*$ and $Δ(φ, θ|x) < 0$ for $x > x^*$. Furthermore, $x^*$ is increasing in $θ$. For $φ = \{0, 1\}$, any $x ∈ [0, 1]$ delivers $Δ(φ, θ|x) = 0$.

**Proof** Fix a $θ ∈ [θ(φ), β(φ)]$. For $φ ∈ (0, 1)$, $\frac{∂Δ(φ, θ|x)}{∂θ} < 0$, there is a unique $x^*$ such that $Δ(φ, θ|x^*) = 0$. Since $\frac{∂Δ(φ, θ|x)}{∂θ} > 0$, $x^*$ is increasing in $θ$ (this is, if $θ' > θ$, $Δ(φ, θ'| x') = 0$ at $x' > x$). Finally, for $φ ∈ \{0, 1\}$, $x$ is irrelevant for updating, hence delivers a $Δ(φ, θ|x) = 0$ for $θ = θ(φ) = β(φ)$.

Q.E.D.

The Proposition follows directly from Lemmas 3 and 4. A cutoff $k(φ)$ is an equilibrium strategy only if it is a best response for any realization of the fundamental $θ$. Take a cutoff $k(φ) = θ^*(φ)$ such that $θ^*(φ) ∈ (θ(φ), β(φ))$. Such a $θ^*(φ)$ is guaranteed by Assumption 4. From cutoff strategies we know that $x(φ, θ) = 0$ for all $θ > θ^*(φ)$ and $x(φ, θ) = 1$ for all $θ < θ^*(φ)$. From Lemma 4, at $θ^*(φ)$, indifference occurs at some $0 ≤ x^*(φ, θ^*) ≤ 1$.

The cutoff $θ^*(φ)$ is an equilibrium because, for all $θ > θ^*(φ)$, $Δ(φ, θ|θ^*) > 0$, and hence it is optimal for the firm to play safe (i.e., $x(φ, θ) = 0$). Similarly, for all $θ < θ^*(φ)$, $Δ(φ, θ|θ^*) < 0$, and hence it is optimal for the firm to take risks (i.e., $x(φ, θ) = 1$). Finally, since the conditions for equilibrium are both $Δ(φ, θ^*| x) > 0$ and $Δ(φ, θ^*| x) < 0$, an arbitrarily close fundamental $θ^* + ε$ (with $ε → 0$) is also an equilibrium cutoff.

A.2 Proof of Proposition 3

**Proof** In a first part, I prove uniqueness in a single period $t$ for arbitrary distributions and a fix set of continuation values. In particular, I assume $z_i = θ + σε_i$, where $ε_i ∼ F$, with density
\( f, \) mean 0, c.d.f. \( F \) increasing over a connected set and a monotone likelihood ratio property, this is for all \( a > b, \frac{f(a-\theta)}{f(b-\theta)} \) is increasing in \( \theta \). Note that conditional on \( \theta \), signals \( z_i \sim F \left( \frac{z_i-\theta}{\sigma} \right) \).

Fundamentals are distributed with mean \( \mu \), a density \( v(\theta) \) and c.d.f. \( V(\theta) \) strictly increasing over a connected set that includes the dominance regions.

For this part, to reduce notation, I do not introduce explicitly the time subscript \( t \). I proceed in four steps. First, I derive the posterior density and distribution of \( \theta \) given a signal \( z \). Second, I prove there is a unique signal \( z^*(\phi) \) that makes a strategic firm \( \phi \) indifferent in expectation between taking risk or not. Third, I show that, for any \( z \), using \( z^*(\phi) \) is a best response when the prior about \( \theta \) follows a uniform distribution on the real line and lenders believe \( z^*(\phi) \) is the equilibrium cutoff. Finally, I show that, as \( \sigma \to 0 \), the best response in a game with any prior distribution of \( \theta \) uniformly converges to following the unique cutoff \( z^*(\phi) \) when other firms follow the same cutoff \( z^*(\phi) \).

In a second part I prove uniqueness in the fully dynamic game, showing value functions are recursively well-defined and that the dynamic game can be solved as a series of static games. In this part, I denote explicitly time periods \( t \).

- **First part of the Proposition: Uniqueness in a single period for arbitrary distributions**
  - **Step 1: Distributions of fundamentals conditional on signals**

**Lemma 5** The posterior density \( f_{\theta|z} \) and distribution \( F_{\theta|z} \) of \( \theta \) given a signal \( z \) are given by

\[
f_{\theta|z}(\eta|z) = \frac{v(\eta)f\left(\frac{z-\eta}{\sigma}\right)}{\int_{-\infty}^{\infty} v(\theta)f\left(\frac{z-\theta}{\sigma}\right) d\theta},
\]

\[
F_{\theta|z}(\eta|z) = \frac{\int_{-\infty}^{\eta} v(\theta)f\left(\frac{z-\theta}{\sigma}\right) d\theta}{\int_{-\infty}^{\infty} v(\theta)f\left(\frac{z-\theta}{\sigma}\right) d\theta} = \frac{\int_{-\infty}^{\eta} v(z-\sigma u)f(u)du}{\int_{-\infty}^{\infty} v(z-\sigma u)f(u)du}.
\]

**Proof** By Bayes’ rule,

\[
f_{\theta|z}(\theta|z) = \frac{v(\theta)f_{z|\theta}(z|\theta)}{f_z(z)},
\]

where \( f_z \) and \( f_{z|\theta} \) are the densities of \( z \) and \( z|\theta \), respectively. Since \( z \) is the sum of \( \theta \) and \( \sigma \varepsilon \), its density is given by the convolution of their densities, i.e., \( v \) and \( f_{\sigma \varepsilon} \). Considering that \( F_{\sigma \varepsilon}(\eta) = F(\eta/\sigma), f_{\sigma \varepsilon}(\eta) = \frac{f(\eta/\sigma)}{\sigma} \), then \( f_z \) can be defined as

\[
f_z(z) = \sigma^{-1} \int_{-\infty}^{\infty} v(\theta)f\left(\frac{z-\theta}{\sigma}\right) d\theta.
\]

We can obtain the distribution of the observed signal \( z \) after observing a fundamental \( \theta \),

\[
F_{z|\theta}(\eta|\theta) = Pr(z \leq \eta|\theta) = F\left(\frac{\eta-\theta}{\sigma}\right),
\]

\[
f_{z|\theta}(\eta|\theta) = \frac{dF_{z|\theta}(\eta|\theta)}{dz} = \sigma^{-1} f\left(\frac{\eta-\theta}{\sigma}\right),
\]

37
Plugging equations (16) and (15) in (14), we obtain equation (12). The posterior distribution is obtained by integrating over the density,

\[ F_{\theta|z}(\eta|z) = \int_{-\infty}^{\eta} f_{\theta|z}(\theta|z) d\theta = \frac{\int_{-\infty}^{\eta} v(\theta) f(\frac{\eta - \theta}{\sigma}) d\theta}{\int_{-\infty}^{\infty} v(\theta) f(\frac{\eta - \theta}{\sigma}) d\theta}, \]

and the expression in equation (13) follows from variable transformation \( u = \frac{z - \theta}{\sigma} \).

**Step 2: Unique equilibrium cutoff** \( z^*(\phi) \).

**Lemma 6** There is a unique cutoff signal for each reputation \( \phi \) such that \( \Delta(\phi, z|z^*) = 0 \) for \( z = z^*(\phi) \), \( \Delta(\phi, z|z^*) > 0 \) for \( z > z^*(\phi) \), and \( \Delta(\phi, z|z^*) < 0 \) for \( z < z^*(\phi) \), where \( \Delta(\phi, z|z^*) \) is defined by equation 4, where \( k_z(\phi) = z^*(\phi) \), and represent the expected differential gains from playing safe for a firm \( \phi \) that observes \( z \) when other firms follow a cutoff \( z^*(\phi) \).

This cutoff \( z^*(\phi) \) is obtained using Laplacian beliefs over the probability the firm plays risky when the fundamental is \( \theta \), where \( \hat{x} = F\left(\frac{z - \theta}{\sigma}\right) \), such that

\[ \Delta(\phi, z^*|z^*) = \int_{\frac{1}{2}}^{1} \left[ \Delta(\phi, \theta|\hat{x}|\hat{x})|z^*\right] d\hat{x} = 0. \]

**Proof** When fundamentals \( \theta \) are not observed directly, differential gains \( \Delta \) turn into expected differential gains conditional on the signal. When the firm observes a signal \( z \) and believes other firms \( \phi \) use a cutoff \( k_z(\phi) \), expected gains from playing safe are as defined in equation 4

\[ \Delta(\phi, z|k_z) = E_{\theta|z}(\Delta(\phi, \theta|\hat{x})|k_z) \]

Introducing noise in the observation of fundamentals pins down the expected fraction of strategic risk-takers \( \hat{x} \) as a function of cutoff beliefs \( k_z(\phi) \) and observed fundamentals \( \theta \).

\[ \hat{x} = F\left(\frac{k_z(\phi) - \theta}{\sigma}\right). \]

Developing the expectation from equation (4), explicitly denoting \( \hat{x} \) is a function of \( \theta \)

\[ \Delta(\phi, z|k_z) = \int_{-\infty}^{\infty} \left[ \Delta(\phi, \theta|\hat{x}(\theta))|k_z\right] dF_{\theta|z}(\theta|z). \]

Note that \( \theta = k_z(\phi) - \sigma F^{-1}(\hat{x}) \). From equation (13), define

\[ \Psi(\hat{x}|z, k_z) = F_{\theta|z}(k_z - \sigma F^{-1}(\hat{x})|z, k_z) = \frac{\int_{\frac{z - k_z}{\sigma} + F^{-1}(\hat{x})}^{\infty} v(z - \sigma u) f(u) du}{\int_{-\infty}^{\infty} v(z - \sigma u) f(u) du}. \]

Changing variables from \( \theta \) to \( \hat{x} \),

\[ \Delta(\phi, z|k_z) = \int_{0}^{1} \left[ \Delta(\phi, \theta|\hat{x})|k_z\right] d\Psi(\hat{x}|z, k_z). \]
Laplacian beliefs arise from
\[ \Psi(x|z, k_z) = Pr(\theta < k_z - \sigma F^{-1}(x)|z) = F\left[\frac{z - k_z}{\sigma} + F^{-1}(x)\right]. \]

In equilibrium \( k_z(\phi) = z^*(\phi). \) Evaluating the expectation at \( z = z^*(\phi), \) \( \Psi(x|z^*, z^*) = \hat{x}. \)

\[ \Delta(\phi, z^*|z^*) = \int_0^1 [\Delta(\phi, \theta(\hat{x})|\hat{x})|z^*]d\hat{x} = 0. \]

By Lemmas 3 and 4, we know there is a unique solution \( z^*(\phi) \) to this equation. Q.E.D.

– **Step 3: Best response with uniform priors over fundamentals**

Now we need to verify that a firm \( \phi \) playing risky if \( z < z^*(\phi) \) and safe if \( z > z^*(\phi) \) indeed constitutes an equilibrium. Signals \( z \) allow firms to have an idea not only about the fundamental but also about the signals other firms have observed. Following Toxvaerd (2008), I first assume \( \theta \) is drawn from a uniform distribution on the real line, hence an improper distribution with infinite probability mass. This assumption allows us to normalize the prior distribution assuming \( v(\theta) = 1, \) simplifying the density to \( f_{\theta|z}(\theta|z) = \sigma^{-1} f\left(\frac{z - \theta}{\sigma}\right) \) and the distribution to \( F_{\theta|z}(\theta|z) = F\left(\frac{z - \theta}{\sigma}\right). \) We will denote \( \Delta(\phi, z|k_z) \) the expected differential gains from safe projects for the special case in which the prior of fundamentals is uniform,

\[ \tilde{\Delta}(\phi, z|k_z) = \int_{-\infty}^{\infty} \left[ \Delta(\phi, \theta|F\left(\frac{k_z - \theta}{\sigma}\right)) \right] |k_z| \sigma^{-1} f\left(\frac{z - \theta}{\sigma}\right) d\theta. \]

Changing variables introducing \( m = \frac{\theta - k_z}{\sigma}, \)

\[ \tilde{\Delta}(\phi, z|k_z) = \int_{-\infty}^{\infty} [\Delta(\phi, \theta|F(-m)) |k_z| \sigma^{-1} f\left(\frac{z - k_z}{\sigma} - m\right) d\theta. \]

We can rewrite it more conveniently, defining \( \tilde{\Delta} \)

\[ \tilde{\Delta}(\phi, z|k_z) = \tilde{\Delta}(\phi, z, z'|k_z) = \int_{-\infty}^{\infty} B(z', m|k_z) D(z, m|k_z) dm, \]

renaming \( \theta \) as \( z', \) to write the expressions in terms of \( m, \) where \( B(z', m|k_z) = \Delta(\phi, z', F(-m)|k_z) \) and \( D(z, m|k_z) = \sigma^{-1} f\left(\frac{z - k_z}{\sigma} - m\right). \) As shown in Athey (2002), because of the monotone likelihood property, \( \tilde{\Delta}(\phi, z, z'|k_z) \) inherits the single crossing property of \( \Delta(\phi, \theta|k_z). \) This means there exists a \( z^*(\phi, k_z, z') \) such that \( \tilde{\Delta}(\phi, z, z'|k_z) > 0 \) if \( z > z^*(\phi, k_z, z') \) and \( \tilde{\Delta}(\phi, z, z'|k_z) < 0 \) if \( z < z^*(\phi, k_z, z'). \) Assuming \( z < z' \) and \( \tilde{\Delta}(\phi, z, z|k_z) = 0, \)

\[ \tilde{\Delta}(\phi, z', z'|k_z) \geq \tilde{\Delta}(\phi, z, z'|k_z) \geq \tilde{\Delta}(\phi, z, z|k_z) = 0, \quad (\text{strict} \ for \ \phi \in (0, 1)). \]

The first inequality comes from the state monotonicity and the second from the single crossing property. A symmetric argument holds for \( z > z'. \) Hence, there exists a best response \( \chi: \mathbb{R} \rightarrow \)
such that

\[ \Delta(\phi, z|k_z) > 0 \quad \text{if } z > \chi(k_z) \]
\[ \Delta(\phi, z|k_z) = 0 \quad \text{if } z = \chi(k_z) \]
\[ \Delta(\phi, z|k_z) < 0 \quad \text{if } z < \chi(k_z) \]

There exists a unique \( z^*(\phi) \) that solves

\[ \Delta(\phi, z^*|z^*) = \int_0^1 \Delta(\phi, \theta(\hat{x}), \hat{x}|z^*) d\hat{x} = 0. \]  \hspace{1cm} (19)

Hence, \( \chi(z^*(\phi)) = z^*(\phi) \), showing that there is a unique equilibrium in cutoff strategies for each \( \phi \) such that

\[ x^*(\phi, z) = \begin{cases} 
0 & \text{if } z > z^*(\phi) \\
1 & \text{if } z < z^*(\phi) 
\end{cases} \] \hspace{1cm} (20)

- Step 4: Best response with general priors over fundamentals

**Lemma 7** \( \Delta(\phi, z|k_z) \rightarrow \overline{\Delta}(\phi, z|k_z) \) uniformly, when \( k_z(\phi) = z - \sigma \xi \), as \( \sigma \rightarrow 0 \).

**Proof** First, \( \Delta(\phi, z|z - \sigma \xi) \rightarrow \overline{\Delta}(\phi, z|z - \sigma \xi) \) continuously as \( \sigma \rightarrow 0 \), this is,

\[ \Psi(\hat{x}|z - \sigma \xi) = \frac{\int_{\xi + F^{-1}(\hat{x})}^{\infty} v(z - \sigma u) f(u) du}{\int_{-\infty}^{\xi + F^{-1}(\hat{x})} v(z - \sigma u) f(u) du} \rightarrow 1 - F(\xi + F^{-1}(\hat{x})) \equiv \overline{\Psi}(\hat{x}|z - \sigma \xi). \]

As in Toxvaerd (2008), we show convergence with respect to the uniform convergence norm, which implies uniform convergence. Uniformity ensures that the equivalence between the games with the two different assumptions about the prior distributions is not the result of a discontinuity at \( \sigma = 0 \).

Pick \( \overline{\xi}(\phi) < \theta(\phi) \) and \( \overline{\xi}(\phi) > \overline{\theta}(\phi) \) and restrict attention to the compact sets \( Z \equiv [\overline{\xi}(\phi), \overline{\xi}(\phi)] \) and \( Z_\sigma \equiv [\overline{\xi}(\phi) - \sigma \xi, \overline{\xi}(\phi) + \sigma \xi] \). Hence, \( \Delta(\phi, z|k_z) \) maps into a compact set.

Define the uniform convergence norm as

\[ \| \Delta(\phi) \| \equiv \sup_{z,k_z} \{|\Delta(\phi, z|k_z)|\}. \]

We can show continuity with respect to the Euclidean metric. Fix \( z' \) and \( k'_z(\phi) \) such that

\[ \forall \epsilon_1 > 0, \exists \delta_1 |z - z'| < \delta_1 \Rightarrow |\Delta(\phi, z|k_z) - \Delta(\phi, z'|k_z)| < \epsilon_1, \forall k_z \]
\[ \forall \epsilon_2 > 0, \exists \delta_2 |k_z - k'_z| < \delta_2 \Rightarrow |\Delta(\phi, z|k_z) - \Delta(\phi, z|k'_z)| < \epsilon_2, \forall z \]
This implies
\[ \sqrt{(z - z')^2 + (k_z - k'_z)^2} < \sqrt{\delta_1^2 + \delta_2^2}. \]

By the triangle inequality,
\[ |\Delta(\phi, z|k_z) - \Delta(\phi, z'|k'_z)| = |\Delta(\phi, z|k_z) - \Delta(\phi, z'|k_z) + \Delta(\phi, z'|k_z) - \Delta(\phi, z'|k'_z)| \]
\[ \leq |\Delta(\phi, z|k_z) - \Delta(\phi, z'|k_z)| + |\Delta(\phi, z'|k_z) - \Delta(\phi, z'|k'_z)| \]
\[ \leq \epsilon_1 + \epsilon_2. \]

Hence, \( \Delta(\phi, z|k_z) \) belongs to the space of continuous functions on \( Z \times \hat{Z} \).

Uniform convergence is equivalent to
\[ \| \Delta(\phi) - \tilde{\Delta}(\phi) \| = \sup_{z,k} \{ \Delta(\phi, z|k_z) - \tilde{\Delta}(\phi, z|k_z) \} \to 0 \]
with respect to the uniform convergence norm, as \( \sigma \to 0 \), after substituting for the functions and taking limits.

• Second part of the Proposition: Uniqueness in the fully dynamic game.

In the last period \( T \), the cutoff \( z^*_T(\phi) \) is unique (under the condition in equation 8) since \( \Delta_T(\phi, z_T|k_z,T) \) is well-defined and \( V_{T+1}(\phi) = 0 \) for all \( \phi \); thus here reputation concerns do not generate multiplicity. Once \( z^*_T(\phi) \) is determined, the equilibrium interest rate at \( T \) for each \( \phi \) is
\[ R_T(\phi|z^*_T) = \frac{\bar{R}}{Pr(c_T)} = \frac{\bar{R}}{(1 - \phi)p_r + \phi[p_r V(z^*_T) + p_s(1 - V(z^*_T))}] \]

Then we can define expected continuation values in \( T \) for each reputation level \( \phi \). For signals \( z_T < z^*_T(\phi) \), firms take risks and for signals \( z_T > z^*_T(\phi) \), firms play it safe. As \( \sigma \to 0 \), errors coming from miscoordination go to zero and, in the limit, expected profits in the last period \( T \) are independent of \( \sigma \)
\[ V_T(\phi) = \int_{-\infty}^{z^*_T(\phi)} p_r[K - \theta_T - R_T(\phi|z^*_T)]v(\theta_T)d\theta_T + \int_{z^*_T(\phi)}^{\infty} p_s[K - R_T(\phi|z^*_T)]v(\theta_T)d\theta_T. \quad (21) \]

Since equilibrium thresholds are well-defined and unique in period \( T \), continuation values \( V_T(\phi) \) are also well-defined and unique for all \( \phi \).

Now consider the decision of a firm \( \phi \) in the next to last period \( T - 1 \). The problem is essentially static, since continuation values \( V_T(\phi) \) are well-defined and unique for all \( \phi \). Furthermore, since fundamentals are i.i.d. over time, \( \theta_{T-1} \) is irrelevant in forecasting \( \theta_T \), and then value functions are independent of \( \theta \). Then, \( \Delta_T(\phi, z_{T-1}|k_z,T-1) \) is also well-defined for all \( \phi \), thus leading to a unique equilibrium cutoff \( z^*_{T-1}(\phi) \) (following Proposition 2) and unique and well-defined \( V_{T-1}(\phi) \) for all \( \phi \).

By straightforward inductive reasoning, we know that, as \( \sigma \to 0 \), a unique sequence of cutoffs \( \{ z^*_t(\phi) \}_{t=0}^T \) and a unique sequence of expected continuation values exist for each reputation level \( \phi \) in each period \( t \), characterized by equation (11). Q.E.D.
A.3 Conditions for Proposition 4

In this Section we discuss the conditions for \( V_t(\phi) \to V(\phi) \) as \( T \to \infty \) (i.e., by backward induction, continuation values converge to a fixed point for all \( \phi \) and periods \( t \) far enough from \( T \)). These fixed points are the bounded limits required to show that there is an infinite horizon equilibrium that is a unique limit of the finite horizon Markov perfect equilibrium.\(^{23}\)

In short, the condition for convergence is that the variance of fundamentals is large enough. I will prove this by steps. First, I discuss the case without reputation formation as a benchmark in which reputation levels do not interact and obtain sufficient conditions for convergence. Then, I introduce reputation formation and show that those conditions are also sufficient.

Step 1: No reputation formation:

This is an artificial and expositional convenient case in which a firm born with a given reputation \( \phi \) and cannot change it (because age cannot be observed, for example). First assume safe projects deliver higher expected continuation values. That is, if commitment were feasible, firms would choose to take safe projects rather than risky projects, regardless of their reputation. This assumption makes sense in our context, since the focus is on the case in which safe projects are almost always the efficient behavior.

We prove this part generalizing any payoff \( \Pi_t(\theta) \) and \( \Pi_s(\theta) \) such that \( \frac{\partial \Pi_t(\theta)}{\partial R} > \frac{\partial \Pi_s(\theta)}{\partial R} \). From Proposition 3 and without reputation formation (i.e., \( V_{t+1}(\phi') = V_{t+1}(\phi) \)),

\[
V_t(\phi) = \mathcal{V}(z^*_t(\phi)) [\beta p_s V_{t+1}(\phi) - p_r R_t(\phi | z^*_t)] + \int_{-\infty}^{z^*_t(\phi)} p_t \Pi_t(\theta_t) v(\theta_t) d\theta_t \\
+ (1 - \mathcal{V}(z^*_t(\phi))) [\beta p_s V_{t+1}(\phi) - p_r R_t(\phi | z^*_t)] + \int_{z^*_t(\phi)}^{\infty} p_t \Pi_s(\theta_t) v(\theta_t) d\theta_t.
\]

Applying the envelope theorem,

\[
\frac{\partial V_t(\phi)}{\partial V_{t+1}(\phi)} = \mathcal{V}(z^*_t(\phi)) \beta p_r + (1 - \mathcal{V}(z^*_t(\phi))) \beta p_s - \frac{\partial R_t(\phi | z^*_t)}{\partial z^*_t} \frac{\partial z^*_t}{\partial V_{t+1}(\phi)} [p_s - \mathcal{V}(z^*_t(\phi))(p_s - p_r)].
\]

The cutoff \( z^*_t \) is determined by \( p_s \Pi_s(z^*_t) - p_r \Pi_r(z^*_t) - (p_s - p_r) R_t(\phi | z^*_t) = -\beta (p_s - p_r) V_{t+1}(\phi) \), since there is no reputation formation. Taking derivatives with respect to \( V_{t+1}(\phi) \),

\[
\frac{\partial z^*_t}{\partial V_{t+1}(\phi)} = - \frac{\beta (p_s - p_r)}{p_s \frac{\partial \Pi_t}{\partial z^*_t} - p_r \frac{\partial \Pi_r}{\partial z^*_t}} < 0.
\]

Also

\[
\frac{\partial R_t(\phi | z^*_t)}{\partial z^*_t} = \frac{\phi \bar{R} (p_s - p_r)}{Pr(c)^2} v(z^*_t) > 0.
\]

Recall \( \frac{\partial V_t(\phi)}{\partial V_{t+1}(\phi)} > 0 \) and \( V_t(\phi) > 0 \) when \( V_{t+1}(\phi) = 0 \). Hence, convergence to a fixed point \( V(\phi) \) happens if \( \frac{\partial V_t(\phi)}{\partial V_{t+1}(\phi)} < 1 \). It is clear this is the case for \( \phi = 0 \) (since \( \frac{\partial R_t(\phi | z^*_t)}{\partial z^*_t} = 0 \)). At

\(^{23}\)I have not yet examined the broader issue of what other equilibria there might be in the infinite horizon game.
Assume the sufficient condition expressed in equation (22) is met. Then, there is a unique

\[ v(\theta) < \frac{1 - \beta p_s}{\beta p_s} \frac{p_r \partial \Pi_s - p_r \partial \Pi_r}{R(p_s - p_r)^2}, \quad \text{for all } \theta \in \mathbb{R}. \]  

(22)

In the case of linear payoffs and normal distributions study in the text, \( \partial \Pi_s = 0, \partial \Pi_r = -1 \) and \( v(\theta) \leq \sqrt{\frac{\pi}{2\pi}} \) for all \( \theta \). Hence the sufficient condition is \( \frac{\alpha}{2\pi} < \left[ \frac{p_r^2}{R(p_s - p_r)^2} \right] \left[ \frac{1 - \beta p_s}{\beta p_s} \right]^2 \) for all \( \theta \in \mathbb{R} \).

In words, the variance of fundamentals should be large enough (or the density low enough) to have convergence in continuation values for all reputation levels when reputation cannot be modified. Recall that this is a really stringent sufficient condition, since the worst combination of parameters we use are not jointly consistent. For example, if \( \phi = 1 \) and \( \mathcal{V}(z^*(\phi)) = 0, Pr(c) \) is not \( p_r \) but \( p_s \), hence convergence conditions are effectively more relaxed.

Step 2: Reputation formation:

Assume the sufficient condition expressed in equation (22) is met. Then, there is a unique \( \nabla(\phi) \) for all \( \phi \) such that considering reputation formation

\[
\nabla(\phi) = \frac{\beta p_s - \mathcal{V}(z^*(\phi)) \beta p_s}{1 - \mathcal{V}(z^*(\phi)) \beta p_r} \nabla(\phi') - \frac{1 - \mathcal{V}(z^*(\phi)) p_r + \mathcal{V}(z^*(\phi)) p_s}{1 - \mathcal{V}(z^*(\phi)) \beta p_s} R(\phi|z^*) \]

\[ + \frac{1}{1 - \mathcal{V}(z^*(\phi)) \beta p_r} \left[ \int_{-\infty}^{z^*(\phi)} \Pi_r(\theta) v(\theta) d\theta + \int_{z^*(\phi)}^{\infty} p_s \Pi_s(\theta) v(\theta) d\theta \right]. \]

Taking derivatives to consider a greater continuation value for playing safe obtained from a higher reputation

\[
\frac{\partial \nabla(\phi)}{\partial \nabla(\phi')} = \frac{\beta p_s - \mathcal{V}(z^*(\phi)) \beta p_s}{1 - \mathcal{V}(z^*(\phi)) \beta p_r} \frac{\partial R(\phi|z^*)}{\partial z^*} \frac{\partial z^*}{\partial \nabla(\phi')} \left[ \mathcal{V}(z^*(\phi)) p_r + (1 - \mathcal{V}(z^*(\phi)) p_s \right].
\]

It is straightforward to see \( \frac{\partial \nabla(\phi)}{\partial \nabla(\phi')} > 0 \). It is also possible to check monotonicity of continuation values, since \( \frac{\partial \nabla(\phi)}{\partial \nabla(\phi')} < 1 \) when the sufficient condition expressed in equation (22) is fulfilled. With and without reputation formation extreme continuation values, \( \nabla(0) \) and \( \nabla(1) \), are the same. Since reputation generates a convex combination between unique values in a compact set, the resulting continuation values \( \nabla(\phi) \) with reputation formation are also unique.

A.4 Proof of Proposition 5

Proof As a first step, assume convergence has been achieved (Proposition 4). From Proposi-
tion 3, for a general $\Pi_s(\theta)$ and $\Pi_r(\theta)$, $z^*(\phi)$ is determined by solving

$$\int_0^1 \Delta(\phi, z^*|z^*, \hat{x})d\hat{x} = p_s\Pi_s(z^*) - p_r\Pi_r(z^*) + (p_s - p_r) \left[ \beta \int_0^1 \mathbf{V}(\phi'(\phi, \hat{x}))d\hat{x} - R(\phi|z^*) \right] = 0.$$ 

Taking derivatives with respect to $\phi$,

$$\frac{dz^*(\phi)}{d\phi} = -\frac{1}{\theta} \frac{\partial \Delta(\phi, z^*|z^*, \hat{x})}{\partial z^*} d\hat{x}, \quad (23)$$

where

$$\frac{\partial \Delta(\phi, z^*|z^*, \hat{x})}{\partial \phi} = (p_s - p_r) \left[ \beta \frac{\partial \mathbf{V}(\phi')}{\partial \phi'} \frac{\partial \phi'}{\partial \phi} |z^* - \partial R(\phi|z^*) \right],$$

and

$$\frac{\partial \Delta(\phi, z^*|z^*, \hat{x})}{\partial z^*} = p_s \frac{\partial \Pi_s}{\partial z^*} - p_r \frac{\partial \Pi_r}{\partial z^*} - (p_s - p_r) \frac{\partial R(\phi|z^*)}{\partial z^*}, \quad \text{for all } \hat{x}.$$ 

From equation (6),

$$\frac{dR(\phi|z^*)}{d\phi} = \frac{\partial R}{\partial \phi} + \frac{\partial R}{\partial z^*} \frac{dz^*}{d\phi}, \quad (24)$$

where $\frac{\partial R}{\partial \phi} = -\frac{\mathbf{r}(p_s - p_r)(1-V(z^*))}{\mathbf{r}(p_s - p_r)} < 0$ and $\frac{\partial R}{\partial z^*} = \frac{\mathbf{r}(p_s - p_r)\mathbf{r}(z^*)}{\mathbf{r}(p_s - p_r)} > 0$ for all $\phi$.

Finally, from equation (11), using the envelope theorem

$$\frac{d\mathbf{V}(\phi)}{d\phi} = \frac{\partial \mathbf{V}}{\partial \phi} + \frac{\partial \mathbf{V}}{\partial z^*} \frac{dz^*}{d\phi}, \quad (25)$$

where

$$\frac{\partial \mathbf{V}}{\partial \phi} = \beta \left[ \mathbf{V}(z^*)p_r \frac{\partial \mathbf{V}(\phi)}{\partial \phi} + (1 - \mathbf{V}(z^*))p_s \frac{\partial \mathbf{V}(\phi')}{\partial \phi} \bigg|_{\phi=0} \right] - (p_s - \mathbf{V}(z^*)p_s - p_r) \frac{\partial R(\phi)}{\partial \phi},$$

and

$$\frac{\partial \mathbf{V}}{\partial z^*} = \beta v(z^*) \left[ p_s \left( \int_0^1 \mathbf{V}(\phi'|\hat{x})d\hat{x} - \mathbf{V}(\phi'|\hat{x} = 0)d\hat{x} \right) - p_r \left( \int_0^1 \mathbf{V}(\phi'|\hat{x})d\hat{x} - \mathbf{V}(\phi)d\hat{x} \right) \right].$$

We are interested in the sign of the derivatives 23-25. To determine them, I solve backward from the last period $T$.

At period $T$, $\frac{\partial \Delta_T(\phi, z^*|z^*, \hat{x})}{\partial \phi} > 0$ for all $\hat{x}$ (since $\mathbf{V}_{T+1} = 0$ for all $\phi$) and $\frac{\partial \Delta_T(\phi, z^*|z^*)}{\partial z^*} > 0$ (from equation 8). Hence, $\frac{dz^*_T}{d\phi} < 0$. From equation (24), $\frac{dR_T(\phi)}{d\phi} < 0$. Finally, from equation (25) (since $\mathbf{V}_{T+1} = 0$ for all $\phi$), $\frac{d\mathbf{V}_T(\phi)}{d\phi} > 0$.

At period $T-1$ we additionally have the effects coming from $\mathbf{V}_T$. From Bayesian learning $\frac{\partial \phi'}{\partial \phi} = \frac{p_r^*p_r + (1 - \mathbf{V}(z^*)p_s - p_r)(1 - \mathbf{V}(z^*)p_s - p_r)}{p_r^*p_r + (1 - \mathbf{V}(z^*)p_s - p_r)(1 - \mathbf{V}(z^*)p_s - p_r)} > 0$ for all $\hat{x}$. Hence, $\frac{d\phi^*_T}{d\phi} > 0$. From results at $T$, $\int_0^1 \frac{\partial \mathbf{V}_T(\phi')}{\partial \phi} \phi' d\hat{x} > 0$. Hence, $\int_0^1 \frac{\partial \Delta_{T-1}(\phi, z^*|z^*, \hat{x})}{\partial \phi} d\hat{x} > 0$ and $\frac{dz^*_{T-1}}{d\phi} < 0$. From equation (24), $\frac{dR_{T-1}(\phi)}{d\phi} < 0$. Finally, it follows
that $\frac{dV_{T-1}(\phi)}{d\phi} > 0$ from equation (25) and from the fact that $\frac{\partial V_T(\phi)}{\partial \phi} > 0$ for all $\phi$ and that $
abla T(\phi'|\bar{x}) d\bar{x}$ can be written as a convex combination between $V_T(\phi)$ and $V_T(\phi'|\bar{x} = 0)$.

Solving backward until convergence, $\frac{dV(z^*(\phi))}{d\phi} < 0 \left( \frac{dz^*(\phi)}{d\phi} < 0 \right)$, $\frac{dR(\phi)}{d\phi} < 0$ and $\frac{\partial V(\phi)}{\partial \phi} > 0$ for all $\phi \in [0, 1]$.

A.5 Proof of Lemma 2

Proof Differentiating equation (23) with respect to $\phi$, we get

$$\frac{d^2 z^*}{d\phi^2} = -\frac{1}{\beta} \left[ \frac{\partial^2 \Delta}{\partial \phi^2} + 2 \frac{\partial^2 \Delta}{\partial \phi \partial z^*} \frac{dz^*}{d\phi} + \frac{\partial^2 \Delta}{\partial z^*^2} \left( \frac{dz^*}{d\phi} \right)^2 \right].$$

(26)

In what follows, I go back to a linear relation between payoffs and fundamentals, so the shape of cutoffs is not just an artifice of the shape of payoffs. The components of equation 26 are,

$$\frac{\partial^2 \Delta}{\partial \phi^2} = (p_s - p_r) \left[ \beta \int_0^1 \left( \frac{\partial \bar{V}}{\partial \phi'} \frac{\partial^2 \phi'}{\partial \phi^2} \right) d\bar{x} - \frac{\partial^2 R}{\partial \phi^2} \right],$$

(27)

and

$$\frac{\partial^2 \Delta}{\partial \phi \partial z^*} = -(p_s - p_r) \frac{\partial^2 R}{\partial \phi \partial z^*},$$

From equation (25)

$$\frac{\partial^2 \bar{V}}{\partial \phi^2} \left( \frac{dz^*}{d\phi} \right) = \bar{V}(z^*) p_r \left[ \beta \frac{\partial^2 \bar{V}(\phi)}{\partial \phi^2} - \frac{\partial^2 R(\phi|z^*)}{\partial \phi^2} \right] + (1 - \bar{V}(z^*)) p_s \left[ \beta \left( \frac{\partial \bar{V}}{\partial \phi'} \frac{\partial^2 \phi'}{\partial \phi^2} \left( \frac{dz^*}{d\phi} \right) \right) - \frac{\partial^2 R(\phi|z^*)}{\partial \phi^2} \right],$$

(28)

where

$$\frac{\partial^2 R(\phi|z^*)}{\partial \phi^2} = \frac{2R(p_s - p_r)^2 (1 - \bar{V}(z^*))^2}{Pr(c)^3} > 0.$$

I proceed in two steps. First, as a benchmark, I solve backward from $T$ when reputation cannot be updated. Then, I show how reputation formation convexifies the schedule of cutoffs.

Step 1: No reputation formation: Assume a firm born with a given reputation $\phi$ and cannot change it. I call the cutoffs in this case $\bar{z}^*(\phi)$. In this case, beliefs $\bar{x}$ do not play any role, $\frac{\partial \phi'}{\partial \phi} = 1$ and $\frac{\partial^2 \phi'}{\partial \phi^2} = 0$ for all $\phi$. Hence, equations 27 and 28 can be rewritten as

$$\frac{\partial^2 \Delta}{\partial \phi^2} = (p_s - p_r) \left[ \beta \frac{\partial^2 \bar{V}(\phi+1)}{\partial \phi^2} - \frac{\partial^2 R(\phi)}{\partial \phi^2} \right].$$

45
and \[
\frac{\partial^2 V_t}{\partial \phi^2} = (V(\bar{z}_t^r)p_r + (1 - V(\bar{z}_t^s))p_s) \left[ \beta \frac{\partial^2 V_{t+1}(\phi)}{\partial \phi^2} - \frac{\partial^2 R_t(\phi|z^s)}{\partial \phi^2} \right].
\]

At period \( T \), since \( V_{T+1} = 0 \) for all \( \phi \), \( \frac{\partial^2 \Delta T}{\partial \phi^2} < 0 \) and \( \frac{\partial^2 V_T}{\partial \phi^2} < 0 \). However, these signs do not guarantee that equation (26) is positive. The sufficient condition for \( \frac{d^2 z^*}{d \phi^2} > 0 \) is the following:
\[
\frac{d^2 z^*}{d \phi} > \frac{V(\bar{z}_T^r)}{\phi} \left[ F(\phi) + (1 - V(\bar{z}_T^s))p_s \right]
\]
or, more generally, the variance of fundamentals is large enough. \(^{24}\)

The condition is more difficult to be fulfilled for low values of \( \phi \).

At period, \( T - 1 \), \( \frac{\partial^2 \Delta T - 1}{\partial \phi^2} < 0 \) and \( \frac{\partial^2 V_{T - 1}}{\partial \phi^2} < 0 \). This means \( \frac{d^2 z^* - 1}{d \phi^2} > 0 \) for a higher range of \( \phi \) values. The same analysis hold until convergence. In this case, \( \frac{\partial^2 V}{\partial \phi^2} = -\frac{\psi(\bar{z}^* - \epsilon)}{1 - \beta \psi(\bar{z}^*)} \frac{\partial^2 \Delta_T}{\partial \phi^2} \) and \( \frac{\partial^2 \Delta_T}{\partial \phi^2} = -\frac{(p_r - p_s)}{1 - \beta \psi(\bar{z}^*)} \frac{\partial^2 \Delta_T}{\partial \phi^2} \), with \( \psi(\bar{z}^*) = V(\bar{z}^*)p_r + (1 - V(\bar{z}^*))p_s \). Without reputation concerns it may be that \( \frac{d^2 z^*}{d \phi^2} > 0 \) for all \( \phi \), but less likely at lower reputation levels.

**Step 2: Reputation formation:** Consider now the full model with reputation formation. This leads to convexity by combining continuation values of different reputation levels. We consider again equations 27 and 28.

At period \( T \), as in step 1, \( \frac{\partial^2 \Delta T}{\partial \phi^2} < 0 \), \( \frac{\partial^2 V_T}{\partial \phi^2} < 0 \), and \( \frac{d^2 z^*}{d \phi^2} = \frac{d^2 z^*}{d \phi^2} \).

At period \( T - 1 \), since \( \frac{\partial^2 \Delta T}{\partial \phi^2} < 0 \), \( \frac{\partial^2 V_{T - 1}}{\partial \phi^2} < 0 \), and \( \frac{d^2 z^* - 1}{d \phi^2} = \frac{d^2 z^* - 1}{d \phi^2} \) for all \( \phi \) in \( [0, 1] \), \( \frac{\partial^2 \Delta T - 1}{\partial \phi^2} < 0 \) and \( \frac{\partial^2 V_{T - 1}}{\partial \phi^2} < 0 \), exactly as in step 1. Furthermore, \( \int_0^1 \frac{\partial N_T}{\partial \phi} \frac{\partial^2 \Delta_T}{\partial \phi^2} \phi \phi \phi d \phi < 0 \) and \( \int_0^1 \frac{\partial^2 V_T}{\partial \phi^2} \phi \phi \phi d \phi \), which means \( \frac{d^2 \Delta T - 1}{d \phi^2} \) and \( \frac{d^2 V_{T - 1}}{d \phi^2} \) are lower than their counterparts without reputation concerns, derived in step 1. This implies that \( \frac{d^2 z^* - 1}{d \phi^2} > \frac{d^2 z^* - 1}{d \phi^2} \) for all \( \phi \).

Solving backward until convergence, reputation formation introduces pressure for concavity of continuation values and hence the convexity of the schedule of cutoffs and interest rates at all reputation levels, leading to \( \frac{d^2 z^*}{d \phi^2} > \frac{d^2 z^*}{d \phi^2} \) for all \( \phi \).

Even more importantly, as reputation formation becomes easier (i.e., signals are more precise), for \( \frac{p_r}{p_s} \to 0 \), \( \frac{\partial \phi}{\partial \phi} \to \infty \) and \( \frac{\partial^2 \phi}{\partial \phi^2} \to \infty \), hence \( \frac{d^2 z^* - 1}{d \phi^2} > 0 \) for all \( \phi \) (since it always convexifies the schedule of cutoffs for low reputation levels, which are the levels of reputation where convexity was more difficult to obtain without reputation formation). Hence, for any reputation \( \phi \), there is always a \( \frac{p_r}{p_s}(\phi) \in (0, 1] \) such that \( \frac{d^2 z^* - 1}{d \phi^2} = 0 \). Furthermore, from the condition in step 1, \( \frac{p_r}{p_s}(\phi) \) is weakly increasing in \( \phi \). \( \quad \text{Q.E.D.} \)

### A.6 Computational Procedure

- Set a large grid of \( \phi \in [0, 1] \).
- Solve full information (FI) environment (efficiency).

\(^{24}\)This condition requires some algebra that is available upon request.
- Guess a $V_{FI,0} = 0$.
- Obtain $\theta_{FI,0}^*$ from $\Delta(\theta)_{FI} = p_s\Pi_s(\theta) - p_r\Pi_r(\theta) + \beta(p_s - p_r)V_{FI,0} = 0$.
- Obtain $V_{FI,1} = \frac{\int_{-\infty}^{\theta_{FI,0}^*} [p_r\Pi_r(\theta) - R(\phi|z^*(\phi)) + \beta(\phi|z^*(\phi))]^{\infty} d\theta}{1 - \beta(p_s - V(\theta_{FI,0}^*))(p_s - p_r)}$.
- Use $V_{FI,1}$ as the new guess and iterate until $V_{FI,i} - V_{FI,i-1} < \varepsilon$.

**Solve the environment without (wo) reputation formation.**
- Guess $V(\phi)_{wo,0} = 0$ and $\theta^*(\phi) = \theta_{FI}^*$ for all $\phi$.
- Obtain $\theta^*(\phi)_1$ from $\Delta(\phi, \theta^*(\phi)_1) = 0$, where
  \[
  \Delta(\phi, \theta) = p_s\Pi_s(\theta) - p_r\Pi_r(\theta) + (p_s - p_r)[\beta V(\phi)_{wo,0} - R(\phi|\theta^*(\phi)_0)].
  \]
- For each $\phi$, obtain
  \[
  V(\phi)_{wo,1} = \frac{\int_{-\infty}^{\theta^*(\phi)_1} p_r[\Pi_r(\theta) - R(\phi|\theta^*(\phi)_1)]v(\theta)d\theta + \int_{\theta^*(\phi)_1}^{\infty} p_s[\Pi_s(\theta) - R(\phi|\theta^*(\phi)_1)]v(\theta)d\theta}{1 - \beta(p_r + V(\theta^*(\phi)_1))(p_s - p_r))}.
  \]
- Use $V(\phi)_{wo,1}$ and $\theta^*(\phi)_1$ as new guesses and iterate until $V(\phi)_{wo,i} - V(\phi)_{wo,i-1} < \varepsilon_1$ and $\theta^*(\phi)_i - \theta^*(\phi)_{i-1} < \varepsilon_2$ for all $\phi$.

**Solve the environment with reputation formation.**
- Guess a $V(\phi)_0 = 0$ and $z^*(\phi)_0 = \theta^*(\phi)$ for all $\phi$.
- Using $V(\phi)_0$, for each belief $\hat{x} \in [0, 1]$ from a large grid of size $N_x$, obtain
  \[
  \Delta(\phi, z, \hat{x}|z^*(\phi)_0) = E_z[p_s\Pi_s(\theta) - p_r\Pi_r(\theta)] + (p_s - p_r)[\beta V(\phi'|\hat{x})_0 - R(\phi|z^*(\phi)_0)].
  \]
  Recall that for $\sigma \to 0$, this expression can be well approximated by
  \[
  \Delta(\phi, z, \hat{x}|z^*(\phi)_0) = p_s\Pi_s(z) - p_r\Pi_r(z) + (p_s - p_r)[\beta V(\phi'|\hat{x})_0 - R(\phi|z^*(\phi)_0)].
  \]
- Solve for $z^*(\phi)_1$ from $\sum_{\hat{x}} \frac{\Delta(z^*(\phi)_1)}{N_x} = 0$.
- For all $\theta < (> )z^*(\phi)_1$, $x(\phi, \theta)_1 = 1 (= 0)$.
  * $R(\phi|z^*(\phi)_1)$ follows from $z^*(\phi)_1$.
  * $\phi'$ follows from $x(\phi, \theta)_1$.
- Obtain $V(\phi)_1$ as
  \[
  V(\phi)_1 = \int_{-\infty}^{z^*(\phi)_1} p_r[\Pi_r(\theta) - R(\phi|z^*(\phi)_1)] + \beta V(\phi)_0]v(\theta)d\theta \\
  + \int_{z^*(\phi)_1}^{\infty} p_s[\Pi_s(\theta) - R(\phi|z^*(\phi)_1)] + \beta V(\phi)_0]v(\theta)d\theta.
  \]
- Use $V(\phi)_1$ and $z^*(\phi)_1$ as new guesses and iterate until $V(\phi)_i - V(\phi)_{i-1} < \varepsilon_1$ and $z^*(\phi)_i - z^*(\phi)_{i-1} < \varepsilon_2$ for all $\phi$. 