Lectures Notes on Public Goods

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1 Introduction

1.1 What is Public Goods?

A good is called a pure public good if "each individual's consumption of such a good leads to no subtraction from any other individual's consumption" (Samuelson 1954, p387) This is commonly referred to as *non-rivaly* in use. We will consider the simplest case with a single private good and a single public good.

1.2 The Model

- n consumers, indexed by $i = 1, ..., n$
- x_i : agent i's consumption of private good and denote $\mathbf{x} = (x_1, ..., x_n)$ as the vector of private consumption
- G: the (common) consumption of public good
- Agent i's preference described by the utility function

$$u_i(x_i, G)$$

  which is differentiable and increasing in both arguments, quasi-concave and satisfies Inada Condition

- w_i : agent i's endowment of private good and

$$W = \sum_{i=1}^{n} w_i$$

is the total endowment of private good; and public good endowment is taken to be zero

- Public good may be produced from the private good according to a production function $f : R_+ \to R_+$ where $f' > 0$ and $f'' < 0$. That is, if z is the total units of private goods that are used as inputs to produce the public good, the level of public good produced will be

$$G = f(z).$$

1.3 Optimal Provision of Pure Public Good

We first ask the normative question of what is the optimal level of pure public good. We assume that the government of a fully controlled economy chooses the level of G, and the allocation of private goods $\mathbf{x} = (x_1, ..., x_n)$ to agents according to the Pareto criterion.

Definition 1 *An allocation* $(\mathbf{x}, G) \in R_+^{n+1}$ *is feasible if there exists some* $z \geq 0$ *s.t.*

- $\sum_{i=1}^{n} x_i + z \leq W$;
- $G \leq f(z)$.

Alternatively we could define that an allocation $(\mathbf{x}, G) \in R_+^{n+1}$ is feasible if

$$\sum_{i=1}^{n} x_i + f^{-1}(G) \leq W.$$

Definition 2 *A feasible allocation* $(\mathbf{x}, G)$ *is Pareto optimal if there exists no other feasible allocation* $(\mathbf{x}', G')$ *s.t.*

$$u_i\left(x_i', G'\right) \geq u_i(x_i, G) \, \forall i = 1, ..., n$$

and for some $i \in \{1, ..., n\}$,

$$u_i\left(x_i', G'\right) > u_i(x_i, G).$$

That is, a feasible allocation $(\mathbf{x}, G)$ is Pareto optimal if there is no way of making an agent strictly better off without making someone else worse off.

Now we can characterize the set of Pareto optimal allocations. It is the solution to the following problem:

$$\begin{aligned} &\max_{\{x,G,z\}} u_1(x_1, G) \\ \text{s.t. } & u_i(x_i, G) - \underline{u}_i \geq 0 \text{ for } i = 2, 3, ..., n, \qquad (\text{multiplier } \gamma_i) \\ & W - \sum_{i=1}^{n} x_i - z \geq 0 \qquad (\text{multiplier } \lambda) \\ & f(z) - G \geq 0 \qquad (\text{multiplier } \mu) \\ & G \geq 0, z \geq 0 \text{ and } x_i \geq 0 \text{ for all } i = 1, ..., n \end{aligned}$$

where $\underline{u}_i$ are treated as parameters of the problem. Inada conditions on the utility function implies that the non-negativity constraints can be ignored. The necessary and sufficient (sufficiency due to quasi-concavity assumption on u and f) Kuhn-Tucker conditions are:

$$\begin{aligned} (x_i:) \qquad & \gamma_i \frac{\partial u_i(x_i, G)}{\partial x_i} - \lambda = 0 \\ (G:) \qquad & \sum_{i=1}^{n} \gamma_i \frac{\partial u_i(x_i, G)}{\partial G} - \mu = 0 \\ (z:) \qquad & -\lambda + \mu f'(z) = 0 \end{aligned} \tag{1}$$

where we have set $\gamma_1 = 1$ by convention.

From the fist n equalities, we obtain

$$\gamma_i = \frac{\lambda}{\partial u_i(x_i, G) / \partial x_i}.$$

From the last equality, we obtain

$$\mu = \frac{\lambda}{f'(z)}$$

Plugging these $n+1$ equalities into the the middle condition regarding G, we get

$$\sum_{i=1}^{n} \frac{\partial u_i(x_i, G) / \partial G}{\partial u_i(x_i, G) / \partial x_i} = \frac{1}{f'(z)}. \tag{2}$$

This condition is referred to as the Samuelson condition, the Lindahl-Samuelson condition, or sometimes even the Bowen-Lindahl-Samuelson Condition and is probably familiar to anyone who have taken an intermediate course in public economics.

Interpretations: The left hand of equation (2) is the sum of the marginal rates of substitutions of the n agents. To see this, note that from agent i's indifference curve, the term

$$\frac{\partial u_i\left(x_i, G\right) / \partial G}{\partial u_i\left(x_i, G\right) / \partial x_i}$$

denotes the quantity of private good agent i is willing to give up for a small unit increase in the level of the public good. The right hand of equation (2) is the amount of private good required to produce an additional unit of public good (also known as the marginal rate of transformation). Hence the Samuelson condition says the following: *Any optimal allocation is such that the sum of the quantity of private goods consumers would be willing to give up for an additional unit of public good must equal to the quantity of private good that is actually required to produce the additional unit of public good.*

If there are more than one private goods, say k private goods; and the public good is produced according to

$$f\left(z_1, ..., z_k\right),$$

then the corresponding Samuelson condition for the optimal level of public goods is given by

$$\sum_{i=1}^{n} \frac{\partial u_i\left(x_{ij}, G\right) / \partial G}{\partial u_i\left(x_{ij}, G\right) / \partial x_{ij}} = \frac{1}{\partial f\left(z_1, ..., z_k\right) / \partial z_j} \text{ for all } j = 1, ..., k.$$

A diagrammatic illustration of the Samuelson condition for the case where there are two individuals and two goods is given in Figure 1. In Figure 1, the upper part shows the indifference curves for citizen I and the production constraint AB. Suppose that we fix citizen I on the indifference curve $\underline{u}_I$, then the possibilities for citizen II are shown in the lower part of Figure 1 by CD (which is the difference between AB and $\underline{u}_I$). Clearly Pareto efficiency requires the marginal rate of substitution of the second individual be equal to the slope of the curve CD (i.e. at point E). But this is just the difference between the marginal rate of transformation (the slope of the production possibilities schedule) and the marginal rate of substitution of the first individual (the slope of his indifference curve). Thus we have

$$MRS^{II} = MRT - MRS^{I}.$$

Implementation of the optimal allocation: If the government is able to levy lump sum taxes both to finance the expenditure and to redistribute income, then it is clear that the optimal allocation above can be achieved. If lump sum taxes are not feasible, the government

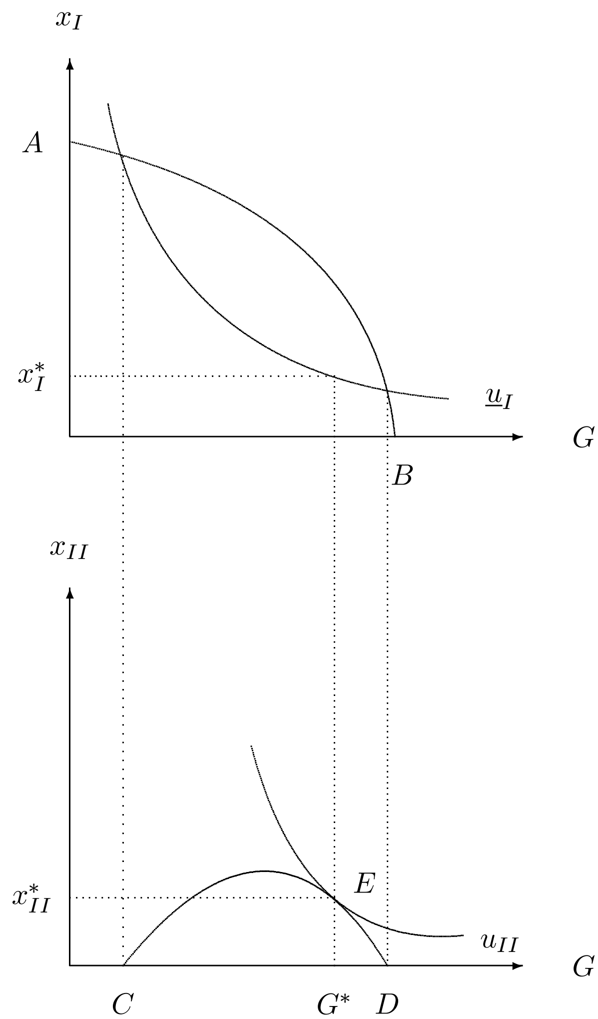


Figure 1: Optimal Provision of Public Goods - The Two Person Example

needs to use distortionary taxes, for example, tax on labor income, to finance the public goods.

2 Can the Optimal Allocation be Decentralized?

Can the optimal allocations characterized by the Samuelson condition be decentralized? Imagine that competitive markets exist for both the private and the public goods. Let the private good be the numeraire.

- Let p denote the price of the public good (in terms of the private good);
- Let g_i denote the quantity of public good purchased by agent i;
- Without loss of generality, we assume that there is a single price-taking profit maximizing firm that operates on the market.

We will make the following (somewhat sloppy) assumption: we assume that every agent are price-takers (i.e. their choice does not affect the price level), but they do feel that their purchase can affect the aggregate level of public goods.

Given the public good purchases by other agents $\tilde{\mathbf{g}}_{-i} = (g_1, ..., g_{i-1}, g_{i+1}, ..., g_n)$, agent i's best response to $\tilde{\mathbf{g}}_{-i}$ given a price p is defined as

$$\begin{aligned} \beta_i(\tilde{\mathbf{g}}_{-i}, p) &= \arg\max_{\{g_i\}} u_i\left(w_i - pg_i, g_i + \sum_{j\neq i} g_j\right) \\ \text{s.t. } g_i &\geq 0 \\ w_i - pg_i &\geq 0 \end{aligned}$$

Assuming that u_i is strictly quasi-concave, there is a unique solution to the maximization problem for the agent given $\tilde{\mathbf{g}}_{-i}$ and p which is fully characterized by

$$\begin{aligned} -\frac{\partial u_i}{\partial x_i}p + \frac{\partial u_i}{\partial G} + \lambda - \mu p &= 0 \\ \lambda g_i &= 0 \\ \mu(w_i - pg_i) &= 0 \end{aligned}$$

Since u_i satisfies Inada condition, $\mu = 0$. Hence we have

$$p \geq \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i}.$$

The profit maximizing supplier of the public good solves, for a given price p, the following problem

$$\max_{z \geq 0} pf(z) - z$$

which yields the condition that

$$p = \frac{1}{f'(z)}.$$

Definition 3 *A competitive equilibrium consists of* $p^*, G^* = (g_1^*, ..., g_n^*)$ *such that*

1. *For each* i, *given* p^* *and* $\tilde{\mathbf{g}}_{-i}^* = \left(g_1^*, ..., g_{i-1}^*, g_{i+1}^*, ..., g_n^*\right)$,

$$g_i^* \in \beta_i\left(\tilde{\mathbf{g}}_{-i}^*, p^*\right)$$

2. *The firm optimizes, i.e.*

$$p^* = \frac{1}{f'\left(f^{-1}\left(\sum_{i=1}^n g_i^*\right)\right)}.$$

Because of the Inada condition on u_i regarding G, we must have that for some $j \in \{1, ..., n\}$,

$$p = \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i}.$$

Together with the firm's optimization condition, we obtain that for some j,

$$\frac{1}{f'(z)} = \frac{\partial u_j / \partial G}{\partial u_j / \partial x_j}.$$

That is, in the competitive equilibrium, it must be the case that

$$\sum_{i=1}^n \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i} > \frac{1}{f'(z)}.$$

Hence there is under-provision of the public good relative to the level prescribed by the Samuelson condition. The intuition is the following: each agent when deciding how much public good to purchase, does not consider the benefit to other agents of the output he purchased. This is true for each agent and consequently as a group the agents purchase less than the amount desirable for Pareto optimality.

Example 1 *Suppose* $u_i(x_i, g) = \gamma \ln g + \ln x_i$, *and* $w_i = W/n$, *and* $f(z) = z$. *Find the egalitarian Pareto optimal allocation; and the competitive equilibrium allocation. It can be shown that the egalitarian Pareto optimal allocation is given by*

$$\hat{G} = \frac{\gamma W}{1+\gamma}, \hat{x}_i = \frac{W}{n(1+\gamma)}, i = 1, ..., n$$

and the competitive equilibrium allocation is

$$G^* = \frac{\gamma W}{n+\gamma}, x_i^* = \frac{W}{n+\gamma}, i = 1, ..., n.$$

It is clear that as n gets larger, the under-provision of the public good gets more severe.

3 Lindahl Equilibria

While the competitive equilibrium with a fixed price of the public good will yield an inefficient allocation, there is a much studied "market institution" that in principle would achieve efficiency. The idea is to think of the amount purchased by each agent as a distinct commodity and have each agent to face a *personalized price* p_i and to have these price chosen in a way such that all agents agree on the level of the public good. Let $s_i \in [0,1]$ be agent i's share of the firm's profit with $\sum_{i=1}^n s_i = 1$

Definition 4 *A Lindahl equilibrium is a vector $\mathbf{p}^* = (p_1^*, ..., p_n^*)$ and an allocation $(x_1^*, ..., x_n^*, G^*)$ such that*

- *The firm maximizes profits, that is,*

$$G^* = \arg\max_{G \geq 0} \left(\sum_i p_i^* \right) G - f^{-1}(G)$$

- *Each consumer maximizes utility, that is,*

$$\begin{aligned} & (x_i^*, G^*) = \arg\max_{x_i, G} u_i(x_i, G) \\ s.t. \qquad & w_i + s_i \left(\sum_i p_i^* G^* - f^{-1}(G^*) \right) - x_i - p_i^* G \geq 0 \end{aligned}$$

- *Market clears, i.e.*

$$\sum_{i=1}^n x_i^* + f^{-1}(G^*) \leq \sum_{i=1}^n w_i.$$

The Lindahl equilibrium is a competitive equilibrium in a fictitious economy where the space of goods has been expanded to $(n+1)$ goods, the private goods and n personalized public goods, that is, the public goods of agent 1 through agent n. These n goods are produced "jointly", so that we must find a vector of prices for which all agents demand equal quantities of the public good. We now show that a Lindahl equilibrium is indeed Pareto optimal. To see this, note that the first order condition for the firm's profit maximization gives

$$\sum_{i=1}^{n} p_i^* = \frac{1}{f'\left(f^{-1}\left(G^*\right)\right)}$$

and the first order condition for individual i's utility maximization is

$$\frac{\partial u_i\left(x_i^*, G^*\right)}{\partial x_i} p_i^* = \frac{\partial u_i\left(x_i^*, G^*\right)}{\partial G} \text{ for all } i = 1, ..., n.$$

Hence

$$\sum_{i=1}^{n} \frac{\partial u_i\left(x_i^*, G^*\right)/\partial G}{\partial u_i\left(x_i^*, G^*\right)/\partial x_i} = \frac{1}{f'\left(f^{-1}\left(G^*\right)\right)}$$

which satisfies the by-now familiar Samuelson condition. Furthermore, all agents' budget sets must hold with equality, which means that market clears with equality. Hence a Lindahl equilibrium is efficient.

Example 2 *Find the Lindahl equilibrium of the economy described by Example 1. Suppose that agent i's personalized price for the public good is p_i. It is easy to solve for i's demand for the public good will be given by*

$$g_i\left(p_i\right) = \frac{1}{p_i} \frac{\gamma W}{\left(\gamma + 1\right) n}.$$

Since the demand of public good must be equal for all the agents in a Lindahl equilibrium it must be the case that in a Lindahl equilibrium, $p_i^* = p_j^*$ for all $i, j \in \{1, ..., n\}$. The firm's profit maximization requires that

$$\sum_{i=1}^{n} p_i^* = 1$$

Hence $p_i^* = 1/n$ for all i. Plugging this individualized price, we obtain that

$$g_i\left(p_i^*\right) = \frac{\gamma W}{\gamma + 1} \text{ for all } i$$

which is the public good level in the egalitarian Pareto-optimal allocation.

The efficiency of Lindahl equilibrium allocation can be established using an argument which is more or less a copy of the textbook proof of the first welfare theorem.

Proposition 1 *Any Lindahl equilibrium is Pareto optimal.*

Proof. Let $(x_1^*, ..., x_n^*, G^*)$ be a Lindahl equilibrium allocation with corresponding prices $(p_1^*, ..., p_n^*)$ and suppose that there exists a feasible allocation $(x_1', ..., x_n', G')$ that Pareto dominates $(x_1^*, ..., x_n^*, G^*)$. That is,

$$\begin{aligned} u_i\left(x_i', G'\right) &\geq u_i\left(x_i^*, G^*\right) \forall i \\ u_i\left(x_i', G'\right) &> u_i\left(x_i^*, G^*\right) \text{ for at least one agent } i \end{aligned}$$

By revealed preference, this implies that (assuming local non-satiation) that

$$\begin{aligned} x_i' + p_i^* G' &\geq x_i^* + p_i^* G^* \ \forall i \\ x_i' + p_i^* G' &> x_i^* + p_i^* G^* \text{ for at least one agent } i \end{aligned}$$

Hence

$$\begin{aligned} \sum_i x_i' + G' \sum_i p_i^* &> \sum_i x_i^* + G^* \sum_i p_i^* = \sum_i w_i + \sum_i s_i \left[\left(\sum_i p_i^*\right) G^* - f^{-1}\left(G^*\right)\right] \\ &= \sum_i w_i + \left(\sum_i p_i^*\right) G^* - f^{-1}\left(G^*\right) \end{aligned}$$

where the first equality follows since each agent has to fulfill her budget constraint with equality in order to maximize utility.

Moreover, from the first profit maximization, we have

$$\left(\sum_i p_i^*\right) G^* - f^{-1}\left(G^*\right) \geq \left(\sum_i p_i^*\right) G' - f^{-1}\left(G'\right)$$

Combining the above two inequalities, we obtain

$$\sum_i x_i' + f^{-1}\left(G'\right) > \sum_i w_i$$

which contradicts the feasibility of the allocation $(x_1', ..., x_n', G')$. ■

3.1 Is Lindahl Equilibrium a Reasonable Market Mechanism?

The Lindahl equilibrium is more a normative prescription for the allocation of public goods than a positive description of the market mechanism. The reason is simple: by the definition of the personalized price in the Lindahl equilibrium, an agent will quickly lean that he should *not* behave competitively (an assumption which has always been justified by the existence of a large number of market participants). He will have incentive to mis-report her desire for the public good. Contrary to the case of private goods, where the incentive to reveal false demand functions decreases with the number of agents, an increase in the number of agents in the case of public good only aggregates the problem. We demonstrate this problem by the following example.

Example 3 *Consider n agents with utility function $u_i(x_i, G) = \ln x_i + \alpha_i \ln G$. We suppose that each agent has an endowment of the private good $w_i = 1$ and no public good. Suppose that the technology is linear, i.e. $f(z) = z$ for all $z \geq 0$. Facing a personalized price p_i, it is clear that agent i will demand public good*

$$p_i g_i(p_i) = \frac{\alpha_i}{1+\alpha_i}$$

Since in a Lindahl equilibrium

$$g_i(p_i) = G \,\forall i$$

we have

$$G \sum p_i = \sum_i \frac{\alpha_i}{1+\alpha_i}$$

For the firm's profit maximization problem to have a solution, it must be that

$$\sum p_i = 1$$

hence

$$G = \sum_i \frac{\alpha_i}{1+\alpha_i}$$

and

$$p_i = \frac{\alpha_i / (1+\alpha_i)}{\sum_j \alpha_j / (1+\alpha_j)}$$

Agent i's consumption of private good is

$$x_i = \frac{1}{1+\alpha_i}$$

Suppose that $n = 3$, and $\alpha_i = 1$ for $i = 1, 2, 3$. The Lindahl equilibrium is then

$$p_i^* = \frac{1}{3}, x_i^* = \frac{1}{2}, G^* = \frac{3}{2}$$

so the equilibrium utility level for agent i is

$$\ln x_i^* + \alpha_i \ln G^* = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)$$

Now make the following thought experiment: suppose Mr. 2 and 3 report truthfully that their types are $\alpha_i = 1$, *but that Mr. 1 lies and claim that* $\alpha_1 = 0$. *If the planner computes the Lindahl price believing all the agents, the corresponding Lindahl prices and allocations will be*

$$p_1 = 0, p_2 = p_3 = \frac{1}{2}, x_1 = 1, x_2 = x_3 = \frac{1}{2}, G = 1.$$

Mr. 1's utility would then be

$$2 \ln (1) = 0.$$

It is easy to see that

$$\ln (1) > \frac{1}{2} \ln \left(\frac{1}{2} \right) + \frac{1}{2} \ln \left(\frac{3}{2} \right)$$

since logarithm is strictly concave. Hence truth telling is not an equilibrium of this game.

While the above example is special, the logic is perfectly general. If agents have to report preferences (or wealth) they will take into consideration that under-reporting means a lower personalized price, so the free-riding problem applies. This does not mean, however, that one can not design more complicated and somewhat contrived mechanisms to implement Lindahl equilibrium allocation.

Exercise 1 *Suppose there are* 3 *agents,* $i = 1, 2, 3$. *Agent i's utility function is given by* $u_i (x_i, G)$ *where* x_i *is the private good and G is the public good. Agent i's endowment of private good is* w_i. *Suppose that all three agents know all preference parameters, wealth etc of the other agents, but the planner is uninformed. Suppose the production function for the public good is* $f (z) = z$. *The planner designs the following mechanism:*

- *Each agent i sends a message from the space*

$$M_i = \left\{ \left(p_1^i, p_2^i, p_3^i, G^i, n^i \right) \mid \sum_j p_j^i = 1, n^i \in N \right\}$$

  where N denotes the set of natural numbers. [That is, the planner asks each agent to report a price vector, a level of public good and a natural number]

- *The planner will choose the following allocation as a function of the message profile*

$\left(m^1, m^2, m^3\right)$:

$$g\left(m^1, m^2, m^3\right) = \begin{cases} \left\langle \left(w_i - \tilde{p}_i \tilde{G}\right), \tilde{G} = \frac{1}{3}\sum_i G^i \right\rangle_{i=1}^{3} & \text{if } \exists \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \text{ s.t. } p_j^i = \tilde{p}_j \\ & \text{for at least two agents } i \\ \sum_i w_i \text{ is divided equally} & \\ \text{among players who} & \text{otherwise.} \\ \text{announced the highest } n^i, & \end{cases}$$

Show that:

1. *Any Lindahl equilibrium allocation is a Nash equilibrium in the game induced by the mechanism* $\langle g, M \rangle$ *constructed above;*

2. *Any interior Nash equilibrium allocation is a Lindahl equilibrium allocation;*

3. *Show by example that the qualifier "interior" is needed above.*

4 Positive Models of Private Provision of Public Goods

So far we have discussed that public goods will in general be under-supplied by voluntary contributions. Still, voluntary contributions of public goods constitutes a large fraction of available resources in the economy (approximately 2% of GDP are private donations to charity). Hence it seems to be of importance to have a reasonable positive theory of private provision of public goods from which one can derive policy implications. We may, for instance, be interested in how private provisions change due to changes in the income distribution and how private donations are affected by public provisions.

4.1 A Static Model of Private Contributions

This section follows the famous paper by Bergstrom, Blume and Varian (henceforth BBV, 1986). From previous analysis, we learn that assuming a concave production function $G = f(z)$ does not add any new insights - so we will here assume linear production technology; and as long as we assume that firms are competitive, decentralizing production does not add any further distortion - so we will assume that private good can be turned into a public good by any agent.

4.1.1 Notations

- A set of consumers $N = \{1, ..., n\}$
- w_i : i's (exogenous) wealth
- x_i : i's consumption of private goods
- g_i : i's contribution toward the public good. For ease of notation, write

$$\begin{aligned} G &= \sum_{i=1}^{n} g_i \\ G_{-i} &= \sum_{j \neq i} g_j \end{aligned}$$

  [Note before, we had $G = f\left(\sum_{i=1}^{n} g_i\right)$ where f is the production function for the public good.]

- Consumer i has utility function $u_i(x_i, G)$, increasing in both arguments.
- Timing of the voluntary contribution game: each agent simultaneously chooses $g_i \in [0, w_i]$.

4.1.2 Nash Equilibrium

A *Nash equilibrium* of the voluntary contribution game is a vector of contributions $(g_1^*, ..., g_n^*)$ such that, for all $i \in N$,

$$g_i^* \in \arg\max_{g_i \in [0, w_i]} u_i\left(w_i - g_i, g_i + G_{-i}^*\right)$$

where

$$G_{-i}^* = \sum_{j \neq i} g_j^*.$$

In the above definition of Nash equilibrium we have used the fact that the budget constraint must hold with equality since the utility function is increasing in both arguments.

We can have an alternative expression of agent i's problem as

$$\begin{aligned} & \max_{x_i, G} u_i(x_i, G) \qquad (3) \\ \text{s.t.} \quad & x_i + G = w_i + G_{-i}^* \\ & G \geq G_{-i}^* \end{aligned}$$

The point of writing the condition for i playing a best response in this more complicated way is that it should be clear that this is just like any ordinary consumer maximization problem where relative prices equal unity and wealth is given by $w_i + G^*_{-i}$.

4.1.3 A Neutrality Theorem

The first result is that, *if all agents contribute*, then small changes in the distribution of income will leave the allocation unchanged. This result was first obtained by Warr (1983) who used an implicit differentiation argument of the first order condition. BBV extended this argument by a non-calculus approach and could then give a more complete description of how equilibria are affected by changes in the wealth distribution.

Proposition 2 (Neutrality Theorem) *Suppose u_i is quasi-concave for all $i \in N$, and let $(g^*_1, ..., g^*_n)$ be the initial equilibrium. Consider a redistribution of income* among contributing agents $C \equiv \{i : g^*_i > 0\}$ *such that no agents loses more income than his original contribution. Let w'_i be i's post-redistribution wealth. Then the post-redistribution NE $\{g'^*_1, ..., g'^*_n\}$ satisfies*

$$g'^*_i - g^*_i = w'_i - w_i$$

Hence,

$$G'^* \equiv \sum_{i=1}^{n} g'^*_i = G^* \equiv \sum_{i=1}^{n} g^*_i.$$

Proof. Suppose that Δw_i is the change in agent i's wealth caused by the redistribution. Suppose that in a post-redistribution equilibrium, every agent other than agent i changes his contribution by the exact amount of his change in wealth. This implies that

$$G'^*_{-i} = G^*_{-i} - \Delta w_i.$$

Agent i's best response is the solution to the following problem

$$\max_{\{x_i, G\}} u_i(x_i, G) \tag{4}$$

$$\text{s.t.} \quad x_i + G = \underbrace{(w_i + \Delta w_i)}_{\text{new wealth}} + \underbrace{(G^*_{-i} - \Delta w_i)}_{\text{others' total contribution}} = w_i + G^*_{-i}$$

$$G \geq G'^*_{-i} - \Delta w_i$$

Note that the first constraint in the above problem is the same as that in (3). The difference between (4) and (3) lies in the second constraint.

We consider two cases:

1. Case I: $\Delta w_i < 0$. In this case, the feasible set in (4) is smaller than that in (3). But since by assumption (no consumer loses more income than his original contribution) $g_i^* + \Delta w_i \geq 0$, we know that (x_i^*, G^*) is in the post-redistribution feasible set. By revealed preference, (x_i^*, G^*) is optimal solution to the problem (4);

2. Case II: $\Delta w_i > 0$. In this case, the feasible set in (4) is larger than that in (3). Suppose to the contrary that there is a choice (x_i', G') such that $u_i(x_i', G') > u_i(x_i^*, G^*)$. Then it follows from quasi-concavity that

$$u_i\left(\lambda x_i' + (1-\lambda)\, x_i^*, \lambda G' + (1-\lambda)\, G^*\right) > u_i\left(x_i^*, G^*\right)$$

for any $\lambda \in (0,1)$. Since $i \in C$ by assumption, $x_i^* < w_i$, so for λ small enough the convex combination is feasible before the wealth redistribution. It follows then G^* could not be optimal initially, a contradiction. ■

To summarize, this neutrality theorem says that in the post-redistribution equilibrium each consumer has precisely the same consumption of the private good and the public good as he had before. The optimal responses of the consumers to the wealth transfer have completely offset the effects of the redistribution.

4.1.4 A General Characterization of the Set of Nash Equilibria

Consider the problem

$$\begin{aligned} &\max u_i(x_i, G) \\ \text{s.t.} \quad & x_i + G = W \end{aligned}$$

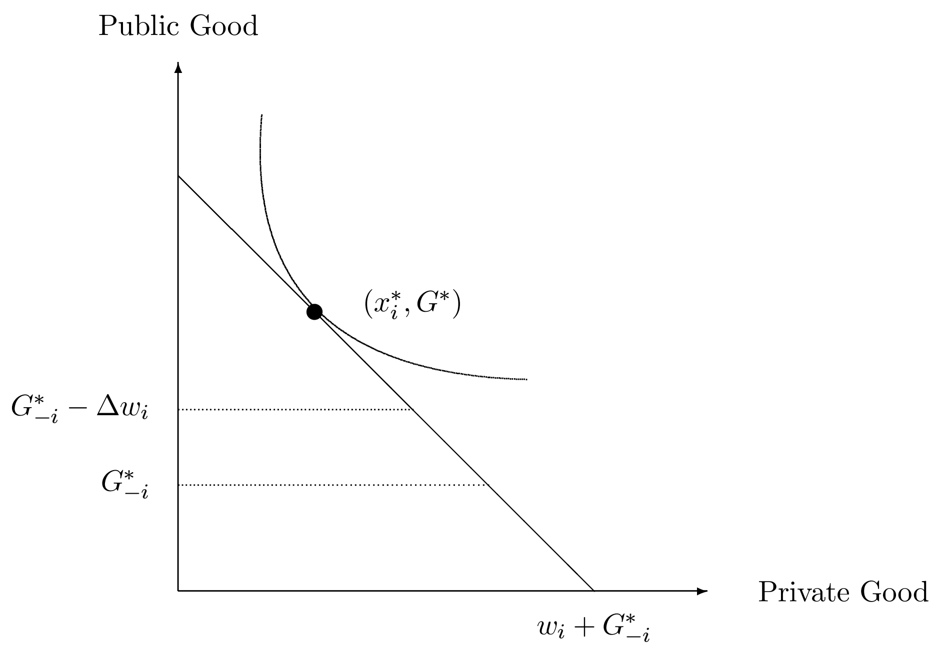
This is a standard consumer optimization problem. Assuming that u_i is strictly quasi-concave, we have a unique solution $f_i(W)$, which in consumer theory language is the demand for the public good G. We make the following assumption:

Assumption: $f_i(\cdot)$ is single-valued, differentiable and satisfy

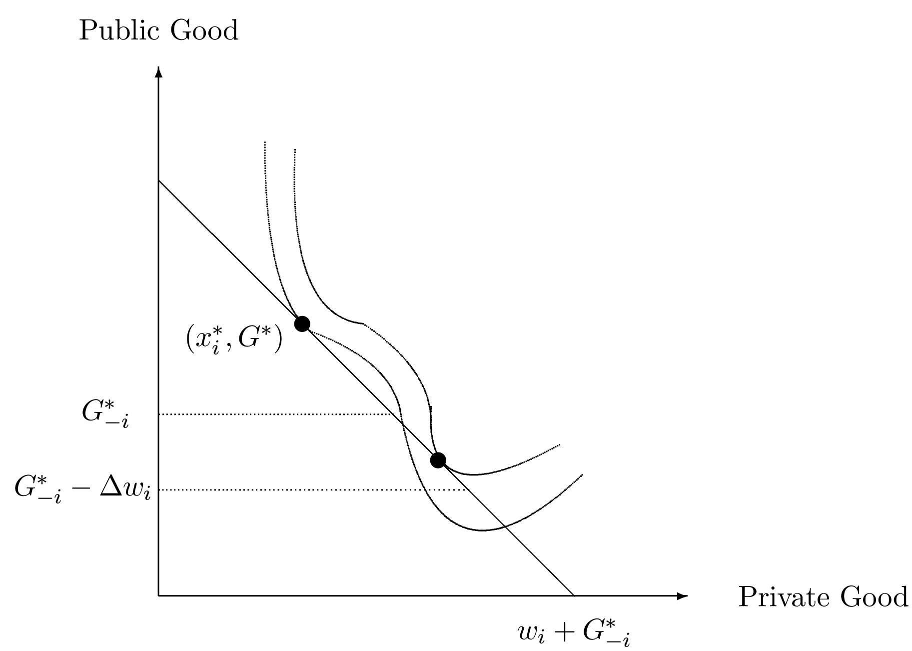
$$f_i'(W) \in (0,1).$$

This assumption requires that both the private and the public goods are normal goods.

The problem that determines i's best response is of course the problem (3). It is clear that the best response function from the problem (3) is (taking i's contribution to the public



(a) Case I: $\Delta w_i < 0$



(b) Case II: $\Delta w_i > 0$: Convexity Rules this Out

Figure 2: Graph in the Proof of the Neutrality Theorem

good, g_i, as the strategic variable), is

$$\beta_i(G_{-i}) = \max\{f_i(w_i + G_{-i}) - G_{-i}, 0\}. \tag{5}$$

Proposition 3 (Existence of Equilibrium) *A Nash Equilibrium exists.*

Proof. Let $\mathcal{W} = \{z \in R^n : z_i \in [0, w_i] \ \forall i\}$. This is a compact and convex set. Note that

$$(\beta_1(G_{-1}), ..., \beta_n(G_{0n}))$$

is a continuous function from $\mathcal{W}$ to itself. By Brouwer's fixed point theorem, there must exists a fixed point, which is the Nash equilibrium of the voluntary contribution game. ∎

Now we investigate a little more of the equilibrium of the voluntary contribution game. Consider an equilibrium $(g_1^*, ..., g_n^*)$. As before, define the set of positive contributors as

$$C^* = \{i \in N : g_i^* > 0\}.$$

The first useful fact simply follows from the best response (5):

FACT 1:

$$\begin{aligned} G^* &= \sum_i g_i^* = f_i\left(w_i + G_{-i}^*\right) \ \forall i \in C^* \qquad (6) \\ G^* &\geq f_j\left(w_j + G_{-j}^*\right) = f_j\left(w_j + G^*\right) \ \forall j \notin C^*. \end{aligned}$$

The important implication of FACT 1 is that once we know the level of the public goods G^*, then the set of positive contributors are unique. To see this, note that for a given G^*, we can ask whether $G^* \geq f_j(w_j + G^*)$ holds for all $j \in N$. If it holds, then $j \notin C^*$; otherwise, $j \in C^*$. There is no ambiguity at all. Also once we know G^*, we know g_i^* for every i. To see this, note that

$$g_i^* = w_i + G^* - f_i^{-1}(G^*) \text{ for } i \in C^*$$

and zero otherwise.

A more subtle result is the following:

FACT 2: There exists a real valued function $F(G^*, C^*)$, differentiable and increasing in G^* such that in a Nash equilibrium,

$$F(G^*, C^*) = \sum_{i \in C^*} w_i.$$

Proof. Since by assumption $f_i' \in (0,1)$, it has a strictly increasing inverse ϕ_i. Moreover, $\phi_i' > 1$. Applying ϕ_i on both sizes of the equality in (6), we obtain

$$\forall i \in C^*, \phi_i(G^*) = w_i + G^*_{-i} = w_i + G^* - g_i^*$$

Summing over all $i \in C^*$, we have

$$\sum_{i \in C^*} \phi_i(G^*) = \sum_{i \in C^*} w_i + (|C^*| - 1) G^*$$

where $|C^*|$ is the cardinality of the set C^*. Now we can define

$$F(G^*, C^*) = \sum_{i \in C^*} \phi_i(G^*) - (|C^*| - 1) G^*.$$

Note that $F(\cdot, C^*)$ is monotonically increasing in G^* since

$$\sum_{i \in C^*} \phi_i'(G^*) > |C^*|. \blacksquare$$

The monotonicity of $F(\cdot, C^*)$ immediately implies that for a fixed set of contributors, there is a unique solution G^* to the equation $F(G^*, C^*) = \sum_{i \in C^*} w_i$.

Combining FACTS 1 and 2, we have the following important conclusion: characterizing the Nash equilibrium allocation is equivalent to characterizing C^* or G^*.

Proposition 4 *[Uniqueness of Equilibrium] Under the assumption that* $f_i' \in (0,1)$, *there is a unique Nash equilibrium given any distribution of wealth.*

Proof. Suppose that there are two Nash equilibria, with public goods quantities G and G', and corresponding positive contribution sets C and C' respectively. Without loss of generality, assume $G' \leq G$.

Since G' is a Nash equilibrium, by FACT 1, we have

$$G' \geq f_i\left(w_i + G'_{-i}\right) \ \forall i \in N$$

Hence $\phi_i(G') \geq w_i + G'_{-i}$ $\forall i \in N$ and in particular for all $i \in C$. Hence

$$\begin{aligned} \sum_{i \in C} w_i &\leq \sum_{i \in C} \phi_i(G') - \sum_{i \in C} \underbrace{\left(G' - g_i'\right)}_{G'_{-i}} \\ &= \sum_{i \in C} \phi_i(G') - |C| G' + \sum_{i \in C} g_i' \end{aligned}$$

Since $\sum_{i\in C} g'_i \leq \sum_{i\in C'} g'_i = G'$ by the definition of C and C', we have

$$\sum_{i\in C} w_i \leq \sum_{i\in C} \phi_i \left(G'\right) - |C|\, G' + G' = F\left(G', C\right)$$

Since $F(G, C) = \sum_{i\in C} w_i$, we have

$$F(G, C) \leq F\left(G', C\right).$$

Since $F(\cdot, C)$ is monotonic, we have $G \leq G'$. Hence $G = G'$. But then FACT 1 implies that $C = C'$ and $g_i = g'_i$. ■

Proposition 4 suggests the following algorithm to calculate the Nash equilibrium for any finite set N and wealth distribution:

- Choose an arbitrary subset $C \subseteq N$;
- Use $F(G, C) = \sum_{i\in C} w_i$ to calculate G [a unique G exists since F is monotonic in G];
- If $G \geq f_j(w_j + G)$ for all $j \notin C$ is satisfied, DONE, WE HAVE FOUND THE UNIQUE EQUILIBRIUM; otherwise, continue with a different set C.

4.1.5 Comparative Statics Regarding Changes in the Wealth Distribution

How does the level of the equilibrium public goods provision responds to the change in the wealth distribution? This comparative statics can lead to testable implications. The following proposition turns out to be crucial:

Proposition 5 *Let $\{g_i\}$ and $\{g'_i\}, i = 1, ..., n$ be Nash equilibria given the wealth distributions $\{w_i\}$ and $\{w'_i\}$ respectively. Let C and C' be the corresponding sets of contributing agents. Then*

$$F\left(G', C\right) - F(G, C) \geq \sum_{i\in C} \left(w'_i - w_i\right).$$

Proof. From FACT 1, we have

$$G' \geq f_i\left(w'_i + G'_{-i}\right) \ \forall i \in C$$

Hence

$$w'_i + G'_{-i} \leq \phi_i\left(G'\right) \ \forall i \in C$$

Sum over $i \in C$,

$$\begin{aligned}\sum_{i\in C} w_i' &\leq \sum_{i\in C} \phi_i\left(G'\right) - |C|\,G' + \sum_{i\in C} g_i' \\ &\leq \sum_{i\in C} \phi_i\left(G'\right) - |C|\,G' + G' = F\left(G', C\right)\end{aligned}$$

Since $\sum_{i\in C} w_i = F\left(G, C\right)$, we have the desired conclusion. ■

Exercise 2 *Prove the following implications of the above proposition:*

1. *Any change in the wealth distribution that leaves unchanged the aggregate wealth of current contributors will either increase or leave unchanged the equilibrium supply of the public good;*

2. *Any change in the wealth distribution that increases the aggregate wealth of current contributions will necessarily increase the equilibrium supply of the public good;*

3. *If a redistribution of wealth among current contributors increase the equilibrium supply of the public good, then the set of contributing agents after the redistribution must be a proper subset of the original set of contributors;*

4. *Any simple transfer of wealth from another agent to a currently contributing agent will either increase or leave constant the equilibrium supply of the public good.*

Proof of claim 3: From FACT 1, we know that for all $j \notin C$, $G - f_j\left(w_j + G\right) \geq 0$. Since $f_j' \in (0,1)$, we know that if $G' > G$, we have $G' - f_j\left(w_j + G'\right) \geq 0$ for all $j \notin C$. Hence $C' \subseteq C$. But if $C' = C$, then if we consider a redistribution from $\{w_i'\}$ to $\{w_i\}$, we will have, by claim 1, $G \geq G'$. A contradiction.

4.1.6 Does Public Provision Crowd Out Private Provision?

It seems intuitively rather obvious that it should be possible to translate the above results regarding income redistribution to results about what happens when the government provides the good publicly, financed by taxes on the citizens. Let

- g_0 be the public provision of the good;

- t_i be the (lump sum) tax on citizen i.

We require budget balance so that $g_0 = \sum_{i=1}^n t_i$. From the single consumer's point of view, it is immaterial whether the good is provided by the government or other consumers, so g_0 affects best responses just like any other voluntary contribution. Also, the lump sum tax t_i is just a reduction in the endowment for i. These two facts suggests that taxing everyone less than they contribute and publicly providing the good would not change anything.

Proposition 6 *Let $(g_1^*, ..., g_n^*)$ be the unique equilibrium levels of contribution in the model with no public intervention. Consider a policy $(g_0, t_1, ..., t_n)$ with $\sum_{i=1}^n t_i = g_0$. Then*

1. *If $t_i \leq g_i^*$ for all $i \in N$, then there is a unique equilibrium in the model with policy, given by $(g_1', ..., g_n')$ such that $g_i' = g_i^* - t_i \forall i \in N$;*
2. *If $t_j > 0$ for some $j \notin C^*$, then although private contribution may decrease, the equilibrium total supply of the public good must increase;*
3. *If for some $i \in C^*, t_i > g_i^*$, then the equilibrium total supply of the public good must increase.*

SKETCH OF PROOF. As before, we can show that under policy $(g_0, t_1, ..., t_n)$, the equilibrium (G, C) must satisfy

$$F(G, C) = \sum_{i \in C} w_i - \sum_{i \in C} t_i + g_0$$

where

$$F(G, C) = \sum_{i \in C} \phi_i(G) - (|C| - 1) G.$$

Claim 1: If $\sum_{i \in C^*} t_i = g_0$, then by revealed preference argument as in the proof of Proposition 2 shows that $G = G^*$. For claim 2, if $g_0 - \sum_{i \in C^*} t_i > 0$, then $F(G, C^*) - F(G^*, C^*) \geq g_0 - \sum_{i \in C^*} t_i > 0$. Hence $G > G^*$. For claim 3: think as follows, first taxing the agent by the exact amount of his contribution, and then taxing him for the extra amount. In the first step, he will reduce his contribution to zero, and leave the total public good unchanged by assertion 1; the second step increases the public good supply by assertion 2.

4.1.7 Guide to the Recent Literature

Bernheim (AER, 1986) Bernheim (1986) extends the neutrality results concerning the voluntary private funding of public goods. The strong and unbelievable neutrality results lead him to question the assumptions in this literature: first, individuals care about the magnitude of their contributions only insofar as these contributions affect the aggregate level of expenditures; second, chains of operative voluntary transfers and contributions link all individuals.

NOTATIONS:

- Individuals $i = 1, ..., n$;
- Each individual chooses labor supply l_i; and private good consumption x_i;
- There are two public goods (G, H);
- Individual i's utility function is given by
$$u_i(l_i, x_i, G, H).$$
- Assume that the units of all goods are chosen so that all prices are unity.

TIMING OF THE MODEL:

- Stage 1: Government picks a policy
$$P = (y, \tau, \gamma_0, \eta_0)$$
where $y \equiv (y_1, ..., y_n)$ is the non-labor income that the government transfers to the agents; $\tau \equiv (\tau_1, ..., \tau_n)$ are a vector of labor income taxes, we allow each τ_i to depend on the vector of labor supply by all the agents $l = (l_1, ..., l_n)$; γ_0, which is also a function of l, is the government's contribution to the first public good; and η_0 is the government's contribution to the second public good. The government's budget constraint is
$$\eta_0(l) + \gamma_0(l) = \left(Y - \sum_{i=1}^{n} y_i\right) + \sum_{i=1}^{n} \tau_i(l)$$
where Y is the government's non-tax resources.
- Stage 2: Each consumer, knowing the government's policy P, simultaneously chooses labor supply; and the government collects its revenue according to τ;
- Stage 3: The government funds to the public goods as prescribed by its policy P and the consumers simultaneously decide how much to contribute to the public good and how much to consume as private good.

NASH EQUILIBRIUM:

Assume that all the consumers contribute to the first public good (G) and none to the second public good (H). Assume that non-negativity constraint on the public good provision is non-binding. Given a policy P^*, consumer i's strategy consists of choosing a labor supply level l_i in stage 2 and a function γ_i which prescribes a level of contribution to the first public good for every potential labor supply vector l. Given P^*, $\{l_i^*, \gamma_i^*\}_{i=1}^n$ is a Nash equilibrium if

- for all l, $\gamma_i^*(l)$ solves
$$\max_{g_i} u_i\left(l_i, y_i^* + l_i - \tau_i^*(l) - g_i, \sum_{j\neq i} \gamma_j^*(l) + g_i, \eta_0^*(l)\right)$$
where $j \neq i$ is understood to include $j = 0$ ($j = 0$ stands for the government).

- For each i, l_i^* solves
$$\max_{l_i} u_i\left(l_i, y_i^* + l_i - \tau_i^*\left(l_i, l_{-i}^*\right) - \gamma_i^*\left(l_i, l_{-i}^*\right), \sum_{j=0}^{n} \gamma_j^*\left(l_i, l_{-i}^*\right), \eta_0^*\left(l_i, l_{-i}^*\right)\right).$$

MAIN RESULT:

Proposition 7 *Consider any two policies, $P = (y, \tau, \gamma_0, \eta_0)$ and $P' = (y', \tau', \gamma_0', \eta_0')$. Suppose that $\eta_0 = \eta_0'$. Then any final allocation sustained as an equilibrium under P can also be sustained as an equilibrium under P'.*

The proof of this result is left as an exercise. The intuition is simple: if all but one individual acts to offset the policy change for each labor supply profile, then the opportunity set of the remaining individual is unchanged: it is therefore optimal for him as well to neutralize any effects. The assumption that the non-negative private contribution to the public good G is non-binding is clear important.

Andreoni (JPE, 1998) In the previous studies on the private provision of public goods, there is no role for fund-raising or seed money, because of the crowding out. But in reality, fund raising and seed money, either from a government grant, or from a group of "leadership givers", can generate additional gift. For more empirical motivations, see the paper. Andreoni (1998)'s paper provides a theoretical basis for fund-raisers and seeds to charity.

THE MODEL:

- n individuals, $i = 1, ..., n$
- x_i : $i's$ consumption of private good
- g_i : $i's$ contribution to the public good;
- w_i : $i's$ wealth
- $u_i(x_i, G)$: $i's$ utility function
- [**major departure**] The public good production function is

$$G = \begin{cases} \sum_{i=1}^{n} g_i \text{ if } \sum_{i=1}^{n} g_i \geq \bar{G} \\ 0 \text{ otherwise} \end{cases}$$

[Clearly if $\bar{G} = 0$, then we are back to BBV model]

Because of the minimum threshold $\bar{G}$, it is possible that the simultaneous contribution game has a unique equilibrium at zero contribution. If some leaders are chosen to make binding pledges of contributions, it is possible that the zero contribution equilibrium is eliminated and the public good is completed. This idea is related to the dynamic contribution to the public good that we will discuss below.

4.1.8 Morgan (ReStud 2000)

Lotteries is a popular and widespread decentralized mechanism for financing public goods. How does lottery compare with standard voluntary contributions mechanisms. Morgan (2000)'s paper shows that, relative to the standard voluntary contribution mechanism, lotteries increases the provision of the public good, and are welfare improving. What is the intuition? The reason for the under-provision of public good in voluntary contribution models is that the positive externality of public good is not properly internalized. When purchasing lotteries, however, agents do not properly internalize a negative externality because an agent's lottery purchase not only increases his chance of winning, but also decrease others' chance of winning. This negative externality leads to over-purchase of lotteries, and acts as a compensating externality to ameliorate the free-rider problem in the provision of the public good.

THE MODEL:

- $N = \{1, ..., n\}$: set of individuals (consumers, bettors);
- w_i : individual i's wealth
- utility function is quasi-linear
$$u_i = x_i + h_i(G)$$
where x_i is the (numeraire) private good, and G is the public good. $h_i' > 0, h_i'' < 0$ for all i.
- The production function for the public good is given by
$$G = f(z) = z.$$

FIRST BEST:

From the Samuelson condition (specialized to the quasi-linear utility function and linear production function), the first best level of public good is determined by

$$\sum_{i=1}^{n} h_i'(G^*) = 1$$

if the public good is desirable, i.e. $G^* > 0$; and 0 if $\sum_{i=1}^{n} h_i'(0) < 1$. It should be clear that voluntary contribution will lead to an under-provision of the public good relative to the first best due to free riding problem.

FIXED-PRIZE RAFFLES

Now suppose that the government or charity chooses a fixed prize R. Each individual chooses a wager $y_i \in [0, w_i]$. Given the wagers of the other contestants, y_{-i}, individual i will win the prize R with probability

$$\pi(y_i, y_{-i}) = \frac{y_i}{\sum_{i=1}^{n} y_i}$$

The government or the charity, will then provide the public good with the total wagers over the prize outlay R. That is,

$$G = \sum_{i=1}^{n} y_i - R.$$

If funds are insufficient to cover the cost of the prize, the raffle will be called off and the wagers will be returned to the contestants. [Also needs a δ amount of financing by the charity to rule out zero wager equilibrium.]

Given the bets $(y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$, individual i under a fixed prize raffle solves

$$\max_{y_i \in [0, w_i]} w_i - y_i + \frac{y_i}{y_i + \sum_{j \neq i} y_j} R + h_i \left(y_i + \sum_{j \neq i} y_j - R \right).$$

One can show that

Proposition 8 *The fixed prize raffle provides more of the public good than the voluntary contributions.*

PARI-MUTUEL RAFFLES:

Alternatively, the government or the charity can designate a percentage of the total wagers to be placed in a prize pool. This is called pari-mutuel raffles: some percentage, p, of the “handle” (the total bets) is rebated in the form of prizes; and the remainder $1-p$ of the handle is used to fund a public good.

Given the bets $(y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$, individual i under a pari-mutuel prize raffle solves

$$\max_{y_i \in [0, w_i]} w_i - y_i + \frac{y_i}{y_i + \sum_{j \neq i} y_j} p \left[y_i + \sum_{j \neq i} y_j \right] + h_i \left((1-p) \left(y_i + \sum_{j \neq i} y_j \right) \right).$$

Proposition 9 *The equilibrium public goods provision in a pari-mutuel raffle is exactly the same as that obtained through voluntary contribution.*

What is the intuition? While in a fixed prize raffle, individuals do not internalize the negative externality effects of reducing the winning probability of others, here in a pari-mutuel raffle, there is also a positive externality that is internalized: the bets also increases the prize pool available to all other bettors! In the quasi-linear utility function case, these two externalities cancel out exactly.

4.2 Dynamic Voluntary Provision of Public Goods

Prior to the literature on private provision of public goods, Thomas Schelling had the following insightful comment on this issue:

> “If each party agrees to send a million dollar to the red cross on condition that the other does, each may be tempted to cheat if the other contributes first,

> and each one's anticipation of the other cheating will inhibit agreement. But if the contributions are divided into consecutive small contributions, each can try the other's good faith at a small price. Furthermore, since each can keep the other on short tether to finish, no one ever need risk more than a small contribution at a time."

The idea that dividing the contributions into small sums may help was surprisingly enough not analyzed until the late 80s.

4.2.1 Admati and Perry (ReStud, 1991)

THE CONTRIBUTION GAME

- Two identical players, 1 and 2;
- The value of a completed project is V to both players;
- The total cost of the completing the project is K. [Note that the project is either completed or not completed];
- Players take turns in making contributions, starting with player 1 in period 1. The project is completed as soon as the total contributions made by both players reach the total cost K.

PRELIMINARIES

- Let c_i^t be the amount of player i's contribution at period t;
- A history at time τ is a sequence of contributions made by agents prior to time τ : $\left\{c_1^t, c_2^t\right\}_{t=1}^{\tau-1}$. If it is not i's turn to move in period t, then $c_i^t = 0$;
- A *strategy* for player i specifies the size of his contribution for each history after which it is i's turn to move;
- Players are impatient and discount rates are both $\delta \in (0,1)$. Let T be the first time at which the total contributions reach K, where $T = \infty$ if it is not completed. An outcome of the game is
$$\left\{T, \left\{c_1^t\right\}_{t=1}^{T}, \left\{c_2^t\right\}_{t=1}^{T}\right\}.$$

Player i's payoff from such an outcome is

$$U_i\left(T, \{c_1^t\}_{t=1}^T, \{c_2^t\}_{t=1}^T\right) = \delta^{T-1}V - \sum_{t=1}^{T} \delta^{t-1} W\left(c_i^t\right)$$

where $W(\cdot)$ is the function which measures the disutility from contributing, $W' > 0, W'' > 0$ and $W(0) = 0$.

The main result of the paper relies on the following variables:

- After a history $\left(\{c_1^t\}_{t=1}^\tau \{c_2^t\}_{t=1}^\tau\right)$, denote X as the total amount of contribution still required for the completion of the project, i.e.,

$$X\left(\{c_1^t\}_{t=1}^\tau \{c_2^t\}_{t=1}^\tau\right) = K - \sum_{t=1}^{\tau} c_1^t - \sum_{t=1}^{\tau} c_2^t.$$

Now we recursively define the amount of contribution that would be made in the equilibrium path if the project is completed in equilibrium, denoted by $\{R_q\}_{q=1}^\infty$

STEP 1: Define R_1 as the maximum amount that a player is willing to contribute now if by doing so he completes the project ; while if he contributes zero then the project will be completed next period by the other player. That is, R_1 solves

$$V - W(R_1) = \delta V.$$

It is clear that if in a subgame $X < R_1$, then it is a dominant strategy for the player whose turn it is to complete the project. For any $X \in [0, R_1]$, let

$$\begin{aligned} U_a^*(X) &= V - W(X) \\ U_b^*(X) &= V. \end{aligned}$$

$U_a^*(X)$ and $U_b^*(X)$ are respectively the payoffs of the first and second player to move if the remaining size of the project is X and the first player completes the project in his turn.

STEP 2: Let R_2 be the contribution level that makes a player indifferent between (i) contributing R_2 right now under the assumption that the project will be completed in the next period by the other player; and (ii). contributing zero right now and completing the project in two periods by contributing R_1 then (with the rest contributed by the other player in the next move). That is, R_2 solves

$$\delta V - W(R_2) = \delta^2 V - \delta^2 W(R_1),$$

which can be rewritten as

$$\delta U_b^* (S_1) - W (R_2) = \delta^2 U_a^* (S_1)$$

where $S_n = \sum_{q=1}^{n} R_q$ [hence $S_1 = R_1$]. It is clear that if in a subgame $R_1 < X < R_1 + R_2$, it is an iteratively dominant strategy for the player with the current move to contribute $X - R_1$ assuming that the other player will follow his dominant strategy to complete the project (with contribution R_1). For $X \in [R_1, R_1 + R_2]$, or equivalently, for $X \in [S_1, S_2]$, define

$$\begin{aligned} U_a^* (X) &= \delta U_b^* (S_1) - W (X - R_1) \\ U_b^* (X) &= \delta U_a^* (S_1) \end{aligned}$$

$U_a^* (X)$ is the current mover's payoff if he contributes just enough, $X - R_1$, so that after his move the required contribution is R_1 and the second mover completes the project in his turn.

STEP n : Define R_n recursively as the amount that makes player i whose turn it is indifferent between (i) contributing R_n now, and obtaining $U_b^* (S_{n-1})$ in the next period; and (ii) contributing zero now and obtaining $U_a^* (S_{n-1})$ in two periods. That is, R_n solves

$$\delta U_b^* (S_{n-1}) - W (R_n) = \delta^2 U_a^* (S_{n-1}) .$$

And for every $X \in [S_{n-1}, S_n]$, define

$$\begin{aligned} U_a^* (X) &= \delta U_b^* (S_{n-1}) - W (X - S_{n-1}) \\ U_b^* (X) &= \delta U_a^* (S_{n-1}) . \end{aligned}$$

The essentially unique equilibrium path of the contribution game is described as follows:

Proposition 10 *Let* $S_\infty = \sum_{q=1}^{\infty} R_q$.

1. *Suppose* $S_\infty > K$.

   (a) *If there exists* $N < \infty$ *such that* $S_{N-1} < K < S_N$*, then the unique equilibrium path is: player 1 contributes* $K - S_{N-1}$ *in period 1, and for* $1 < t \leq N$*, the amount contributed in period* t *is* R_{N-t+1}*. Thus the project is completed in* N *rounds.*

(b) *If there exists* $N < \infty$ *such that* $S_N = K$*, then there are two equilibrium paths. In addition to the path described above, the other equilibrium path is as follows: player 1 contributes zero in period 1, and for* $2 \leq t \leq N$*, the amount contributed in period* t *is* R_{N-t+2}*. Thus the project is completed in* $N+1$ *periods.*

2. *If* $S_\infty \leq K$*, then the unique equilibrium path is* $c_i^t = 0$ *for all* i *and all* t*.*

Now we consider a special linear contribution cost function $W(c) = bc$. The equilibrium path described in the above proposition remains an equilibrium path (though uniqueness is lost). For the linear case, we can explicitly solve for S_n, which turns out to be

$$S_n = \frac{V\left(1 - \delta^{2n-1}\right)}{b},$$

hence $S_\infty = V/b$. Hence Proposition 10 implies that the project is completed in that equilibrium if and only if $K < V/b$, which is equivalent to $V > bK$. That is, in a linear case a necessary and sufficient condition for the completion of the project in the above equilibrium is that each player would complete the project immediately if he was the only player. The inefficiency due to delay is stark. However, there *may* be other more efficient equilibria.

More generally, Admati and Perry show that:

Proposition 11 *If* $V \leq W'(0) K$*, then the project is not completed in equilibrium.*

4.2.2 Marx and Matthews (ReStud 2000)

Marx and Matthews (2000) revisit the dynamic voluntary contribution game. The made the following changes to the Admati and Perry (1991) model.

- [One Departure] $N = \{1, ..., n\}$ is the set of agents, $n \geq 2$.
- Time is discrete, periods are indexed by $t = 0, 1, 2, ...$
- [Another Departure] Players can contribute towards a public project in each period. Let $z_i(t)$ denote i's contribution in period t.

- Write

$$\begin{aligned} z(t) &= (z_1(t), ..., z_n(t)) \\ Z(t) &= \sum_{j=1}^{n} z_j(t) \\ Z_{-i}(t) &= \sum_{j \neq i} z_j(t) \end{aligned}$$

  Denote i's cumulative contribution up to time t by

$$x_i(t) = \sum_{\tau=1}^{t} z_i(\tau)$$

  and the aggregate cumulative contribution by

$$X(t) = \sum_{j=1}^{n} x_j(t).$$

- [One More Departure] The agents are assumed to receive utility from the public good in each period. If the aggregate cumulative contribution in period t is X and agent i contributes $z_i(t)$, then agent i's period t payoff is $(1-\delta) f(X) - z_i(t)$, where

$$f(X) = \begin{cases} \lambda X \text{ if } X < K \\ V \text{ if } X \geq K. \end{cases}$$

- More generally, given a sequence of contributions $z = \{z(t)\}_{t=0}^{\infty}$, player i's payoff is

$$U_i(z) = \sum_{t=0}^{\infty} \delta^t [(1-\delta) f(X) - z_i(t)].$$

The main results of Marx and Matthews are roughly as follows:

1. Allowing for contributions to be split up in small pieces may alleviate the inefficiencies of static contribution games: equilibria where a project is implemented exists in the dynamic game for parameterization where the unique equilibrium in the static game has nobody contributing;

2. There is still an inefficiency due to delay;

3. However, this inefficiency may be rather inconsequential if the time period between contributions are small.

5 Provision of Public Goods with Private Information

So far we have assumed perfect information, that is, we assume that the individuals know of others preferences and wealth etc. This is not likely to be the case in reality. When agents' valuations from a public good project are private information, will there exist mechanisms to insure the efficient provision of public good? If not, what is the probability of provision of public goods under the optimal mechanism? The main reference is Mailath and Postlewaite (ReStud, 1990).

5.1 An Illustrative Example: two agent case

- 2 individuals, $i = 1, 2$
- v_i : i's valuation for the public good, and assume that $v_i \in \{L, H\}$ and ex ante $v_i = L$ or $v_i = H$ occurs with equal probability; v_1 and v_2 are independent. Let $L = 0$, $H = 10$. v_i is agent i's private information.
- Cost of providing the public good is 8.

Ideally we would like to have a mechanism that has the following three properties:

1. Efficiency: produce the public good if and only if it is efficient to do so. [In our example, to provide the public good unless both agents have valuation L]
2. Individual rationality: participation must be voluntary;
3. Incentive compatibility: each agent has incentive to truthfully report her privately known valuation.

We assume for now that the cost is shared if both agents announces H. Does there exist a mechanism with the above three properties? Suppose that agent 2 tells the truth, consider agent 1 with type H. Her payoff matrix if under such a mechanism with the efficiency property is as follows:

		Agent 2's type	
		L	H
Agent 1's announcement	$\tilde{L}$	0	10
	$\tilde{H}$	2	6

The expected payoff for agent 1 when announcing $\tilde{L}$ is $0.5 \times 0 + 0.5 \times 10 = 5$ while announcing $\tilde{H}$ is $0.5 \times 2 + 0.5 \times 6 = 4$. Hence truth telling is not a Bayesian Nash equilibrium.

This simple example illustrates that there may not exist a mechanism that simultaneously satisfy efficiency, individual rationality and incentive compatibility. The intuition is simply the free riding problem: in cases when an individual's announcement will not change the probability of the public good being provided (due to the requirement of efficiency), he will have incentive to report a lower valuation so as to be taxed less. Now it is natural to ask the following question: what efficiency level can we achieve by mechanisms that satisfy IR and IC constraints? In this example, suppose that the public good provision probabilities are given by

$$\rho(L, L) = 0, \rho(H, L) = \rho(L, H) = \rho, \rho(H, H) = 1$$

where $\rho(H, L)$ denotes the probability of the public good being provided when agent 1 announces H and agent 2 announces L. The tax rule remains the same as before. Hence we have the following table:

Announcement Profile	Probability of providing the public good	Taxes if the public good is provided
(L, L)	0	n.a.
(L, H)	ρ	$(0, 8)$
(H, L)	ρ	$(8, 0)$
(H, H)	1	$(4, 4)$.

Suppose that the other agent tells the truth. We first verify the IC constraints for L type:

$$u\left(\tilde{L}|L\right) = 0 > u\left(\tilde{H}|L\right) = \frac{1}{2}(-8-4) = 6$$

hence the IC constraints for the low type is satisfied for all p. It is intuitive because if your valuation is 0, why would you lie upward?

Now we verify the IC constraints for H type:

$$\begin{aligned} u\left(\tilde{H}|H\right) &= \frac{1}{2}[\rho(10-8) + (10-4)] = \rho + 3 \\ u\left(\tilde{L}|H\right) &= \frac{1}{2}(10\rho) = 5\rho \end{aligned}$$

IC for type H hence requires that

$$u\left(\tilde{H}|H\right) \geq u\left(\tilde{L}|H\right) \Leftrightarrow \rho \leq \frac{3}{4}.$$

Hence IR, IC mechanism will have to sacrifice efficiency.

5.2 Impossibility Result for Large Economies

It is desirable to have a mechanism that is ex ante efficient, that is, that provides the public good whenever the cost was less than the sum of the benefits to the agents in the economy. When agents' valuations for the public good are privately known, this may be difficult, as we showed in the two agent example above: if agents' cost shares do not depend on their announced values, agents whose valuations exceed their cost shares will have incentive to overstate their valuations; on the other hand, if agents' cost shares *do* depend on their announcements, they may have an incentive to understate their valuations. If we impose an individual rationality constraint that prohibits taxing an agent more than his announced valuation for the public good, we will be in the latter case where agents may have an incentive to understate their valuations. It is intuitive that the understatement problem gets more sever when the number of agents get larger.

NOTATIONS:

- n agents, $i = 1..., n$;
- v_i : $i's$ valuation for the public good, $v_i \in \{H, L\}$ which is private information, and ex ante $\Pr(v_i = H) = p \in (0, 1)$.
- Agent i's utility when the public good is provided and he pays tax t is $v_i - t$;
- The cost of providing the public good in an n person economy is cn [i.e. per capita cost of the public good is constant];
- Assume that $H > c > L$.

DIRECT REVELATION MECHANISM:

In the direct revelation mechanism, the message space for agent i is $M_i = M = \{H, L\}$. The mechanism needs to specify an outcome function (ξ, ρ) where:

- $\xi : M^n \to R^n_+$ is the vector of taxes the agents pay as a function of the announced valuation profile *if* the public good is produced;
- $\rho : M^n \to [0, 1]$ is the probability that the public good is produced as a function of the agents' announced valuation profile.

We can further restrict (without loss of generality) that $\rho(\cdot)$ depends only on k, the number of agents announcing L, and ignore which specific agents announced L. It can also be shown easily that the tax scheme that provides the greatest incentive to agents with high value H to announce truthfully is

$$\xi_i(v) = \begin{cases} L \text{ if } v_i = L \text{ and } kL + (n-k)H \geq nc \\ \frac{nc-kL}{n-k} \text{ if } v_i = H \text{ and } kL + (n-k)H \geq nc. \end{cases}$$

It is trivial to see that the incentive compatibility constraint for L is satisfied with this tax rule. The incentive constraint for H is

$$\sum_{k=0}^{n-1} p(k)\rho(k)\left(H - \frac{nc-kL}{n-k}\right) \geq \sum_{k=0}^{n-1} p(k)\rho(k+1)(H-L)$$

where $p(\cdot)$ is the probability that there are exactly k agents of type L among the remaining $n-1$ agents [note that the number of type L agents among the $n-1$ remaining agents have a Binomial distribution with parameters $n-1$ and p] [convince yourself that the left (respectively, right) hand side is the agent's expected utility of announcing H (respectively announcing L) assuming all other agents tell truth. This can be re-written as

$$\sum_{k=0}^{n-1} p(k)[\rho(k) - \rho(k+1)][H-L] \geq \sum_{k=0}^{n-1} p(k)\rho(k)\left(\frac{nc-nL}{n-k}\right).$$

We will show that the left hand side must go to zero as n goes to infinity; and that the probability of provision of the public good in the n person problem must be smaller than the right hand side.

STEP 1: Show that the right hand side goes to zero as n goes to infinity. Because $p(\cdot)$ is the binomial density, it is maximized at an integer $t \in ((n+1)p - 1, (n+1)p]$. For this t, $p(k-1) < p(k)$ for $k < t$ and $p(k-1) > p(k)$ for $k > t$ and $p(t) \geq p(t-1)$.

CLAIM 1: $\sum_{k=0}^{n-1} p(k)[\rho(k) - \rho(k+1)]$ is maximized with $\rho(k) = 1$ for $k \leq t$ and $\rho(k) = 0$ for $k > t$.

Proof. Let $\rho^*(\cdot)$ be the maximizer of the expression. First we show that $\rho^*(t) = 1$ and $\rho^*(t+1) = 0$. Suppose not, let $\rho'(\cdot) = \rho^*(\cdot)$ except let $\rho'(t) = 1$ and $\rho'(t+1) = 0$. Then

$$\begin{aligned} & \sum_{k=0}^{n=1} p(k)\left[\rho'(k) - \rho'(k+1)\right] - \sum_{k=0}^{n=1} p(k)[\rho^*(k) - \rho^*(k+1)] \\ = \; & [1 - \rho^*(t)][p(t) - p(t-1)] - \rho^*(t+1)[p(t+1) - p(t)] > 0. \end{aligned}$$

A contradiction. Now we can claim that $\rho^*(k) = 1$ for all $k \leq t-1$. Suppose not, then there exist $k \leq t-1$ with $\rho(k) < \rho(k+1)$. Suppose that the smallest such k is k'. Let $\rho'(\cdot) = \rho^*(\cdot)$ except at this k' and let $\rho'(k') = 1$. Then

$$\begin{aligned} & \sum_{k=0}^{n=1} p(k)\left[\rho'(k) - \rho'(k+1)\right] - \sum_{k=0}^{n=1} p(k)\left[\rho^*(k) - \rho^*(k+1)\right] \\ = & \left[1 - \rho^*(k')\right]\left[p(k') - p(k'-1)\right] > 0 \end{aligned}$$

a contradiction. Similarly it can be shown that $\rho(k) = 0$ for all $k > t$. ■

The implication of this claim is that

$$\begin{aligned} \sum_{k=0}^{n-1} p(k)\left[\rho(k) - \rho(k+1)\right][H-L] & \leq p(t)(H-L) \\ & = (H-L)\frac{(n-1)!}{(n-1-t)!t!}p^t(1-p)^{n-1-t} \end{aligned}$$

Using Sterling's formula which states that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}n^{n+1/2}e^{-n}} = 1$$

and using the approximation that $p \approx t/(n-1)$ when n is large, we get

$$\frac{(n-1)!}{(n-1-t)!t!}p^t(1-p)^{n-1-t} \to \frac{1}{\sqrt{2\pi(n-1)p(1-p)}} \to 0 \text{ as } n \to \infty.$$

Hence the left hand side of the incentive constraint goes to zero.

Now we examine the RHS of the incentive constraint:

$$\begin{aligned} \sum_{k=0}^{n-1} p(k)\rho(k)\left(\frac{nc - nL}{n-k}\right) & = (c-L)\sum_{k=0}^{n-1} p(k)\rho(k)\frac{n}{n-k} \\ & \geq (c-L)\sum_{k=0}^{n-1} p(k)\rho(k). \end{aligned}$$

Since the LHS goes to zero as n goes to infinity, the right hand side goes to zero as well. Hence

$$\sum_{k=0}^{n-1} p(k)\rho(k) \to 0 \text{ as } n \to \infty.$$

But $\sum_{k=0}^{n-1} p(k)\rho(k)$ is simply the expected probability of the public good being provided.

6 Local Public Goods

The under-provision of public goods problem we discussed so far is mainly due to the fact that the government can not elicit the citizens' (voters') preferences truthfully. Tiebout (1956) suggests that lots of public goods are provided by local expenditures, and if there were enough communities, individuals would reveal their true preferences for public goods by the choice of community in which to live (in much the same way as individuals reveal their preferences for private goods by their choices), Where there is a wide range of choices, all those deciding to live in the same community would have essentially the same taste, and there would be no problem of reconciling conflicting preferences. This is an intriguing idea since it suggests that the invisible had can solve the important problem of under-provision of public goods. Local public goods are public goods that can be enjoyed only by residents in the local community: for example, local public school, beaches, parks, etc. are typical examples of public goods. The crucial idea in Tiebout's hypothesis is residential mobility, which creates competition among communities.

6.1 A Class of Tiebout Models (Bewley Econometrica 1981)

Bewley (1981) formalizes Tiebout's idea in Arrow-Debreu general equilibrium models:

- Decision Makers in the Model: Consumers, Firms, and Local Governments
- n distinct regions of habitation (communities), $j = 1, ..., n$.
- Each region has a government which provides local public goods, and collects local taxes to pay for them;
- **Crucial Assumption:** There is perfect consumer mobility between regions;
- Consumers are fully informed about prices, taxes, and public goods in each region and choose to live in the region where they can enjoy the highest level of utility;
- Consumers behave competitively, i.e., they do not believe that prices, taxes, or the provision of public goods are influenced by their own choices.
- Local government cannot discriminate among consumers by name or according to taste when levying taxes (otherwise, Lindahl prices may be charged); local government knows about the initial endowments of residents when levying taxes;

- Suppose that there is no spillover effects between regions; the inhabitants of one region are affected by what happens in other regions only through prices and migration;
- **COST OF LOCAL PUBLIC GOODS**: We can think of three possibilities:
  1. The cost is independent of population (called "pure public goods case");
  2. The cost is proportional to the population, hence the per capita cost is constant (called "pure public service case");
  3. The per capita cost of public goods is a U-shaped function of population.
- How does production take place? Two possibilities are considered:
  1. Autarkic Regions Case: all production takes place inside regions and there is no trade between them;
  2. Free Trade Case: production is completely independent of the regional distribution of populations.
- Assume that regions do not directly affect utility or production and there is no land;
- What is the motivation of the government? Two possibilities are considered:
  1. Democratic governments seek to maximize the welfare of their own citizens;
  2. Entrepreneurial governments have objectives which are independent of the welfare of their citizens, may try to repel some inhabitants or attract new ones.
- Decisions: Consumers choose consumption bundles and regions of residence; Firms choose their input-output vectors; Local governments choose the bundle of public goods or services and a tax system: a tax system specifies each inhabitant's tax payment as a function of his initial endowment.

An *allocation* in a Tiebout model specifies the following items:

1. The consumption bundle of each consumer;
2. The input-output vector of each firm;
3. The bundle of public goods provided by each regional government;

4. The region of residence of each consumer.

An allocation is *feasible* if the goods absorbed by consumers and governments may be produced or supplied directly from the consumers' initial endowments. A feasible allocation is *Pareto optimal* if there exists no other feasible allocations which makes every consumer at least as well off and some strictly better off.

A *Tiebout equilibrium* consists of a feasible allocation, a price for each commodity, and a tax system for each region such that the following conditions are satisfied:

1. Each consumer's consumption bundle satisfies his budget constraint given his residence choice and it maximizes the consumer's utility in this budget set;

2. Each consumer chooses the region he prefers (When a consumer compares regions, he assumes that tax systems, public expenditures and prices would not change if he moved);

3. Each firm maximizes profit;

4. Each regional government balances budget (i.e. public goods provided by a regional government must be equal to its tax revenue);

5. Each government's expenditures, tax system and choice of inhabitants are consistent with whatever its objective may be.

6.2 Are Tiebout Equilibria Pareto Efficient?

Tiebout equilibria may not be Pareto efficient. Bewley (1981) proceeds by providing a series of examples which highlight the various perspectives which lead to the inefficiency of Tiebout equilibria.

6.2.1 Democratic Government

Example 4 (Democratic Government, Pure Public Goods) *There are two identical consumers and two regions. One public good and one private good which is leisure. Each consumer is endowed with one unit of leisure. The consumer's utility function is* $u(l, g) = g$ *where* g *is the level of local public good in the region he inhabits and* l *is leisure. The production technology for public goods is* $g_j = L_j$ *where* g_j *is the public good provided in region* j *and* L_j *is the quantity of labor.*

One may wonder, why do consumers behave competitively? If I move from region 1 to 2, shouldn't I expect that the government in region 2 will change its provision of public good. It is true if there are indeed 2 consumers, so we should imagine that there are continuum of consumers with measure two.

One can check that the following is a Tiebout equilibrium: The prices of labor and public good are both 1. The tax system is that a tax of 1 is imposed on any initial endowment. One consumer lives in each region, each consumer sells his leisure to the producer of the public goods and one unit of public good is provided in each region. This Tiebout equilibrium is inefficient since both consumers can be better off if they lived together in one region and two units of public gods are provided in that region and none in the other. The latter is also a Tiebout equilibrium. The punch line: there may be multiple Tiebout equilibria and some Tiebout equilibrium may not be Pareto efficient.

The feature of this example is that the cost of public good provision is independent of the population size and in the inefficient Tiebout equilibrium, the economies of scale is not properly exploited.

The next example illustrates that economies of scale is not necessary for the inefficiency to result:

Example 5 (Democratic Government, Pure Public Service) *Two identical regions, four consumers, and four types of public services, and one private good which is leisure. Each consumer is endowed with one unit of leisure. Regions are labeled by 1 and 2, consumers and public services are labeled by A, B, C, D. Consumer k's utility function is given by*

$$\begin{aligned} u_A(l, g_A, g_B, g_C, g_D) &= 2g_A + g_B \\ u_B(l, g_A, g_B, g_C, g_D) &= g_A + 2g_B \\ u_C(l, g_A, g_B, g_C, g_D) &= 2g_C + g_D \\ u_D(l, g_A, g_B, g_C, g_D) &= g_C + 2g_D \end{aligned}$$

The public service production function in region j is

$$g_{jA} + g_{jB} + g_{jC} + g_{jD} = \frac{L_j}{n_j}$$

where n_j is the number of consumers in region j.

The following situation describes a Tiebout equilibrium: consumer A and C live in region 1 and consumer B and D live in region 2, $(g_{1A}, g_{1B}, g_{1C}, g_{1D}) = (1, 0, 1, 0)$ and

$(g_{2A}, g_{2B}, g_{2C}, g_{2D}) = (0, 1, 0, 1)$. The prices of labor and each type of public service are both 1 in both regions. The tax system in each region is to tax away each consumer's endowment. This is an equilibrium because in this allocation, each consumer gets a utility 2 and if he moves to the other region his utility is reduced to 1.

This Tiebout equilibrium is not Pareto optimal. The following allocation (which is also a Tiebout equilibrium allocation) dominates it: consumers A and B live in region 1, C and D in region 2 and $(g_{1A}, g_{1B}, g_{1C}, g_{1D}) = (1, 1, 0, 0)$ and $(g_{2A}, g_{2B}, g_{2C}, g_{2D}) = (0, 0, 1, 1)$. In this allocation each consumer obtains 3 units of utility.

6.2.2 Entrepreneurial Government

In the above examples local governments do not initiate changes in the supply of public goods when agents move. In Example 4, if one of the regional government proposes a tax system of 1 for every resident and provides 2 units of public goods, such a region will attract all the consumers and the resulting allocation will be efficient; in Example 5, if a local government proposes a tax of one unit for every resident and provides a public service bundle $(g_{1A}, g_{1B}, g_{1C}, g_{1D}) = (1, 1, 0, 0)$, it will attract all the type A and B consumers. Again efficiency will be restored. Is it the case that introducing such entrepreneurial government solves the inefficiency problem? The answer is not necessarily. Here we consider such governments.

One possible motivation for an entrepreneurial government is to maximize the population of the region they control; another possible motivation is to maximize profits: their revenues come from taxes and their expenses are the cost of public goods. The following difficulties must be resolved: First, should we assume that all the regional governments simultaneously choose a public goods/service and tax system mix? Bewley (1981) assumes that each government believes that no other government will change policy when it changes its own. Second, what happens if a change in policy leads to multiple competitive equilibrium in the competitive good market?

Example 6 (Profit-maximizing governments, pure public goods) *Two regions, two consumers, one private good which is leisure, and one public good. Each consumer is endowed with one unit of leisure. Production function for the public good is $g = L$. The utility function of consumer 1 is $u_1(l, g) = g$, while the utility function for consumer 2 is $u_2(l, g) = 3l + g$.*

The following situation is a Tiebout equilibrium of this example. Consumer i lives in region i. In region 1, consumer 1 devotes all his labor to the production of the public good and so consumes no leisure and one unit of the public good; consumer 2 devotes no labor to the production of the public good and consumes one unit of leisure and none of the public good. In each region the price of each good is 1. The tax system in region 1 requires that all consumers pay a tax of 1 and in region 2 no tax is levied. The utility level in this equilibrium of consumer 1 is 1 and that of consumer 2 is 3. Why is this a Tiebout equilibrium? Clearly each consumer optimizes. Suppose that a new regional regime were to established involving both consumers. They would have to pay the same taxes since their initial endowment is the same. Let τ be the tax rate. Then 2τ units of the public good will be provided. To attract consumer 1, it must be the case that $2\tau \geq 1$, hence $\tau \geq 1/2$; to attract consumer 2, it must be the case that $3(1-\tau)+2\tau \geq 3$, which implies that $\tau \leq 0$. Hence there exists no such τ that can attract both consumers away from the above postulated situation.

This Tiebout equilibrium allocation is not Pareto efficient: it is dominated by the following allocation: consumers live together, consumer 1 pays all his income in taxes and consumer 2 pays none. One unit of public good would be provided, consumer 1 would consume no leisure and consumer 2 would consume one unit of leisure. The utility of consumer 1 in this allocation is 1, as was in the previous equilibrium; but the utility of consumer 2 would be 4, higher than before. The reason that this is not an equilibrium is that we assumed that it is impossible to tax the consumers at different rate.

6.3 A Tiebout Model with Efficient Equilibrium

- There are L private goods, and N public goods; The public goods have constant per capita unit provision cost of 1.
- There are J regions;
- There is free trade in both public and private goods, hence the price system is an economy-wide price system. It is given by

$$(p,q) \in \Delta^{L+N-1} = \left\{ p \in R_+^L, q \in R_+^N : \sum_{k=1}^{L} p_k + \sum_{k=1}^{N} q_k = 1 \right\}.$$

- There are T types of consumers: a type t consumer is endowed with $\omega_t \in R_+^L$; type t consumer's utility function is given by $u_t(x,g)$; There are I_t consumers of type t;

- Consumers are indexed by (t,i), where t denotes the type and i distinguishes consumers of the same type. Let $\mathcal{I} = \{(t,i) : t = 1, ..., T, i = 1, ..., I_t\}$.
- A tax system for region j is a function $\tau_j : R^L_+ \to [0,\infty)$, recall that τ_j is a function of the endowment of an individual, that is, if a type t consumer lives in region j, he will be taxed $\tau_j(\omega_t)$;
- The budget set of a consumer of type t if he lives in a region j is

$$B_t(p,\tau_j) = \left\{x \in R^L_+ : p \cdot x \leq p \cdot \omega_t - \tau_j(\omega_t)\right\}.$$

- The production function for the public good is constant returns to scale; hence there are no profits in equilibrium and we do not need to worry about the distribution of profits. The production possibility set is described by Y.

An allocation for the economy is of the form (r,x,g,y) where

- $r : \mathcal{I} \to \{1, ..., J\}$ is a function which assigns consumers to their regions of residence;
- $x = \left(x_{(t,i)}\right)_{(t,i)\in\mathcal{I}}$ is the vector which describes the allocation of private goods, where $x_{(t,i)} \in R^L_+$ for all (t,i);
- $g = (g_1, ..., g_J)$ describes the allocation of public services, where $g_j \in R^N_+$;
- $y \in Y$ is the social production vector.

Given a residence function r, we use n^r_j to denote the number of consumers assigned to region j according to r, i.e.,

$$n^r_j = |(t,i) : r(t,i) = j|.$$

An allocation (r,x,g,y) is feasible if

$$\left(\sum_{(t,i)\in\mathcal{I}} \left(x_{(t,i)} - \omega_t\right), \sum_{j=1}^{J} n^r_j g_j\right) = y.$$

In the above expression, $\left|\sum_{(t,i)\in\mathcal{I}} \left(x_{(t,i)} - \omega_t\right)\right|$ is the excess of the private goods endowments after consumption in the economy; and $\sum_{j=1}^{J} n^r_j g_j$ is the total production cost in the economy.

A Tiebout equilibrium consists of a feasible allocation (r,x,g,y), a price system $(p,q) \in \Delta^{L+N-1}$, and tax system $(\tau_1, ..., \tau_J)$ that satisfies the following conditions:

1. (Profit Maximization): $(p,q)\cdot y = \max\{(p,q)\cdot z : z \in Y\};$

2. (Utility Maximization and Free Mobility): For all $(t,i) \in \mathcal{I}$,

$$u_t\left(x_{(t,i)}, g_{r(t,i)}\right) = \max\{u_t(x, g_j) : x \in B_t(p, \tau_j), j = 1, ..., J\}$$

and

$$x_{(t,i)} \in B_t\left(p, \tau_{r(t,i)}\right) \text{ for all } (t,i) \in \mathcal{I}.$$

3. (Perfectly Competitive Governments, or zero profit condition on the government):

$$n_j^r q \cdot g_j = \sum_{\{(t,i):r_{(t,i)}=j\}} \tau_j(\omega_t).$$

4. (Government Profit Maximization) No local government can propose a tax system τ_j' and a fiscally feasible vector of public services g_j' that makes all of its existing consumers better off than they are under the postulated allocation (r,x,g,y).

We make the following assumptions:

A1. $T \leq J$. That is, there are at least as many regions as types of consumers;

A2. $u_t : R_+^{L+N} \to R$ is continuous, strictly quasi-concave and strictly monotone, for all t.

A3. For all t, $\omega_t \in R_+^L$ and $\omega_t \neq \mathbf{0}$.

A4. $Y \subset R^{L+N}$ and Y is a closed convex cone with apex zero;

A5. $Y \cap R_+^{L+N} = \{\mathbf{0}\};$

A6. There exists $\bar{y} \in Y$ such that every component of

$$\bar{y} + \left(\sum_{t=1}^{n} \omega_t, 0\right)$$

is positive.

Proposition 12 *Under A1-A6, there exists a Tiebout equilibrium; moreover, every Tiebout equilibrium allocation is Pareto optimal.*

The idea of the proof is to define a corresponding general equilibrium economy with no regions or public services. First, apply a standard existence theorem to show that the new economy has a competitive equilibrium, then show that a Tiebout equilibrium corresponds to this competitive equilibrium.

6.4 Empirical Studies Related to Tiebout Hypothesis

There are quite a few empirical papers that attempt to test the implications of Tiebout Hypothesis.

Oates (JPE 1969) Oates (JPE 1969) used a cross section data of 53 municipalities in New Jersey to estimate the effect of local public finance on property values. The so-called Tiebout hypothesis (which is not clearly spelled out) is as follows: If a community increases its property rate in order to expand its output of public services, then because consumers shop around different communities offering various packages of local public services and selects as a residence the community which offers the tax-expenditure program which best suited to his taste, it is possible that property value needs not decrease, and may well increase, as a result of the increase in the property tax rate. His findings are as follows:

1. Local property values bear a significant negative relationship to the effective tax rate and a significant positive relationship with expenditure per pupil in the public schools;

2. The size of the coefficients suggests that, for an increase in property taxes unaccompanied by an increase in the output of local public services, the bulk of the rise in taxes will be capitalized in the form of reduced property values; on the other hand, if a community increases its tax rates and employs the receipts to improve its school system, the increased benefits from the expenditure side of the budget will roughly offset the depressive effect of the higher tax rates on local property values.

It is a good idea to read this paper and write down your comments and criticisms of this paper. See Epple, Zelenitz and Visscheer (June 1978, JPE) for a systematic study of the testable implications of the Tiebout Hypothesis.

Gramlich and Rubinfeld (JPE 1982) Gramlich and Rubinfeld (1982) test two implications from a Tiebout model: first, if consumers sort themselves according to their taste of public goods, or public services, then in a Tiebout equilibrium, citizens should have grouped themselves together with others with similar tastes, and hence, the variance of local spending demands within a community should be smaller than the variance throughout the whole state; second, communities should actually respond to the citizens' desired level of public goods (the median voter hypothesis), since otherwise, there is no point for the voters to sort themselves.

Gramlich and Rubinfeld (1982) used a micro survey on demands on public spending from 2001 Michigan households, which also includes information on fiscal, demographic, voting and attitudes etc. To test the first hypothesis, they compare the variance of local spending demands within a community with those throughout the state, after controlling for other factors that may influence spending in all districts. To test the second hypothesis, they compare the desired level of public spending with the actual level of public spending and see if the communities supply the level of public expenditures desired by the median voter in their community. Their results are supportive of Tiebout Hypothesis.