A Generalized Solution of the Monopolistic Competition Model with Heterogeneous Firms and a Linear Demand (Melitz-Ottaviano)*

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Abstract

This note solves the monopolistic competition model with linear demand for multiple countries, but without an outside sector. The purpose is to compare the model's predictions regarding firm sales and their distribution with those arising from a model in which consumers are characterized by symmetric CES preferences.

1 Introduction

In this note, we solve a version of the monopolistic competition model with heterogeneous firms and linear demand (Melitz Ottaviano (2008)). We extend the model to a multi-country setting, and allow for income effects by dispensing with the outside sector. The main results of Melitz-Ottaviano still hold; in particular, firm mark-ups depend on firm size. However, we highlight a number of new results for the model. First, the distribution of sales of small firms is more skewed than the simple Pareto distribution, which is what the constant elasticity of substitution (CES) model would imply. This result is similar to Arkolakis (2008). Second, when compared to the CES framework, the model lacks an additional degree of freedom in order to match both bilateral aggregate trade across countries and the distribution of sales. Finally, the equilibrium number of entrants is a function of the population and other constants, but not of bilateral trade costs. This result is the same as the one found by Arkolakis, Demidova, Klenow and Rodriguez-Clare (2008) for the CES model.

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2 Solving for the FOCs

Assume a measure L of identical consumers, where each one of them is endowed with 1 unit of labor and does not value leisure. Preferences of a representative consumer over a continuum of products $\omega \in \Omega$ are given by

$$U = \alpha \int_{\Omega} q^c(\omega)\, d\omega - \frac{1}{2}\gamma \int_{\Omega} \left(q^c(\omega)\right)^2 d\omega - \frac{1}{2}\eta \left(\int_{\Omega} q^c(\omega)\, d\omega \right)^2$$

where α, η, γ are all positive and $q^c(\omega)$ is the quantity consumed. The consumer maximizes this utility function subject to the budget constraint

$$\int_{\Omega} q^c(\omega) p(\omega) d\omega = w,$$

where w is the unit wage and $p(\omega)$ is the price of good ω. The FOCs of the above problem yield $(\forall q^c(\omega) > 0)$:

$$\lambda p(\omega) = \alpha - \gamma q^c(\omega) - \eta \int_{\Omega} q^c(\omega)\, d\omega. \tag{1}$$

where λ is the Lagrangian multiplier. Also, from (1), we can derive:

$$q^c(\omega) = \frac{1}{\gamma} \left(\alpha - \lambda p(\omega) - \eta \int_{\Omega} q^c(\omega)\, d\omega \right). \tag{2}$$

Let $\Omega^* \subset \Omega$ represent consumed varieties, and let M be the measure of this set. Defining:

$$\bar{q^c} := \frac{1}{M} \int_{\Omega^*} q^c(\omega) d\omega, \qquad \bar{p} := \frac{1}{M} \int_{\Omega^*} p(\omega) d\omega,$$

and integrating (1) over all $\omega \in \Omega^*$ yields:

$$\begin{aligned} \lambda \bar{p} &= \alpha - \gamma \bar{q^c} - \eta M \bar{q^c} \implies \\ \bar{q^c} &= \frac{\alpha - \lambda \bar{p}}{\gamma + \eta M}. \end{aligned}$$

Following (2), demand for variety ω is:

$$q(\omega) = \frac{L}{\gamma} \left(\alpha - \lambda p(\omega) - \eta \int_{\Omega} q^c(\omega)\, d\omega \right).$$

Defining $\bar{q} := L\bar{q^c} = \frac{1}{M} \int_{\Omega^*} L q^c(\omega) d\omega = \frac{1}{M} \int_{\Omega^*} q(\omega) d\omega$ (by definition of $q(\omega)$) and substituting the expression for $\bar{q^c}$ into (2) yields:

$$\begin{aligned}
q(\omega) &= \frac{L}{\gamma}\left(\alpha - \lambda p(\omega) - \eta M \bar{q^c}\right) \implies \\
q(\omega) &= \frac{L}{\gamma}\alpha - \lambda\frac{L}{\gamma}p(\omega) - \frac{L}{\gamma}\eta M\bar{q} \implies \\
q(\omega) &= \frac{L}{\gamma}\alpha - \lambda\frac{L}{\gamma}p(\omega) - \frac{L}{\gamma}\eta M\frac{\alpha - \lambda\bar{p}}{\gamma + \eta M} \implies \\
q(\omega) &= \frac{\alpha L}{\gamma + \eta M} - \lambda\frac{L}{\gamma}p(\omega) + \frac{\eta L}{\gamma}\frac{M\bar{p}}{\gamma + \eta M}\lambda.
\end{aligned}$$

It follows that $q(\omega) = 0$ exactly when

$$\lambda p(\phi^*) = \lambda p^* := \alpha + \eta M\lambda\frac{\bar{p} - \alpha}{\gamma + \eta M} = \frac{\gamma\alpha + \eta M\bar{p}\lambda}{\gamma + \eta M}.$$

The profit maximization problem of the firm with productivity draw ϕ is

$$\begin{aligned}
&\max_{q(\phi)} p(\phi)\, q(\phi) - \frac{w}{\phi}q(\phi) \\
= \quad &\max p(\phi)\left(\frac{\alpha L}{\gamma + \eta M} - \frac{L}{\gamma}\lambda p(\phi) + \frac{\eta L}{\gamma}\frac{M\bar{p}\lambda}{\gamma + \eta M}\right) - \frac{w}{\phi}\left(\frac{\alpha L}{\gamma + \eta M} - \frac{L}{\gamma}\lambda p(\phi) + \frac{\eta L}{\gamma}\frac{M\bar{p}\lambda}{\gamma + \eta M}\right),
\end{aligned}$$

which implies the FOC

$$\begin{aligned}
\frac{\alpha L}{\gamma + \eta M} - 2\frac{L}{\gamma}\lambda p(\phi) + \frac{\eta L}{\gamma}\frac{M\bar{p}}{\gamma + \eta M} + \frac{w}{\phi}\frac{L}{\gamma}\lambda &= 0 \implies \\
\frac{L}{\gamma}\left(\lambda p(\phi) - \lambda\frac{w}{\phi}\right) &= q(\phi). \qquad (3)
\end{aligned}$$

The FOCs also imply that:

$$\begin{aligned}
\frac{1}{2}\frac{\gamma}{L}\left(\frac{\alpha L}{\gamma + \eta M} + \frac{\eta L}{\gamma}\frac{M\bar{p}}{\gamma + \eta M} + \frac{w}{\phi}\frac{L}{\gamma}\right) &= \lambda p(\phi) \implies \\
\frac{1}{2}\left(\lambda p^* + \lambda\frac{w}{\phi}\right) &= \lambda p(\phi) \qquad (4)
\end{aligned}$$

by definition of p^*.

Recall that the marginal firm must satisfy

$$\begin{aligned}
p^* &= \frac{w}{\phi^*} \implies \\
\lambda p(\phi^*) &= \lambda p^* = \lambda\frac{w}{\phi^*}
\end{aligned}$$

which yields:

$$\frac{1}{2}\left(\lambda\frac{w}{\phi^*} + \lambda\frac{w}{\phi}\right) = \lambda p(\phi) \implies$$

$$\frac{1}{2}\left(\frac{w}{\phi^*}+\frac{w}{\phi}\right)=p\left(\phi\right) \tag{5}$$

Substituting (4) into (3) yields:

$$\frac{L}{2\gamma}\lambda\left(\frac{w}{\phi^*}-\frac{w}{\phi}\right)=q\left(\phi\right) \tag{6}$$

Also (4) yields the following mark-up:

$$\frac{1}{2}\left(\frac{w}{\phi^*}-\frac{w}{\phi}\right)= \qquad \mu\left(\phi\right) \tag{7}$$

Using (5) and (6), total revenue is:

$$\begin{aligned} p\left(\phi\right)q\left(\phi\right) &= \lambda\left(\frac{w}{\phi^*}+\frac{w}{\phi}\right)\frac{1}{2}\frac{L}{\gamma}\frac{1}{2}\left(\frac{w}{\phi^*}-\frac{w}{\phi}\right) \\ &= \lambda\frac{L}{4\gamma}\left(\left(\frac{w}{\phi^*}\right)^2-\left(\frac{w}{\phi}\right)^2\right) \end{aligned} \tag{8}$$

and profits are:

$$\begin{aligned} \pi\left(\phi\right) &= \lambda\frac{L}{4\gamma}\left(\left(\frac{w}{\phi^*}\right)^2-\left(\frac{w}{\phi}\right)^2\right)-\lambda\frac{w}{\phi}\frac{L}{2\gamma}\left(\frac{w}{\phi^*}-\frac{w}{\phi}\right) \\ &= \lambda\frac{L}{4\gamma}\left(\frac{w}{\phi^*}-\frac{w}{\phi}\right)^2 \end{aligned} \tag{9}$$

3 Equilibrium

Assume the Pareto distribution of productivities such that the firms pay an entry cost and draw their productivities from

$$G\left(\Phi\leq\phi\right)=1-\frac{b^\theta}{\phi^\theta}$$

We assume that the firms pay an entry cost f_e to enter in the market. After paying the cost they learn their productivity. Let G_{ϕ^*} be the distribution of productivity conditional on entry, so $G_{\phi^*}(\phi) = 1 - \left(\frac{\phi^*}{\phi}\right)^\theta$. Then expected profits, conditional on entry, are

$$
\begin{aligned}
\int_{\phi^*}^{\infty} \pi(\phi) dG_{\phi^*}(\phi) &= \int_{\phi^*}^{\infty} \pi(\phi)\theta(\phi^*)^{\theta}\phi^{-\theta-1}d\phi \\
&= \theta(\phi^*)^{\theta}\lambda \int_{\phi^*}^{\infty} \frac{L}{4\gamma}\left(\frac{w}{\phi^*}-\frac{w}{\phi}\right)^2 \phi^{-\theta-1}d\phi \\
&= \frac{L\theta(\phi^*)^{\theta}w^2}{4\gamma}\lambda \int_{\phi^*}^{\infty}\left[\left(\frac{1}{\phi^*}\right)^2+\left(\frac{1}{\phi}\right)^2-\frac{2}{\phi^*\phi}\right]\phi^{-\theta-1}d\phi \\
&= \frac{L\theta(\phi^*)^{\theta}w^2}{4\gamma}\lambda\left[\frac{1}{\theta(\phi^*)^{\theta+2}}+\frac{1}{(\theta+2)(\phi^*)^{\theta+2}}-\frac{2}{(\theta+1)(\phi^*)^{\theta+2}}\right] \\
&= \frac{L\theta w^2}{4\gamma}\lambda\left[\frac{1}{\theta(\phi^*)^2}+\frac{1}{(\theta+2)(\phi^*)^2}-\frac{2}{(\theta+1)(\phi^*)^2}\right] \\
&= \frac{Lw^2}{2\gamma(\phi^*)^2}\lambda\left[\frac{1}{(\theta+1)(\theta+2)}\right]
\end{aligned}
$$

Therefore the free entry condition implies

$$
\begin{aligned}
&(1-G(\phi^*))\lambda\frac{Lw^2}{2\gamma(\phi^*)^2}\left[\frac{1}{(\theta+1)(\theta+2)}\right] = wf_e \\
\Leftrightarrow &\left(\frac{b}{\phi^*}\right)^{\theta}\frac{L}{2\gamma(\phi^*)^2}w\lambda\left[\frac{1}{(\theta+1)(\theta+2)}\right] = f_e \\
\Leftrightarrow &(\phi^*)^{\theta+2} = \frac{Lb^{\theta}}{2\gamma f_e}w\lambda\left[\frac{1}{(\theta+1)(\theta+2)}\right] \\
\Leftrightarrow &\phi^* = (w\lambda)^{\frac{1}{\theta+2}}\left(\frac{Lb^{\theta}}{2\gamma f_e}\left[\frac{1}{(\theta+1)(\theta+2)}\right]\right)^{\frac{1}{\theta+2}}.
\end{aligned}
\tag{10}
$$

Labor market clearing, which in this case is equivalent to the consumers spending their entire labor income and is also equivalent to labor market clearing implies

$$
\begin{aligned}
wL &= \int_{\phi^*}^{\infty} r(\phi)M dG_{\phi^*}(\phi) \\
&= \lambda\int_{\phi^*}^{\infty}\frac{M}{4\gamma}\left(p^*+\frac{w}{\phi}\right)\left(p^*-\frac{w}{\phi}\right)dG_{\phi^*}(\phi) \\
&= \frac{LM\theta(\phi^*)^{\theta}}{4\gamma}\lambda\int_{\phi^*}^{\infty}\left(\left(\frac{w}{\phi^*}\right)^2-\left(\frac{w}{\phi}\right)^2\right)\phi^{-\theta-1}d\phi \\
&= \frac{LM\theta}{4\gamma}\lambda\left(\frac{w^2}{(\phi^*)^2\theta}-\frac{w^2}{(\theta+2)(\phi^*)^2}\right) \\
&= \frac{LM}{2\gamma}\lambda\frac{w^2}{(\phi^*)^2}\frac{1}{\theta+2}.
\end{aligned}
$$

This gives the wage times the multiplier λ as a function of productivity cutoff:

$$w\lambda = \frac{2\gamma(\theta+2)(\phi^*)^2}{M}. \tag{11}$$

Substituting (11) into (10):

$$\begin{aligned}
\phi^* &= \left(w\lambda \frac{Lb^\theta}{2\gamma f_e} \left[\frac{1}{(\theta+1)(\theta+2)} \right] \right)^{\frac{1}{\theta+2}} \\
&= \left(\frac{2\gamma(\theta+2)(\phi^*)^2}{M} \frac{Lb^\theta}{2\gamma f_e} \left[\frac{1}{(\theta+1)(\theta+2)} \right] \right)^{\frac{1}{\theta+2}} \\
&= \left(\frac{L(\phi^*)^2 b^\theta}{(\theta+1)M f_e} \right)^{\frac{1}{\theta+2}} \\
\Leftrightarrow (\phi^*)^\theta &= \frac{Lb^\theta}{M(\theta+1)f_e} \implies \\
M &= \frac{b^\theta}{(\phi^*)^\theta} \frac{L}{(\theta+1)f_e} \qquad (12)
\end{aligned}$$

which is the expression for ϕ^* in terms of primitives and the equilibrium number of firms. In other words the number of entrants is $\frac{L}{(\theta+1)f_e}$ and out of them a fraction $\frac{b^\theta}{(\phi^*)^\theta}$ survives. Thus, the Krugman effect of the number of firms entering the market is linear in the population of the market is still here, as far as it concerns the number of firms that try to enter the market. However, how many operate will depend on how ϕ^* changes with L.

Now to determine the equilibrium number of firms,recall that for $\eta \neq 0$ we have:

$$\begin{aligned}
\frac{\lambda w}{\phi^*} &= \alpha + \eta M \frac{\bar{p} - \alpha}{\gamma + \eta N} \\
\Leftrightarrow \frac{\lambda w}{\phi^*}(\gamma + \eta M) &= \gamma\alpha + \eta M \bar{p} \\
\Leftrightarrow M &= \frac{\gamma\left(\alpha - \frac{\lambda w}{\phi^*}\right)}{\eta\left(\frac{\lambda w}{\phi^*} - \bar{p}\right)}. \qquad (13)
\end{aligned}$$

First, note that, using (11) and (12),

$$\begin{aligned}
\lambda \frac{w}{\phi^*} &= \frac{2\gamma(\theta+2)(\phi^*)^2}{M\phi^*} \\
&= \frac{2\gamma(\theta+2)}{M} \left(\frac{Lb^\theta}{M(\theta+1)f_e} \right)^{\frac{1}{\theta}} \qquad (14)
\end{aligned}$$

and

$$
\begin{aligned}
\bar{p} &:= \int_{\phi^*}^{\infty} p(\phi) dG_{\phi^*}(\phi) \\
&= \int_{\phi^*}^{\infty} \frac{1}{2}\left(\frac{w}{\phi^*} + \frac{w}{\phi}\right) \theta(\phi^*)^{\theta} \phi^{-\theta-1} d\phi \\
&= \frac{w\theta(\phi^*)^{\theta}}{2} \int_{\phi^*}^{\infty} \frac{\phi^{-\theta-1}}{\phi^*} + \phi^{-\theta-2} d\phi \\
&= \frac{w\theta}{2\phi^*}\left(\frac{1}{\theta} + \frac{1}{\theta+1}\right) \\
&= \left(\frac{w}{2\phi^*}\right)\left(\frac{2\theta+1}{\theta+1}\right) \qquad (15)
\end{aligned}
$$

So using (14) and (15) in (13) yields

$$
M = \frac{(\theta+1)^{\frac{\theta+1}{\theta}} \alpha f_e^{\frac{1}{\theta}}}{\eta(\theta+2) b L^{\frac{1}{\theta}}} M^{\frac{\theta+1}{\theta}} - \frac{2\gamma(\theta+1)}{\eta}
$$

The LHS is a line with slope 1, the RHS is convex in M s.t. at M=0 it's negative, and as M tends to infinity, it also goes to infinity. Thus, the two curves must intersect and an equilibrium exists. Given that we find M we can also recover ϕ^* from 12.

4 The case $\eta = 0$

By the definition of $w\lambda = a\phi^*$ and using equation (11) then we have

$$
a = \frac{2\gamma(\theta+2)\phi^*}{M} \ . \qquad (16)
$$

Now using the equation which substituting in (12)

$$
\begin{aligned}
M &= \frac{b^{\theta}}{(\phi^*)^{\theta}} \frac{L}{(\theta+1) f_e} \implies \\
\frac{2\gamma(\theta+2)\phi^*}{a} &= \frac{b^{\theta}}{(\phi^*)^{\theta}} \frac{L}{(\theta+1) f_e} \implies \\
\phi^* &= \left(\frac{a b^{\theta} L}{2\gamma(\theta+2)(\theta+1) f_e}\right)^{1/(\theta+1)} \qquad (17)
\end{aligned}
$$

and of course

$$
M = \frac{2\gamma(\theta+2)}{a} \left(\frac{a b^{\theta} L}{2\gamma(\theta+2)(\theta+1) f_e}\right)^{1/(\theta+1)} \qquad (18)
$$

which implies that total output for the case that $\eta = 0$ is given only by labor income since

$$\begin{aligned}
M\int_{\phi^*} p(\phi)\, q(\phi)\, d\phi &= M\lambda \frac{Lw^2}{4\gamma}\int_{\phi^*}^{\infty}\left(\left(\frac{1}{\phi^*}\right)^2 - \left(\frac{1}{\phi}\right)^2\right)\frac{\theta\left(\phi^*\right)^{\theta}}{\phi^{\theta+1}}d\phi \\
&= M\lambda \frac{Lw^2\theta}{4\gamma}\left(\left(\frac{1}{\phi^*}\right)^2\frac{1}{\theta} - \frac{1}{\theta+2}\left(\frac{1}{\phi^*}\right)^2\right) \\
&= M\lambda \frac{L\theta}{4\gamma}\frac{w^2}{\left(\phi^*\right)^2}\left(\frac{2}{\theta}\frac{1}{\theta+2}\right) \\
&= wa\phi^*\frac{L}{4\gamma}\frac{M}{\left(\phi^*\right)^2}\theta\left(\frac{2}{\theta}\frac{1}{\theta+2}\right) \\
&= wL\ ,
\end{aligned}$$

where we used (11), (16) in the last equality. Notice, that even though average productivity increases with L also the number of firms is not as large so in the end GDP depends linearly in L.

4.1 Discussion of results for $\eta = 0$

There are not so many different things related to the equilibrium expressions (17), (18). While b enters ϕ^*, N and L enters N in the same way as in the CES model, a difference can be seen with respect to the relation L and ϕ^*. In particular, with linear demand higher L implies higher ϕ^*. However, higher productivity does not change the relationship of aggregate GDP with L since it is also linear (given that the number of firms is not growing as fast as L increases).

Regarding the distribution of sales we have

$$\begin{aligned}
p(\phi)\, q(\phi) &= \lambda\left(\frac{w}{\phi^*}+\frac{w}{\phi}\right)\frac{1}{2}\frac{L}{\gamma}\frac{1}{2}\left(\frac{w}{\phi^*}-\frac{w}{\phi}\right) \\
&= \lambda\left(\frac{w}{\phi^*}\right)^2\frac{L}{4\gamma}\left(1-\left(\frac{\phi^*}{\phi}\right)^2\right)
\end{aligned}$$

Using $w\lambda = a\phi^*$,

$$a = \frac{2\gamma(\theta+2)\phi^*}{M}$$

and from the Pareto distribution that

$$\frac{\phi^*}{\phi} = (1-\mathrm{Pr})^{1/\theta}$$

we have

$$p(\phi)\, q(\phi) = \lambda\left(\frac{w}{\phi^*}\right)^2\frac{L}{4\gamma}\left(1-(1-\mathrm{Pr})^{2/\theta}\right)$$

and when we divide by average sales

$$\int_{\phi^*} p(\phi)\, q(\phi)\, dG_{\phi^*}(\phi) = \lambda \frac{L}{4\gamma} \frac{w^2}{(\phi^*)^2} \left(\frac{2}{\theta+2}\right)$$

we get the percentile sales normalized by mean sales

$$\frac{y}{\bar{y}} = \frac{1-(1-\Pr)^{2/\theta}}{\frac{2}{\theta+2}}$$

which has the desirable property that is independent of market size.

Finally, notice that trade shares are given by

$$\begin{aligned}
\lambda_{ij} &= \frac{J_i \frac{b_i^\theta}{(\phi_{ij}^*)^\theta} \theta \frac{L_j}{\mu_j 4\gamma} \left(\frac{2}{\theta} \frac{1}{(\theta+2)}\right)}{\sum_\upsilon J_\upsilon \frac{b_\upsilon^\theta}{(\phi_{\upsilon j}^*)^\theta} \theta \frac{L_j}{\mu_j 4\gamma} \left(\frac{2}{\theta} \frac{1}{(\theta+2)}\right)} \\
&= \frac{J_i \frac{b_i^\theta}{(\phi_{ij}^*)^\theta}}{\sum_\upsilon J_\upsilon \frac{b_\upsilon^\theta}{(\phi_{\upsilon j}^*)^\theta}} \implies \qquad (19)
\end{aligned}$$

$$\lambda_{ij} = \frac{J_i b_i^\theta w_i^{-\theta} \tau_{ij}^{-\theta}}{\sum_\upsilon J_\upsilon b_\upsilon^\theta w_\upsilon^{-\theta} \tau_{\upsilon j}^{-\theta}}$$

4.2 Using the Linear Demand for quantitative analysis

Notice that the only coefficient affecting the distribution of sales is the curvature of the Pareto, θ. This fact highlights one of the main drawbacks of using the simple linear demand model for quantitative analysis in trade: the same parameter will be the only one appearing in the gravity equations, not allowing enough degrees of freedom to both match the observed distribution of firm sales and the total trade among countries (see the appendix in the original Melitz-Ottaviano paper for the derivation of total trade flows as a function of trade costs and wages and also expression (19)). On the contrary, the model with CES demand features that additional degree of freedom since both θ and the elasticity of demand, σ, appear in the expression for the distribution of sales. Thus, a generalization of the CES framework such as the framework in Arkolakis (2008) allows both for a micro-foundation of the deviations from CES demand (so that there is a theoretical underpinning for a non-strictly CES demand structure) and a quantitative successful framework for predicting trade flows across countries.[1] Of course the model with linear demand may be extended in the

[1] An additional feature of the CES model with market specific costs of entry is that it can achieve the log-linear association between number of entrants in a market and the size of the market. This emprical pattern may not be replicated in a straightforward way in the linear demand model without market specific costs of entry. Goksel (2008) considers the predictions of the linear demand model for bilateral trade flows under these different specifications.

future to other forms of demand that incorporate linear demand as a subcase. In addition, the linear demand framework is successful in terms of delivering variable markups something that a model based on CES is not easy to get.[2]

5 Multi-Country Model

We can construct a multi-country version of the model

$$U_j = \alpha \int_{\Omega_j} q_j^c(\omega)\, d\omega - \frac{1}{2}\gamma \int_{\Omega_j} \left(q_j^c(\omega)\right)^2 d\omega - \frac{1}{2}\eta \left(\int_{\Omega_j} q_j^c(\omega)\, d\omega \right)^2$$

where α, η, γ are all positive and $q^c(\omega)$ is the quantity consumed. The consumer maximizes this utility function subject to the budget constraint

$$\int_{\Omega_j} q_j^c(\omega) p_j(\omega) d\omega = w_j,$$

where w is the unit wage and $p(\omega)$ is the price of good ω. The FOCs of the above problem yield $(\forall q^c(\omega) > 0)$:

$$\mu_j p_j(\omega) = \alpha - \gamma q_j^c(\omega) - \eta \int_{\Omega_j} q_j^c(\omega)\, d\omega. \tag{20}$$

where μ_j is the Lagrangian multipliers. Also, we can derive:

$$q_j^c(\omega) = \frac{1}{\gamma} \left(\alpha - \mu_j p_j(\omega) - \eta \int_{\Omega_j} q_j^c(\omega)\, d\omega \right). \tag{21}$$

Let $\Omega_j^* \subset \Omega_j$ represent consumed varieties, and let M be the measure of this set. Defining:

$$\bar{q_j^c} := \frac{1}{M_j} \int_{\Omega_j^*} q_j^c(\omega) d\omega, \qquad \bar{p}_j := \frac{1}{M_j} \int_{\Omega_j^*} p_j(\omega) d\omega,$$

and integrating (1) over all $\omega \in \Omega^*$ yields:

$$\begin{aligned} \mu_j \bar{p}_j &= \alpha - \gamma \bar{q_j^c} - \eta M_j \bar{q_j^c} \implies \\ \bar{q_j^c} &= \frac{\alpha - \mu_j \bar{p}_j}{\gamma + \eta M_j}. \end{aligned}$$

[2] Simonovska (2008) proposes a model with non-homothetic preferences that features variable mark-ups, allowing her to capture a positive relationship between prices of tradeables and per-capita income levels across countries. She also tests the quantitative predictions of the model analyzed in this note regarding prices.

Following (21), demand for variety ω for a country with a continuum of consumers of measure L_j is:

$$\frac{L_j}{\gamma}\left(\alpha - \mu_j p_j(\omega) - \eta \int_{\Omega_j} q_j^c(\omega)\, d\omega\right).$$

We will consider a symmetric equilibrium where all the firms from source country i with productivity ϕ choose the same equilibrium variables. It follows that $q_{ij}(\phi) = 0$ exactly when

$$\mu_j p_{ij}(\phi_{ij}^*) = \mu_j p_{ij}^* := \alpha + \eta M_j \mu_j \frac{\bar{p}_j - \alpha}{\gamma + \eta M_j} = \frac{\gamma\alpha + \eta M_j \bar{p}_j \mu_j}{\gamma + \eta M_j}. \tag{22}$$

The profit maximization problem of the firm with productivity draw ϕ is

$$\begin{aligned} \pi_{ij}(\phi) &= \max_{q_{ij}(\phi), p_{ij}(\phi),} p_{ij}(\phi)\, q_{ij}(\phi) - \tau_{ij}\frac{w_i}{\phi} q_{ij}(\phi) \\ &= \max p(\phi)\left(\frac{\alpha L_j}{\gamma + \eta M_j} - \frac{L_j}{\gamma}\mu_j p_{ij}(\phi) + \frac{\eta L_j}{\gamma}\frac{M_j \bar{p}_j \mu_j}{\gamma + \eta M_j}\right) - \tau_{ij}\frac{w_i}{\phi}\left(\frac{\alpha L_j}{\gamma + \eta N_j} - \frac{L_j}{\gamma}\lambda p_{ij}(\phi) + \frac{\eta L_j}{\gamma}\frac{M_j \bar{p}}{\gamma + \cdots}\right. \end{aligned}$$

where we have replaced for the iceberg transportation costs and the production

function for a firm ϕ in the cost function of producing and shipping the good abroad. The above problem implies the FOC

$$\begin{aligned} \frac{\alpha L_j}{\gamma + \eta M_j} - 2\frac{L_j}{\gamma}\mu_j p_{ij}(\phi) + \frac{\eta L_j}{\gamma}\frac{M_j \bar{p}_j}{\gamma + \eta M_j} + \tau_{ij}\frac{w_i}{\phi}\frac{L_j}{\gamma}\mu_j &= 0 \implies \\ \frac{L_j}{\gamma}\left(\mu_j p_{ij}(\phi) - \mu_j \tau_{ij}\frac{w_i}{\phi}\right) &= q_{ij}(\phi). \end{aligned} \tag{23}$$

The FOCs also imply that:

$$\begin{aligned} \frac{L_j}{\gamma}\left(\mu_j p_{ij}\left(\phi_{ij}^*\right) - \mu_j \tau_{ij}\frac{w_i}{\phi_{ij}^*}\right) &= 0 \implies \\ p_{ij}(\phi^*) &= \tau_{ij}\frac{w_i}{\phi_{ij}^*} \end{aligned}$$

>From the FOC we also have

$$\begin{aligned} \frac{\alpha L_j}{\gamma + \eta N_j} + \frac{\eta L_j}{\gamma}\frac{M_j \bar{p}_j}{\gamma + \eta M_j} + \tau_{ij}\frac{w_i}{\phi^*}\frac{L_j}{\gamma}\mu_j &= 2\frac{L_j}{\gamma}\mu_j p_{ij}^* = 2\frac{L_j}{\gamma}\mu_j \tau_{ij}\frac{w_i}{\phi_{ij}^*} \implies \\ \frac{\alpha L_j}{\gamma + \eta M_j} + \frac{\eta L_j}{\gamma}\frac{M_j \bar{p}_j}{\gamma + \eta M_j} &= \frac{L_j}{\gamma}\mu_j \tau_{ij}\frac{w_i}{\phi_{ij}^*} \end{aligned}$$

and thus

$$\frac{\alpha L_j}{\gamma + \eta N_j} - \frac{L_j}{\gamma}\mu_j p_{ij}(\phi) + \frac{\eta L_j}{\gamma}\frac{M_j \bar{p}_j}{\gamma + \eta M_j} + \tau_{ij}\frac{w_i}{\phi}\frac{L_j}{\gamma}\mu_j = \frac{L_j}{\gamma}\mu_j p_{ij}(\phi)$$

$$\frac{1}{2}\frac{\gamma}{L_j}\left(\frac{\alpha L_j}{\gamma+\eta M_j}+\frac{\eta L_j}{\gamma}\frac{M_j\bar{p}_j}{\gamma+\eta M_j}+\tau_{ij}\frac{w_i}{\phi}\frac{L_j}{\gamma}\mu_j\right)=\mu_j p_{ij}\left(\phi\right) \implies$$

$$p_{ij}\left(\phi\right)=\frac{1}{2}\left(p^*_{ij}+\tau_{ij}\frac{w_i}{\phi}\right) \tag{24}$$

and therefore using (25) and (24) the quantity is given by

$$q_{ij}\left(\phi\right)=\frac{L_j}{\gamma}\left(\frac{1}{2}\mu_j p^*_{ij}-\frac{1}{2}\mu_j\tau_{ij}\frac{w_i}{\phi}\right) \tag{25}$$

The sales of the firm using the above two equations are

$$p_{ij}\left(\phi\right)q_{ij}\left(\phi\right)=\left(\tau_{ij}\frac{w_i}{\phi^*_{ij}}+\tau_{ij}\frac{w_i}{\phi}\right)\frac{1}{2}\mu_j\frac{L_j}{\gamma}\frac{1}{2}\left(\tau_{ij}\frac{w_i}{\phi^*_{ij}}-\tau_{ij}\frac{w_i}{\phi}\right) \implies$$

$$p_{ij}\left(\phi\right)q_{ij}\left(\phi\right)=\mu_j\frac{L_j}{4\gamma}\left(\left(\tau_{ij}\frac{w_i}{\phi^*_{ij}}\right)^2-\left(\tau_{ij}\frac{w_i}{\phi}\right)^2\right) \tag{26}$$

and profits are:

$$\begin{aligned}
\pi_{ij}\left(\phi\right)= &\quad \mu_j\frac{L_j}{4\gamma}\left(\left(\tau_{ij}\frac{w_i}{\phi^*_{ij}}\right)^2-\left(\tau_{ij}\frac{w_i}{\phi}\right)^2\right)-\mu_j\tau_{ij}\frac{w_i}{\phi}\frac{L_j}{2\gamma}\left(\tau_{ij}\frac{w_i}{\phi^*_{ij}}-\tau_{ij}\frac{w_i}{\phi}\right)\\
= &\quad \mu_j\frac{L_j}{4\gamma}\left(\tau_{ij}\right)^2\left[\left(\left(\frac{w_i}{\phi^*_{ij}}\right)^2-\left(\frac{w_i}{\phi}\right)^2\right)-2\left(\frac{w_i}{\phi}\frac{w_i}{\phi^*_{ij}}-\left(\frac{w_i}{\phi}\right)^2\right)\right] \qquad (27)
\end{aligned}$$

$$\pi_{ij}\left(\phi\right)=\mu_j\frac{L_j}{4\gamma}\left(\tau_{ij}\right)^2\left[\left(\frac{w_i}{\phi^*_{ij}}\right)-\left(\frac{w_i}{\phi}\right)\right]^2$$

5.1 Total sales

We have that

$$\mu_j p_{ij}(\phi^*_{ij})=\alpha+\eta M_j\mu_j\frac{\bar{p}_j-\alpha}{\gamma+\eta N_j}$$

and looking at the case $\eta=0$

$$\begin{aligned}
\mu_j p_{ij}(\phi^*_{ij}) &= \alpha \implies\\
\mu_j w_i &= \frac{\alpha\phi^*_{ij}}{\tau_{ij}}
\end{aligned}$$

and thus using equation (26) can be written as

$$\begin{aligned}
p_{ij}\left(\phi\right)q_{ij}\left(\phi\right) &= \mu_j\frac{L_j}{4\gamma}\left(\left(\tau_{ij}\frac{w_i}{\phi^*_{ij}}\right)^2-\left(\tau_{ij}\frac{w_i}{\phi}\right)^2\right)\\
&= \frac{w_iL_j}{4\gamma}\left(\tau_{ij}\right)^2\left(\frac{\alpha\phi^*_{ij}}{\tau_{ij}}\frac{1}{\left(\phi^*_{ij}\right)^2}-\left(\frac{\tau_{ij}}{\phi}\right)^2\frac{\alpha\phi^*_{ij}}{\tau_{ij}}\right)\\
&= \frac{w_iL_j}{4\gamma}\alpha\tau_{ij}\left(\frac{1}{\phi^*_{ij}}-\frac{\phi^*_{ij}}{\phi^2}\right)
\end{aligned}$$

and thus the average sales are

$$\begin{aligned}
\overline{p_{ij}q_{ij}} &= \int_{\phi_{ij}^*} \frac{w_i L_j}{4\gamma}\alpha\tau_{ij}\left(\frac{1}{\phi_{ij}^*}-\frac{\phi_{ij}^*}{\phi^2}\right)\theta\frac{\left(\phi_{ij}^*\right)^\theta}{\phi^{\theta+1}}d\phi \\
&= \theta\frac{w_i L_j}{4\gamma}\alpha\tau_{ij}\left(\frac{1}{\theta\phi_{ij}^*}-\frac{1}{(\theta+2)\,\phi_{ij}^*}\right) \\
&= \frac{\tau_{ij}}{\phi_{ij}^*}\theta\frac{w_i L_j}{4\gamma}\alpha\left(\frac{1}{\theta}-\frac{1}{(\theta+2)}\right) \\
&= \theta\alpha^2\frac{L_j}{\mu_j 4\gamma}\left(\frac{2}{\theta}\frac{1}{(\theta+2)}\right) .
\end{aligned}$$

$$\frac{1}{\frac{J_j\frac{w_j f_e(\theta+1)(\theta+2)2\gamma}{\alpha^2}}{L_j}}\sum_{\upsilon} J_\upsilon\frac{b_\upsilon^\theta}{\left(\phi_{\upsilon j}^*\right)^\theta}=\mu_j$$

$$\begin{aligned}
XXX &= \theta\alpha^2\frac{L_j}{\frac{1}{\frac{J_j\frac{w_j f_e(\theta+1)(\theta+2)2\gamma}{\alpha^2}}{L_j}}\sum_\upsilon J_\upsilon\frac{b_\upsilon^\theta}{\left(\phi_{\upsilon j}^*\right)^\theta}4\gamma}\left(\frac{2}{\theta}\frac{1}{(\theta+2)}\right) \\
&= .\frac{w_j}{\sum_\upsilon J_\upsilon\frac{b_\upsilon^\theta}{\left(\phi_{\upsilon j}^*\right)^\theta}}f_e(\theta+1)J_j \\
&= \frac{w_j L_j}{\sum_\upsilon N_{\upsilon j}}XXXX
\end{aligned}$$

Notice that the last line implies that average sales per firm, are not source country specific. The number of firms from source i selling to country j is

$$N_{ij}=J_i\frac{b_i^\theta}{\left(\phi_{ij}^*\right)^\theta}$$

and therefore total sales are given

$$T_{ij}=J_i\frac{b_i^\theta}{\left(\phi_{ij}^*\right)^\theta}\theta\alpha^2\frac{L_j}{\mu_j 4\gamma}\left(\frac{2}{\theta}\frac{1}{(\theta+2)}\right)$$

where J_i can be determined using free entry and labor market clearing. In particular budget constraint (which is equivalent to labor market clearing) implies that

$$\begin{aligned}
w_i L_i &= \sum_j J_i\frac{b_i^\theta}{\left(\phi_{ij}^*\right)^\theta}\int_{\phi^*}^\infty p_{ij}(\phi)q_{ij}(\phi)\,dG_{\phi_{ij}^*}(\phi) \\
w_i L_i &= \sum_j J_i\frac{b_i^\theta}{\left(\phi_{ij}^*\right)^\theta}\int_{\phi^*}^\infty \mu_j\frac{L_j}{4\gamma}\left(\left(\tau_{ij}\frac{w_i}{\phi_{ij}^*}\right)^2-\left(\tau_{ij}\frac{w_i}{\phi}\right)^2\right)dG_{\phi_{ij}^*}(\phi) \\
w_i L_i &= \sum_j J_i\frac{b_i^\theta}{\left(\phi_{ij}^*\right)^\theta}\frac{L_j\theta}{4\gamma}\mu_j\frac{w_i^2}{(\phi_{ij}^*)^2}\tau_{ij}^2\left(\frac{1}{\theta}-\frac{1}{\theta+2}\right) \qquad (28)
\end{aligned}$$

profits are given by

$$\sum_{\upsilon}\left(\frac{\phi_{ii}^{*}}{\phi_{i\upsilon}^{*}}\right)^{\theta}\int_{\phi_{i\upsilon}^{*}}^{\infty}\pi_{i\upsilon}(\phi)dG_{\phi^{*}}(\phi) = \sum_{\upsilon}\int_{\phi^{*}}^{\infty}\pi_{i\upsilon}(\phi)\theta(\phi^{*})^{\theta}\phi^{-\theta-1}d\phi$$

$$= \theta(\phi_{ii}^{*})^{\theta}\sum_{\upsilon}\mu_{\upsilon}\frac{L_{\upsilon}}{4\gamma}\int_{\phi^{*}}^{\infty}\left[\left(\tau_{i\upsilon}\frac{w_i}{\phi_{i\upsilon}^{*}}\right)-\left(\tau_{i\upsilon}\frac{w_i}{\phi}\right)\right]^{2}\phi^{-\theta-1}d\phi$$

$$= \theta(\phi_{ii}^{*})^{\theta}\sum_{\upsilon}\mu_{\upsilon}\frac{L_{\upsilon}}{4\gamma}(\tau_{i\upsilon})^{2}\int_{\phi^{*}}^{\infty}\left[\left(\frac{w_i}{\phi_{i\upsilon}^{*}}\right)-\left(\frac{w_i}{\phi}\right)\right]^{2}\phi^{-\theta-1}d\phi$$

$$= \theta w_i^{2}\sum_{\upsilon}\frac{\mu_{\upsilon}}{(\phi_{i\upsilon}^{*})^{2}}\frac{L_{\upsilon}}{4\gamma}\left(\frac{\phi_{ii}^{*}}{\phi_{i\upsilon}^{*}}\right)^{\theta}(\tau_{i\upsilon})^{2}\left[\frac{1}{\theta}+\frac{1}{\theta+2}-\frac{2}{\theta+1}\right]$$

Therefore the free entry condition implies

$$\frac{b_i^{\theta}}{(\phi_{ii}^{*})^{\theta}}\theta w_i^{2}\sum_{\upsilon}\frac{\mu_{\upsilon}}{(\phi_{i\upsilon}^{*})^{2}}\frac{L_{\upsilon}}{4\gamma}\left(\frac{\phi_{ii}^{*}}{\phi_{i\upsilon}^{*}}\right)^{\theta}(\tau_{i\upsilon})^{2}\left[\frac{1}{\theta}+\frac{1}{\theta+2}-\frac{2}{\theta+1}\right] = w_i f_e \implies$$

$$\frac{b_i^{\theta}}{(\phi_{ii}^{*})^{\theta}}\sum_{\upsilon}\left(\frac{\phi_{ii}^{*}}{\phi_{i\upsilon}^{*}}\right)^{\theta}\frac{L_{\upsilon}}{2\gamma}\mu_{\upsilon}\frac{w_i}{(\phi_{i\upsilon}^{*})^{2}}(\tau_{i\upsilon})^{2}\left[\frac{1}{(\theta+1)(\theta+2)}\right] = f_e$$

now replacing the above equation inside (28)

$$\begin{aligned} w_i L_i &= \sum_{\upsilon} J_i \frac{b_i^{\theta}}{(\phi_{i\upsilon}^{*})^{\theta}}\frac{L_{\upsilon}\theta}{4\gamma}\mu_{\upsilon}\frac{w_i^{2}}{(\phi_{i\upsilon}^{*})^{2}}\tau_{i\upsilon}^{2}\left(\frac{1}{\theta}-\frac{1}{\theta+2}\right) \\ w_i L_i &= J_i\sum_{\upsilon}\left(\frac{b_i}{\phi_{i\upsilon}^{*}}\right)^{\theta}\frac{L_{\upsilon}\theta}{4\gamma}\mu_{\upsilon}\frac{w_i^{2}}{(\phi_{i\upsilon}^{*})^{2}}\tau_{i\upsilon}^{2}\left(\frac{1}{\theta}-\frac{1}{\theta+2}\right) \implies \end{aligned}$$

$$J_i = \frac{L_i}{(\theta+1)\,f_e} \tag{29}$$

Thus, the number of entrans is independent of tariffs and trade in general.

6 Bibliography

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6.1 Appendix

$$\frac{p_{ij}(\omega)-\tau_{ij}\frac{w_i}{\phi(\omega)}}{q_{ij}(\omega)}=\frac{1}{\mu_j}\frac{\gamma}{L_j}$$

Which implies that for ω and ω' we have

$$\begin{aligned}
\frac{p_{ij}(\omega)-\tau_{ij}\frac{w_i}{\phi(\omega)}}{q_{ij}(\omega)} &= \frac{p_{ij}(\omega')-\tau_{ij}\frac{w_i}{\phi(\omega')}}{q_{ij}(\omega')} \implies \\
\frac{p_{ij}(\omega)-\tau_{ij}\frac{w_i}{\phi(\omega)}}{q_{ij}(\omega)}\int_{\Omega^*_{\upsilon j}} q_{ij}(\omega')\,d\omega' &= \int_{\Omega^*_{\upsilon j}}\left(p_{ij}(\omega')-\tau_{ij}\frac{w_i}{\phi(\omega')}\right)d\omega' \implies \\
\frac{p_{ij}(\omega)-\tau_{ij}\frac{w_i}{\phi(\omega)}}{q_{ij}(\omega)}\sum_{\upsilon}\int_{\Omega^*_{\upsilon j}} q_{ij}(\omega')\,d\omega' &= \sum_{\upsilon}\int_{\Omega^*_{\upsilon j}} p_{ij}(\omega')\,d\omega'-\tau_{ij}w_i\sum_{\upsilon}\int_{\Omega^*_{\upsilon j}}\frac{1}{\phi(\omega')}d\omega'
\end{aligned}$$

define

$$\begin{aligned}
Q_j &= \sum_{\upsilon}\int_{\Omega^*_{\upsilon j}} q_{ij}(\omega')\,d\omega' \\
P_j &= \sum_{\upsilon}\int_{\Omega^*_{\upsilon j}} p_{ij}(\omega')\,d\omega' \\
\Phi_j &= \sum_{\upsilon}\int_{\Omega^*_{\upsilon j}} \frac{1}{\phi(\omega')}d\omega'
\end{aligned}$$

$$p_{ij}(\omega)-\tau_{ij}\frac{w_i}{\phi(\omega)}=\frac{P_j-\tau_{ij}w_i\Phi_j}{Q_j}q_{ij}(\omega)$$

$$q_{ij}(\omega)=Q_j\frac{p_{ij}(\omega)-\tau_{ij}\frac{w_i}{\phi(\omega)}}{P_j-\tau_{ij}w_i\Phi_j}$$