1 Characterization of the equilibrium

Consider a world economy with two symmetric countries. Wages are normalized to one in each country.

- Demand:
$$q\left(p,p^{*}\right)=d(\ln p-\ln p^{*})$$
- Prices:
$$p=\frac{-d'\left(\ln p-\ln p^{*}\right)}{-1-d'\left(\ln p-\ln p^{*}\right)}c.$$

  So domestic and import prices, $p_d(z,p)$ and $p_x(z,p^*)$, are given by the implicit solution of
$$\begin{aligned}p_d(z,p^*) &= \frac{-d'\left(\ln p_d(z,p^*)-\ln p^*\right)}{-1-d'\left(\ln p_d(z,p^*)-\ln p^*\right)}\frac{1}{z}\\ p_x(z,p^*) &= \frac{-d'\left(\ln p_x(z,p^*)-\ln p^*\right)}{-1-d'\left(\ln p_x(z,p^*)-\ln p^*\right)}\frac{\tau}{z}\end{aligned}$$
- Sales:
$$\begin{aligned}x_d(z,p^*) &= p_d(z,p^*)e^{d(\ln p_d(z,p^*)-\ln p^*)}\\ x_x(z,p^*) &= p_x(z,p^*)e^{d(\ln p_x(z,p^*)-\ln p^*)}\end{aligned}$$
- Profits:
$$\begin{aligned}\pi_d(z,p^*) &= \frac{p_d(z,p^*)-\frac{1}{z}}{p_d(z,p^*)}x_d(z,p^*)\\ \pi_x(z,p^*) &= \frac{p_x(z,p^*)-\frac{\tau}{z}}{p_x(z,p^*)}x_x(z,p^*)\end{aligned}$$
- Productivity cut-off z_d^* and z_x^* are given by the implicit solution of:
$$\pi_d(z_d^*,p^*) = f_d \quad (1)$$
$$\pi_x(z_x^*,p^*) = f_x \quad (2)$$
- Total number of entrants is pinned down by labor market clearing:
$$N=\frac{L}{f_e+\int_{z_d^*}\left(f_d+\frac{e^{d(\ln p_d(z,p^*)-\ln p^*)}}{z}\right)g(z)dz+\int_{z_x^*}\left(f_x+\frac{e^{d(\ln p_x(z,p^*)-\ln p^*)}}{z/\tau}\right)g(z)dz} \quad (3)$$
- Choke price/Lagrange multiplier is pinned down by budget constraint (where we have used free entry to argue that income was equal to the wage):
$$N\left[\int_{z_d^*}x_d(z,p^*)g(z)dz+\int_{z_x^*}x_x(z,p^*)g(z)dz\right]=1 \quad (4)$$
- An equilibrium corresponds to (z_d^*,z_x^*,N,p^*) that solves equations 1-4

2 Welfare Analysis

- Changes in expenditure are given by:

$$
\begin{aligned}
d\ln e &= -\left[\lambda_d S_d + \lambda_x S_x\right] d\ln N + \lambda_d s_d^* dz_d^* + \lambda_x s_x^* dz_x^* \\
&\quad + \rho d\ln p^* + (1-\rho)\,\lambda_x d\ln\tau
\end{aligned}
\tag{5}
$$

where:

$$\lambda_d \equiv \int_{z_d^*} x_d(z,p^*)g(z)dz, \tag{6}$$

$$\lambda_x \equiv \int_{z_x^*} x_x(z,p^*)g(z)dz, \tag{7}$$

$$S_d \equiv \int_{z_d^*} \left[\frac{p_d(z,p^*) - p^* u\left(q_d\left(z,p^*\right)\right)/q_d\left(z,p^*\right)}{p_d(z,p^*)}\right] \frac{x_d(z,p^*)g(z)}{\int_{z_d^*} x_d(z,p^*)g(z)dz} dz \tag{8}$$

$$S_x \equiv \int_{z_x^*} \left[\frac{p_x(z,p^*) - p^* u\left(q_x\left(z,p^*\right)\right)/q_x\left(z,p^*\right)}{p_x(z,p^*)}\right] \frac{x_x(z,p^*)g(z)}{\int_{z_x^*} x_x(z,p^*)g(z)dz} dz \tag{9}$$

$$s_d^* \equiv \left[\frac{p_d(z_d^*,p^*) - p^* u\left(q_d\left(z_d^*,p^*\right)\right)/q_d\left(z_d^*,p^*\right)}{p_d(z_d^*,p^*)}\right] \frac{x_d(z_d^*,p^*)g(z)}{\int_{z_d^*} x_d(z,p^*)g(z)dz} \tag{10}$$

$$s_x^* \equiv \left[\frac{p_x(z_x^*,p^*) - p^* u\left(q_x\left(z_x^*,p^*\right)\right)/q_x\left(z_x^*,p^*\right)}{p_x(z_x^*,p^*)}\right] \frac{x_x(z_x^*,p^*)g(z)}{\int_{z_x^*} x_x(z,p^*)g(z)dz} \tag{11}$$

$$\rho \equiv \int_{z_d^*} \left[\frac{d\ln p_d(z,p^*)}{d\ln p^*}\right] \frac{x_d(z,p^*)g(z)}{\int_{z_d^*} x_d(z,p^*)g(z)dz} dz = \int_{z_x^*} \left[\frac{d\ln p_d(z,p^*)}{d\ln p^*}\right] \frac{x_x(z,p^*)g(z)}{\int_{z_x^*} x_x(z,p^*)g(z)dz} \tag{12}$$

[Note that I have subsitute the Lagrange multiplier in the expenditure minimization program by p^*. This is OK if p^* is defined as the inverse of the Lagrange multiplier associated with the budget constraint in the utilitization maximization program.]

- The alternative is to use a known expenditure function in order to compute equivalent variation for large changes. In the case of generalized CES the demand function is

$$
\begin{aligned}
x(p,p^*) &= p \times \alpha \left(\left(\frac{p}{p^*}\right)^{-\sigma} - 1\right) \\
&= p \times a \left[e^{-\sigma(\ln p - \ln p^*)} - 1\right] \\
&= p \times e^{\ln a + \ln\left[e^{-\sigma\left(\ln p - \ln p^*\right)} - 1\right]}
\end{aligned}
$$

the expenditure function is given by

$$\begin{aligned}
\mathbf{e}(\mathbf{p},u) &= \left[\int_{\Omega^*} p_\omega^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}} \left(u^{\frac{\sigma-1}{\sigma}} + a^{\frac{\sigma-1}{\sigma}} \int_{\Omega^*} d\omega\right)^{\frac{\sigma}{\sigma-1}} - a \int_{\Omega^*} p_\omega d\omega \\
&= \left[\sum_i N_i \int_{z_{ij}^*} p\left(z\right)^{1-\sigma} g\left(z\right) dz\right]^{\frac{1}{1-\sigma}} \left(u^{\frac{\sigma-1}{\sigma}} + a^{\frac{\sigma-1}{\sigma}} \sum_i N_i \int_{z_{ij}^*} g\left(z\right) dz\right)^{\frac{\sigma}{\sigma-1}} \\
&\quad -a \sum_i N_i \int_{z_{ij}^*} p\left(z\right) g\left(z\right) dz
\end{aligned}$$

and accordingly because of symmetry $z_{ii}^* = z_d^*$ and $z_{ij}^* = z_x$ for $i \neq j$ and where u is the utility level in the equilibrium, i.e.

$$\begin{aligned}
u &= \left(\int_\Omega \left[(q_\omega + a)^{\frac{\sigma-1}{\sigma}} - a^{\frac{\sigma-1}{\sigma}}\right] d\omega\right)^{\frac{\sigma}{\sigma-1}} \\
&= \left(\sum_i N_i \int_{z_{ij}^*} \left[(q\left(z\right) + a)^{\frac{\sigma-1}{\sigma}} - a^{\frac{\sigma-1}{\sigma}}\right] g\left(z\right) dz\right)^{\frac{\sigma}{\sigma-1}}
\end{aligned}$$

3 Algorithm

1. For given (θ, f, τ) solve for (z_d^*, z_x^*, N, p^*) using equations 1-4
2. Compute $(\lambda_d, \lambda_x, S_d, S_x, s_d^*, s_x^*, \rho)$ using 6-12
3. For the same (θ, f) but a different $\tilde{\tau}$, solve for $\left(\tilde{z}_d^*, \tilde{z}_x^*, \tilde{N}, \tilde{p}^*\right)$ using equations 1-4
4. Compute $d \ln N = \ln \tilde{N} - \ln N$, $dz_d^* = \tilde{z}_d^* - z_d^*$, $dz_x^* = \tilde{z}_x^* - z_x^*$, $d \ln p^* = \ln \tilde{p}^* - \ln p^*$, $d \ln \tau = \ln \tilde{\tau} - \ln \tau$
5. Compute $d \ln e$ using 5 as well as the values of $(\lambda_d, \lambda_x, S_d, S_x, s_d^*, s_x^*, \rho)$ and $(d \ln N, dz_d^*, dz_x^*, d \ln p^*, d \ln \tau = \ln \tilde{\tau} - \ln \tau)$
6. Repeat the same procedure for the same θ, the same change from τ to $\tilde{\tau}$, but a different value of f