Appendix to "Gains from Trade with Endogenous Variety"

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Abstract

In this appendix we solve the monopolistic competition model with heterogeneous firms and free entry. Our objective is to evaluate the importance of the free entry condition (compared to a predetermined number of potential entrants as in the Chaney version of the Melitz model) in the determination of aggregate variables of the model. The main result is that the model with free entry generates outcomes that are observationally equivalent to the ones of a model with no free entry and consequently the two models generate identical welfare predictions.

1 Model

We consider a monopolistic competition model with heterogeneous firms. We denote the exporting country by i and the importing country by j, where $i, j = 1, ..., N$.

1.1 Demand

We assume that each firm offers a different good (monopolistic competition) and that firms have potentially different productivity ϕ in producing that good. Due to symmetry we assume that all firms with the same productivity from a given source country i choose the same price for their goods in a given destination country. Given a measure of L_j representative consumers in country j, the demand for a firm with productivity ϕ from country i charging a price $p_{ij}(\phi)$ in country j is:

$$x_{ij}(\phi) = \frac{(p_{ij}(\phi))^{-\sigma}}{P_j^{1-\sigma}} w_j L_j,$$

where w_j is the wage per capita, $\sigma > 1$ is the elasticity of substitution, and

$$P_j^{1-\sigma} = \sum_{v} \int_0^{\infty} p_{vj}(\phi)^{1-\sigma} M_{vj} \mu_{vj}(\phi) \, d\phi.$$

$\mu_{ij}(\phi)$ is the distribution of productivities of firms originating from country i conditional on selling to country j and M_{ij} is the measure of firms from country i selling to country j.

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1.2 Firm

Each firm must pay an i, j specific fixed cost of entry and also incurs iceberg transportation costs of trade $\tau_{ij} > 1$, $\tau_{ii} = 1$ in terms of labor. Assuming that the fixed costs require destination country labor, we have that

$$\pi_{ij}(\phi) = \max_{p_{ij}} \left\{ \frac{p_{ij}^{1-\sigma}}{P_j^{1-\sigma}} w_j L_j - w_j L_j \tau_{ij} \frac{p_{ij}^{-\sigma} w_i}{P_j^{1-\sigma} \phi} - w_j f_{ij}, 0 \right\},$$

and the FOCs give

$$p_{ij} = \frac{\sigma}{\sigma - 1} \frac{w_i}{\phi} \tau_{ij},$$

$$\pi_{ij}(\phi) = \left(\frac{\sigma}{\sigma - 1} \frac{w_i}{\phi} \tau_{ij} \right)^{1-\sigma} \frac{1}{\sigma} \frac{w_j L_j}{P_j^{1-\sigma}} - w_j f_{ij}.$$

The threshold productivity of operation is

$$\left(\phi_{ij}^*\right)^{\sigma-1} = \frac{f_{ij}}{\left(\frac{\sigma}{\sigma-1} \tau_{ij} w_i\right)^{1-\sigma} \frac{1}{\sigma} \frac{L_j}{P_j^{1-\sigma}}}. \tag{1}$$

1.3 Firm Entry

We assume that firms have to pay a fixed entry cost, f_e, in advance in order to enter the market and draw a productivity realization. New entrants draw their productivity from a Pareto distribution with shape parameter $\theta > \sigma - 1$, c.d.f. $G(\phi; b_i) = 1 - \frac{b_i^\theta}{\phi^\theta}$, and support $[b_i, +\infty)$, where b_i can be interpreted as a technology parameter. If a firm gets a productivity draw that is below ϕ_{ii}^*, then it exits immediately without operating.[1] Thus, because of free entry, in the equilibrium, the expected profits of a firm must equal to entry costs.[2] Using the free entry condition and the assumed functional forms, we have[3]

$$\sum_v \int_{\phi_{iv}^*} \frac{\left(\frac{\sigma}{\sigma-1} \frac{\tau_{iv} w_i}{\phi}\right)^{1-\sigma}}{P_v^{1-\sigma} \sigma} w_v L_v \theta \frac{\left(\phi_{iv}^*\right)^\theta}{\phi^{\theta+1}} \frac{\left(\phi_{ii}^*\right)^\theta}{\left(\phi_{iv}^*\right)^\theta} d\phi - \sum_v \int_{\phi_{iv}^*} w_v f_{iv} \theta \frac{\left(\phi_{iv}^*\right)^\theta}{\phi^{\theta+1}} \frac{\left(\phi_{ii}^*\right)^\theta}{\left(\phi_{iv}^*\right)^\theta} d\phi = \frac{w_i f_e}{\frac{b_i^\theta}{\left(\phi_{ii}^*\right)^\theta}} \Leftrightarrow$$

$$\sum_v w_v f_{iv} \frac{\theta}{\theta - \sigma + 1} \frac{\left(\phi_{ii}^*\right)^\theta}{\left(\phi_{iv}^*\right)^\theta} - \sum_v w_v \frac{\left(\phi_{ii}^*\right)^\theta}{\left(\phi_{iv}^*\right)^\theta} f_{iv} = \frac{w_i f_e}{\frac{b_i^\theta}{\left(\phi_{ii}^*\right)^\theta}} \Leftrightarrow$$

[1] We assume that the parameters of the model are such that the lower productivity threshold $\phi_{ij}^* > \phi_{ii}^* > b_i$, $\forall i, j$, $i \neq j$.

[2] Essentially, we assume that there exists a perfect capital market, which requires firms to pay a fixed entry cost before drawing a productivity realization. Consequently, we multiply the LHS by $1 - G(\phi_{ii}^*, b_i)$, the probability of obtaining the average profit, since firms with profits below this average necessarily exit the market. Alternatively, we could have specified a more general case with intertemporal discounting, δ. In this case the expected profits from entry should equal the discounted entry cost in the equilibrium.

[3] An implication of free entry is that in the equilibrium all the profits are accrued to labor for the production of the entry cost.

$$\sum_{\upsilon} w_\upsilon f_{i\upsilon} \frac{(\phi_{ii}^*)^\theta}{(\phi_{i\upsilon}^*)^\theta} \left(\frac{\theta}{\theta-\sigma+1} - 1 \right) = \frac{w_i f_e}{\frac{b_i^\theta}{(\phi_{ii}^*)^\theta}} \Leftrightarrow$$

$$\sum_{\upsilon} \frac{w_\upsilon}{w_i} f_{i\upsilon} \frac{(\phi_{ii}^*)^\theta}{(\phi_{i\upsilon}^*)^\theta} \frac{\sigma-1}{\theta-\sigma+1} = \frac{f_e}{\frac{b_i^\theta}{(\phi_{ii}^*)^\theta}}. \tag{2}$$

2 Solving for Equilibrium

The equilibrium number of firms producing in country i, N_i, is determined by the following labor market clearing condition:

$$N_i \left(\sum_{\upsilon} \int_{\phi_{i\upsilon}^*} \frac{\left(\frac{\sigma}{\sigma-1} \frac{\tau_{i\upsilon} w_i}{\phi} \right)^{-\sigma}}{P_\upsilon^{1-\sigma}} \frac{\tau_{i\upsilon}}{\phi} w_\upsilon L_\upsilon \theta \frac{(\phi_{i\upsilon}^*)^\theta}{\phi^{\theta+1}} \frac{(\phi_{ii}^*)^\theta}{(\phi_{i\upsilon}^*)^\theta} d\phi + \frac{f_e}{(1-G(\phi_{ii}^*, b_i))} \right) + \sum_{\upsilon} N_\upsilon \frac{(\phi_{\upsilon\upsilon}^*)^\theta}{(\phi_{\upsilon i}^*)^\theta} f_{\upsilon i} = L_i \Longrightarrow$$

$$N_i \left(\sum_{\upsilon} (\sigma-1) \frac{w_\upsilon}{w_i} f_{i\upsilon} \frac{(\phi_{ii}^*)^\theta}{(\phi_{i\upsilon}^*)^\theta} \frac{\theta}{\theta-\sigma+1} + \frac{f_e}{\frac{b_i^\theta}{(\phi_{ii}^*)^\theta}} \right) + \sum_{\upsilon} N_\upsilon \frac{(\phi_{\upsilon\upsilon}^*)^\theta}{(\phi_{\upsilon i}^*)^\theta} f_{\upsilon i} = L_i. \tag{3}$$

Substituting out equation (2), we obtain

$$N_i \left(\theta \frac{f_e}{\frac{b_i^\theta}{(\phi_{ii}^*)^\theta}} + \frac{f_e}{\frac{b_i^\theta}{(\phi_{ii}^*)^\theta}} \right) + \sum_{\upsilon} N_\upsilon \frac{(\phi_{\upsilon\upsilon}^*)^\theta}{(\phi_{\upsilon i}^*)^\theta} f_{\upsilon i} = L_i,$$

which together with the price index, which implies that (see appendix B for derivations)

$$w_i L_i = \frac{\theta\sigma}{\theta-\sigma+1} \left(\sum_{\upsilon} N_\upsilon \frac{(\phi_{\upsilon\upsilon}^*)^\theta}{(\phi_{\upsilon i}^*)^\theta} w_i f_{\upsilon i} \right),$$

implies that

$$N_i \frac{f_e}{\frac{b_i^\theta}{(\phi_{ii}^*)^\theta}} (\theta+1) = L_i - L_i \frac{\theta-\sigma+1}{\theta\sigma} \Longrightarrow$$

$$N_i = \frac{(\sigma-1)}{\theta\sigma \frac{f_e}{\frac{b_i^\theta}{(\phi_{ii}^*)^\theta}}} L_i\ , \tag{4}$$

which completes the derivation of the number of operating firms.[4]

[4] With a slightly altered proof the same results hold under the assumption that fixed costs are paid in terms of domestic labor.

Notice that total export sales from country i to j are[5]

$$T_{ij} = \underbrace{\left(\frac{\phi_{ii}^*}{\phi_{ij}^*}\right)^{\theta} N_i}_{\text{firms}} \quad \underbrace{w_j f_{ij} \frac{\sigma\theta}{\theta-\sigma+1}}_{\text{average sales of operating firms}} . \tag{5}$$

Define the fraction of total income of country j spent on goods from country i by λ_{ij}. Using the definition of total sales from i to j and equations (1) and (4), we have

$$\lambda_{ij} = \frac{T_{ij}}{\sum_{\upsilon} T_{\upsilon j}},$$

which gives that

$$\lambda_{ij} = \frac{L_i b_i^{\theta} \left(\tau_{ij} w_i\right)^{-\theta} f_{ij}^{1-\theta/(\sigma-1)}}{\sum_{\upsilon} L_{\upsilon} b_{\upsilon}^{\theta} \left(\tau_{\upsilon j} w_{\upsilon}\right)^{-\theta} f_{\upsilon j}^{1-\theta/(\sigma-1)}}. \tag{6}$$

It is quite remarkable that even with free entry the equation determining market shares, (6), turns out to be quite similar to the one introduced by Eaton and Kortum (2002). In particular, market share appears to be affected by the cost factors τ_{ij} and w_i with an elasticity of $-\theta$. In fact, this relationship is the same as in Chaney (2007), and Arkolakis (2006) in the case, in which $f_{ij} = f_j$ and the number of potential entrants is equal to L_i (Notice that Chaney assumes the particular relation between the number of entrants and population, while here we get it as an equilibrium result).

3 Variety Effects of Trade Liberalization

In this section we limit our analysis to a two country model. Using equation (5) and the definition of λ_{ij}, we obtain

$$\begin{aligned} T_{ij} &= \lambda_{ij} w_j L_j \implies \\ \lambda_{ij} &= \frac{\left(\frac{\phi_{ii}^*}{\phi_{ij}^*}\right)^{\theta} N_i w_j f_{ij} \frac{\theta\sigma}{\theta-\sigma+1}}{w_j L_j}. \end{aligned}$$

It follows that the measure of firms from country i selling to j, M_{ij}, normalized by λ_{ij} is

$$\begin{aligned} \frac{M_{ij}}{\lambda_{ij}} &= \frac{\left(\frac{\phi_{ii}^*}{\phi_{ij}^*}\right)^{\theta} N_i}{\frac{\left(\frac{\phi_{ii}^*}{\phi_{ij}^*}\right)^{\theta} N_i f_{ij} \frac{\sigma\theta}{\theta-\sigma+1}}{L_j}} \implies \\ M_{ij} &= \lambda_{ij} \frac{L_j}{f_{ij} \frac{\sigma\theta}{\theta-\sigma+1}}. \end{aligned}$$

[5] Average sales of firms from i conditional on operating in j are the same in the model with free entry and the one with a predetermined number of entrants.

Thus,

$$
\begin{aligned}
M_{ij} + M_{jj} &= \lambda_{ij} \frac{L_j}{f_{ij} \frac{\sigma\theta}{\theta-\sigma+1}} + \lambda_{jj} \frac{L_j}{f_{jj} \frac{\sigma\theta}{\theta-\sigma+1}} \\
&= \frac{L_j}{\frac{\sigma\theta}{\theta-\sigma+1}} \left(\frac{\lambda_{ij}}{f_{ij}} + \frac{\lambda_{jj}}{f_{jj}} \right),
\end{aligned}
$$

which, given that $\lambda_{ij} = 1 - \lambda_{jj}$, gives a generalization of the Baldwin and Forslid (2007) result. We can think of λ_{jj} representing the "closeness" of country j and $1 - \lambda_{jj}$ its "openness". In particular, lowering λ_{jj} (trade liberalization) will have an anti-variety effect iff $f_{ij} > f_{jj}$. If $f_{ij} = f_{jj}$, trade liberalization has no impact in the measure of consumed varieties.

Notice that the analysis in this section does not depend on any symmetry assumption across countries, but only requires the assumptions of CES demand, CRS production technology, and Pareto distribution of productivities. Similar results could also be derived in a context without free entry of firms such as the one of Chaney (2007).

4 Welfare

In this class of models welfare for each representative consumer is given by

$$
C_j = \frac{w_j}{P_j},
$$

which does not depend on the assumption of free entry. The price index is

$$
P_j^{1-\sigma} = N_j \int_{\phi_{jj}^*}^{+\infty} \left(\frac{\sigma}{\sigma-1} \frac{w_i}{\phi} \right)^{1-\sigma} \theta \frac{\left(\phi_{jj}^*\right)^\theta}{\phi^{\theta+1}} d\phi + \sum_{\upsilon \neq j} N_\upsilon \frac{\left(\phi_{\upsilon\upsilon}^*\right)^\theta}{\left(\phi_{\upsilon j}^*\right)^\theta} \int_{\phi_{\upsilon j}^*}^{+\infty} \left(\frac{\sigma}{\sigma-1} \frac{\tau_{\upsilon j} w_\upsilon}{\phi} \right)^{\sigma-1} \theta \frac{\left(\phi_{\upsilon j}^*\right)^\theta}{\phi^{\theta+1}} d\phi.
$$

Notice that using (1) in the expression for the price index, we have that

$$
\begin{aligned}
P_j^{1-\sigma} &= \sum_\upsilon \left(\frac{\sigma}{\sigma-1} \tau_{\upsilon j} w_\upsilon \right)^{1-\sigma} N_\upsilon \frac{\left(\phi_{\upsilon\upsilon}^*\right)^\theta}{\left(\phi_{\upsilon j}^*\right)^\theta} \int_{\phi_{\upsilon j}^*} \phi^{\sigma-1} \theta \frac{\left(\phi_{\upsilon j}^*\right)^\theta}{\phi^{\theta+1}} d\phi \implies \\
P_j^{-\theta} &= \sum_\upsilon N_\upsilon \frac{\left(\phi_{\upsilon\upsilon}^*\right)^\theta}{\left(\frac{1}{\sigma}\right)^{-\theta/(\sigma-1)}} \left(\frac{\sigma}{\sigma-1} \tau_{\upsilon j} w_\upsilon \right)^{-\theta} \frac{\theta\sigma}{\theta-\sigma+1} \frac{\left(f_{\upsilon j}\right)^{1-\theta/(\sigma-1)}}{\left(L_j\right)^{1-\theta/(\sigma-1)}} \implies \\
P_j^{-\theta} &= \sum_\upsilon \frac{b_\upsilon^\theta}{f_e} \frac{L_\upsilon}{\left(\frac{1}{\sigma}\right)^{-\theta/(\sigma-1)}} \left(\frac{\sigma}{\sigma-1} \tau_{\upsilon j} w_\upsilon \right)^{-\theta} \frac{\sigma-1}{\theta-\sigma+1} \frac{\left(f_{\upsilon j}\right)^{1-\theta/(\sigma-1)}}{\left(L_j\right)^{1-\theta/(\sigma-1)}}. \qquad (7)
\end{aligned}
$$

Using equation (6), we can express wages as

$$
w_j^\theta = \frac{1}{\lambda_{jj}} \frac{L_j b_j^\theta f_{jj}^{1-\theta/(\sigma-1)}}{\sum_\upsilon L_\upsilon b_\upsilon^\theta \left(\tau_{\upsilon j} w_\upsilon\right)^{-\theta} f_{\upsilon j}^{1-\theta/(\sigma-1)}}.
$$

Thus, a reduction in λ_{jj} resembles a trade liberalization episode. In fact, in a proper calibration exercise, looking at the effects of a trade liberalization involves matching foreign market shares

$(1-\lambda_{jj}$, and thus $\lambda_{jj})$ before and after the trade liberalization. Finally, the welfare is given by

$$\begin{aligned}\frac{w_j}{P_j} &= \left(\frac{\frac{1}{\lambda_{jj}} \frac{L_j b_j^\theta f_{jj}^{1-\theta/(\sigma-1)}}{\sum_\upsilon L_\upsilon b_\upsilon^\theta (\tau_{\upsilon j} w_\upsilon)^{-\theta} f_{\upsilon j}^{1-\theta/(\sigma-1)}}}{\frac{1}{\sum_\upsilon \frac{b_\upsilon^\theta}{f_e} \frac{L_\upsilon}{\left(\frac{1}{\sigma}\right)^{-\theta/(\sigma-1)}} \left(\frac{\sigma}{\sigma-1} \tau_{\upsilon j} w_\upsilon\right)^{-\theta} \frac{\sigma-1}{\theta-\sigma+1} \frac{(f_{\upsilon j})^{1-\theta/(\sigma-1)}}{(L_j)^{1-\theta/(\sigma-1)}}}} \right)^{1/\theta} \\ &= \lambda_{jj}^{-1/\theta} L_j^{1/(\sigma-1)} \left(\frac{b_j^\theta f_{jj}^{1-\theta/(\sigma-1)}}{f_e \left(\frac{\sigma}{\sigma-1}\right)^\theta (\sigma)^{\theta/(\sigma-1)}} \frac{\sigma-1}{\theta-\sigma+1} \right)^{1/\theta}.\end{aligned}$$

In this model, λ_{jj} influences welfare in exactly the same way as in Eaton and Kortum (and in fact as in Chaney '07 and Arkolakis '06), with an elasticity of $-1/\theta$. Thus, when the models with or without free entry are calibrated to match domestic and foreign market shares before and after trade liberalization, they deliver the same predictions for the welfare implications of "openness". Another point worth noticing is that the implications of a larger population size to total welfare of a country are not affected by θ because the number of offered goods increases proportionately with the population of the market. Thus, consumers in a larger country are not forced to consume a higher share of varieties produced with lower productivities as in models with no free entry.

5 Conclusion

We show that a model of monopolistic competition with heterogeneous firms and free entry is isomorphic to a model with a predetermined number of entrants along key dimensions: trade shares, trade effects on offered varieties, and welfare implications of trade.

6 Bibliography

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7 Additional Notes

7.1 Additional Appendix A

Using the solution to the first order conditions of the firm, profit becomes

$$
\begin{aligned}
\pi_{ij}(\phi) &= \left(\frac{\sigma}{\sigma-1}\frac{w_i}{\phi}\right)^{1-\sigma}\frac{1}{\sigma}\frac{w_j L_j}{P_j^{1-\sigma}} - w_j f_{ij} \\
&= w_j f_{ij}\left(\frac{\phi}{\phi_{ij}^*}\right)^{\sigma-1} - w_j f_{ij} \\
&= w_j f_{ij}\left(\left(\frac{\phi}{\phi_{ij}^*}\right)^{\sigma-1} - 1\right).
\end{aligned}
$$

Notice that the ratio of average variable profits in a market to the fixed cost of operation is constant since:

$$
\int_{\phi_{ij}^*}^{+\infty} w_j f_{ij}\left(\left(\frac{\phi}{\phi_{ij}^*}\right)^{\sigma-1} - 1\right)\theta\frac{\left(\phi_{ij}^*\right)^{\theta}}{\phi^{\theta+1}}d\phi = w_j f_{ij}\frac{\theta}{\theta-\sigma+1}.
$$

and

$$
\int_{\phi_{ij}^*}^{+\infty} w_j f_{ij}\theta\frac{\left(\phi_{ij}^*\right)^{\theta}}{\phi^{\theta+1}}d\phi = w_j f_{ij}.
$$

7.2 Additional Appendix B: Definition of Equilibrium

In order to determine the equilibrium variables

$$
\begin{aligned}
N_i \text{ , for } i &= 1,...,N, \\
w_i \text{ , for } i &= 1,...,N, \\
\phi_{ij}^*, \text{ for } i,j &= 1,...,N,
\end{aligned}
$$

we require that (given the definition of P_j in terms of these variables):

a) productivity cutoffs satisfy

$$
\left(\phi_{ij}^*\right)^{\sigma-1} = \frac{f_{ij}}{\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\frac{1}{\sigma}\frac{L_j}{P_j^{1-\sigma}}}, \forall i,j = 1,...,N,
$$

b) labor market clears

$$
N_i = \frac{(\sigma-1)}{\theta\sigma\frac{f_e}{\frac{b_i^\theta}{\left(\phi_{ii}^*\right)^\theta}}}L_i, \quad \forall i = 1,...,N,
$$

c) zero expected profits due to free entry

$$
\sum_{\upsilon}\frac{w_\upsilon}{w_i}f_{i\upsilon}\frac{\left(\phi_{ii}^*\right)^\theta}{\left(\phi_{i\upsilon}^*\right)^\theta}\frac{\sigma-1}{\theta-\sigma+1} = \frac{f_e}{\frac{b_i^\theta}{\left(\phi_{ii}^*\right)^\theta}}, \quad \forall i = 1,...,N,
$$

In fact, given

$$\lambda_{ij} = \frac{N_i \left(\frac{\phi^*_{ii}}{\phi^*_{ij}}\right)^{\theta} f_{ij} \frac{\sigma\theta}{\theta-\sigma+1}}{L_j},$$

condition c) is equivalent to the trade balance equation

$$w_i L_i = \sum_{\upsilon} \lambda_{i\upsilon} w_{\upsilon} L_{\upsilon}, \quad i = 1, ..., N.$$

Also it can be shown that the following expression is equivalent to the price index,

$$w_i L_i = \frac{\sigma\theta}{\theta - \sigma + 1} \left(\sum_{\upsilon} N_{\upsilon} \frac{(\phi^*_{\upsilon\upsilon})^{\theta}}{(\phi^*_{\upsilon i})^{\theta}} w_i f_{\upsilon i} \right).$$

To see this, note that

$$w_i L_i = \frac{\sigma\theta}{\theta - \sigma + 1} \left(\sum_{\upsilon} N_{\upsilon} \frac{(\phi^*_{\upsilon\upsilon})^{\theta}}{(\phi^*_{\upsilon i})^{\theta}} w_i f_{\upsilon i} \right) \Leftrightarrow$$

$$\begin{aligned}
P_j^{1-\sigma} &= \sum_{\upsilon} N_{\upsilon} \frac{(\phi^*_{\upsilon\upsilon})^{\theta}}{\left(\phi^*_{\upsilon j}\right)^{\theta}} \frac{f_{\upsilon j}}{\left(\frac{\sigma}{\sigma-1}\tau_{\upsilon j} w_{\upsilon}\right)^{1-\sigma} \frac{1}{\sigma}\frac{L_j}{P_j^{1-\sigma}}} \left(\frac{\sigma}{\sigma-1}\tau_{\upsilon j} w_{\upsilon}\right)^{1-\sigma} \frac{\theta}{\theta - \sigma + 1} \Leftrightarrow \\
P_j^{1-\sigma} &= \sum_{\upsilon} N_{\upsilon} \frac{(\phi^*_{\upsilon\upsilon})^{\theta}}{\left(\phi^*_{\upsilon j}\right)^{\theta}} \left(\phi^*_{\upsilon j}\right)^{\sigma-1} \left(\frac{\sigma}{\sigma-1}\tau_{\upsilon j} w_{\upsilon}\right)^{1-\sigma} \frac{\theta}{\theta - \sigma + 1} \Leftrightarrow
\end{aligned}$$

$$P_j^{1-\sigma} = \sum_{\upsilon} N_{\upsilon} \frac{(\phi^*_{\upsilon\upsilon})^{\theta}}{\left(\phi^*_{\upsilon j}\right)^{\theta}} \int_{\phi^*_{\upsilon j}} \left(\frac{\sigma}{\sigma-1} \frac{\tau_{\upsilon j} w_{\upsilon}}{\phi}\right)^{1-\sigma} \theta \frac{\left(\phi^*_{\upsilon j}\right)^{\theta}}{\phi^{\theta+1}} d\phi.$$