A Unified Theory of Firm Selection and Growth*

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Abstract

This paper develops a theory of firm selection and growth and embeds it into an international trade framework of balanced growth. I assume that firm-level growth is the result of idiosyncratic productivity improvements while there is continuous arrival of new potential producers. Firms can also pay an increasing market penetration cost to sell more to a given market. The model is consistent with a set of salient regularities of firm and exporter selection and growth, as well as the observed distribution of sales. I calibrate the model parameters that determine firm dynamics by looking at the exit rates of a cohort at the US census and the elasticity of trade in Eaton and Kortum. This elasticity is regulated in the model by the firm-level growth process. The calibrated model can account for almost all the turnover and growth of US census cohorts over two decades. It can also account for a large part of the turnover and growth of Colombian exporters in individual destinations.

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1 Introduction

In the last few decades, economists have performed a systematic investigation of the empirical patterns of turnover and growth of firms. A series of salient features of the data has been uncovered, indicating an inverse relationship between size and growth of smaller firms and a robust cross-sectional distribution of firm sales.[1] Recent work by Eaton, Kortum, and Kramarz (2008) and Eaton, Eslava, Kugler, and Tybout (2008) reports these phenomena when looking at the sales of exporting firms to individual destinations. While theoretical research has dealt separately with subsets of these facts, with most prominent recent examples being those of Klette and Kortum (2004) and Luttmer (2007), a unified framework that explains turnover and growth and is consistent with the observed cross-sectional distribution of firm sales has yet to be developed.

This paper develops a theory of firm-sales dynamics and integrates this theory into an analytically tractable, multi-country general equilibrium framework. It argues that this simple framework, based on idiosyncratic firm productivity shocks, is promising for studying firm behavior in the domestic market *and* in individual exporting destinations. The theory can qualitatively and quantitatively (to a large extent) account for the main facts on selection and growth of firms in the domestic and exporting destinations. Additionally, this paper shows that the cross-sectional phenomena of firm sales are intimately linked to the dynamic ones. The parameterization of the theory that can explain the dynamic phenomena can also account for the observed cross-sectional distribution of sales: this distribution for the largest firms closely approximates the Pareto while there are many more small firms than the Pareto would predict.

Following Kortum (1997) and Eaton and Kortum (2001), I assume that the rate at which new ideas arrive at each country is exogenously given. I associate each idea to a monopolistically competitive firm that can produce a differentiated good and (potentially) earn profits. After an idea is "born", its productivity is expected to increase over time. Deviations from the expected growth rate follow a Brownian motion. This assumption has recently been used by Gabaix (1999) (to explain city size), and Luttmer (2007) and implies independence of the growth rate and the size of the productivity of ideas, i.e. Gibrat's law of proportionate growth.

The setup of entry (and/or random exit) of ideas that I adapt is similar to the one of Reed

[1] See Audretsch (1995), Sutton (1997), Caves (1998) and Axtell (2001) for a discussion of these findings.

(2001) and is used to generate a stationary productivity distribution in lieu of the common assumption of a lower exit barrier. A key technical contribution of this setup is that it brings into light the celebrated results of Yule (1925) and Simon (1955). They demonstrated that random entry and Gibrat's law are two minimal sufficient conditions that give rise to a cross-sectional distribution with Pareto right tails, a feature also shared by this model. Pareto distributed productivities is a common assumption in static trade models such as the workhorse Melitz (2003)-Chaney (2008) framework and its extension by Arkolakis (2008).

Consumer's preferences are assumed to be of the standard Constant Elasticity of Substitution (CES) form. In addition, I model market entry costs as developed by Arkolakis (2008), where firms have to pay a market penetration cost to reach additional consumers in each country. If a firm is productive enough, it chooses to pay the entry cost to reach a positive fraction of consumers in a market. The most productive firms optimally choose to penetrate a given market to a larger extent, and reach almost all consumers. Small firms may exhibit substantial differences in the number of consumers reached, depending on the convexity of the function that governs the cost to reach additional consumers.

Given the assumptions of CES demand and constant marginal costs of production, the Pareto distribution of productivities implies Pareto distributed sales for the largest exporters in a market. However, the demand of the small exporters in each market exhibits departures from the CES demand setup due to the market penetration technology. Thus, the model implies that there are many more small firms in each market than the Pareto distribution would imply.

Going a step beyond explaining the cross-sectional facts, the model can also account for the dynamic aspects of firm sales in individual destinations or firm gross sales, if it is considered as an one country model.[2] In particular, the model can deliver the inverse relationship between the growth rate and the size of firm's sales in a market for two reasons: the effects of selection and the market penetration technology. Due to selection, small firms with negative growth rates do not sell, leading to an upward bias of the growth of the small firms that survive. Moreover, increasing market penetration costs make it more difficult for a firm to expand when it is large compared to when it is small in a given market. Thus, in the model with increasing market

[2] In an appendix, available on my website, I extend the setup with universal productivity improvements and show that the predictions of the model with imperfectly correlated productivity (or demand) improvements accross markets are consistent with the above results. Notice that given the modeling of product differentiation in this model productivity and demand improvements in firm sales are isomorphic.

penetration costs even if one were to eliminate the statistical bias arising from selection, the sales of smaller firms in a given market would show a higher growth rate. This implication holds for the sales of a firm in each market and is consistent with empirical findings for domestic and individual exporting destinations.

The model is solved in a balanced growth path equilibrium and develops the first dynamic model based on the Melitz (2003)-Chaney (2008) multi-country setup, retaining though the key aggregate properties of trade models with CES demand.[3] At each snapshot of time, this model is identical to its static counterpart. To quantitatively assess the predictions of the model, I fully exploit the cross-sectional restrictions that the modeling of international trade imposes. Therefore, I use the same parameterization as in the related static calibration. In addition, the drift and the variance of the stochastic process governing firm growth have to be determined. These two parameters are calibrated to match two moments in the data: the elasticity of bilateral country trade with respect to trade costs and the survival rate of a cohort of firms from the US census. The elasticity of trade in the model is the shape parameter of the Pareto distribution which is a function of the drift and the variance.

This procedure implies that the parameters of the model governing the growth of firms can be calibrated without using information on firm growth. In turn, I illustrate the ability of the calibrated model to predict firm turnover (entry and exit) and growth. The model with calibrated domestic productivity improvements can quantitatively capture the turnover of US census firms over time and a large part of the turnover of Colombian exporting firms into individual exporting markets. An important finding is that the model calibrated to match the domestic firm exit cannot account for 1/3 of the first year turnover in foreign markets. Additionally, the model can predict cohort market shares by simultaneously delivering the increase in the average size of surviving firms and the sales of new firms that are initially small but grow fast overtime, conditional on survival.

These findings draw three notable conclusions from this research. First, as it is illustrated above, an intimate relationship between understanding cross-sectional and dynamic facts exists. Second, this model is consistent with facts on firm exporting dynamics without appealing to the

[3] Trade models with CES demand such as those of Anderson (1979), Krugman (1980), Eaton and Kortum (2002), the Chaney version of Melitz (2003), as well as its extension by Arkolakis (2008), have identical gravity structure (see for example the discussion in Arkolakis, Demidova, Klenow, and Rodriguez-Clare (2008), Eaton and Kortum (2005) and Chen and Novy (2008)).

common assumption of sunk costs of entry or other indivisibilities into the entry or production costs. In fact, I argue that assuming indivisibilities in entry costs will imply a growth rate and size distribution for the smaller firms in a market which are not in line with the empirical findings. Third, a series of aggregate facts can be delivered by a model that is consistent with firm-level behavior. In this sense, the theory that looks in a unified way at the micro process of firm selection and growth is also properly "unified" with the workhorse macroeconomic gravity trade framework.

Firm dynamics with a continuum of heterogeneous firms are examined in one-country models by Jovanovic (1982), Hopenhayn (1992), Klette and Kortum (2004), Luttmer (2007) among others and in the two-country extension of the last framework by Irarrazabal and Opromolla (2006) and Atkeson and Burstein (2009).[4] In Jovanovic (1982), growth of the firm depends primarily on age rather than on size, which is a key difference from the framework of this paper.[5] In contrast to Jovanovic (1982) and Hopenhayn (1992) but similarly to Klette and Kortum (2004) the model in this paper offers analytical relationships for the turnover and growth of firms.[6] In contrast to their work, it also implies a Pareto distribution for the sales of large firms and is amenable to multi-country analysis. Extending the results of Luttmer (2007) the model delivers a cross-sectional distribution that exhibits Pareto tails and provides analytically tractable relationships that can account for the growth of firms. The two-country extensions of Irarrazabal and Opromolla (2006) and Atkeson and Burstein (2009) carry the merits and limitations of Luttmer's framework but lose analytical tractability in a multi-country setup.[7]

The rest of the paper is organized as follows. Section 2 summarizes the evidence on firm turnover, growth, and size distribution. In section 3, I develop the multi-country framework and

[4] Growth conditional on size and age is discussed by Cooley and Quadrini (2001) and Arellano, Bai, and Zhang (2008) in a framework with financial constraints. Additionally, Rossi-Hansberg and Wright (2007) develop a model where plant growth and size are negatively related in different industries due to mean reversion in the accumulation of industry-specific human capital. This related work, while complementary to this paper is a considerable departure from the heterogeneous firm, analytically tractable framework that I consider.

[5] However, in Jovanovic (1982) growth rates can increase or decrease with size depending on the shape of the cost function of the firms.

[6] Lentz and Mortensen (2008) and Bernard, Redding, and Schott (2009) develop models of firm dynamics extending the theories of Klette and Kortum (2004) and Hopenhayn (1992) respectively. In turn, their models borrow many of the qualitative features of these theories.

[7] In recent work Irarrazabal and Opromolla (2009) adapt a framework of entry and exit similar to the one in this paper, in that it does not assume the free entry of firms in order to characterize the multi-country equilibrium of the model. The authors retain the main assumptions of the fixed cost framework (without assuming sunk costs of exporting) and study the theoretically implied entry-exit patterns of exporters into individual destinations.

in section 4 I provide an analytical characterization of the theoretical predictions of the model. In sections 5 and 6, I calibrate the model and evaluate its predictions with the US census and Colombian export data. Section 7 contains the concluding remarks.

2 Evidence on Firm Turnover, Growth and Size Distribution

This section summarizes the findings of a set of studies that present empirical regularities regarding the turnover of firms, their growth, and their cross-sectional size distribution in domestic and export data.

2.1 Entry and Exit of firms

The entry-exit behavior of firms has been extensively analyzed in the Industrial Organization literature. Researchers find that firms of smaller size have a lower probability of selling next period in the market, when compared to larger firms. In addition, conditional on survival, hazard rates decline with age. Another robust empirical finding regarding firm entry is that new entrants and exitors are typically small firms.

Evidence for the first two findings is summarized by Caves (1998). Eaton, Eslava, Kugler, and Tybout (2008) (henceforth EKKT) present similar evidence for the entry and exit of Colombian exporters to individual exporting destinations. The findings regarding size of entrants and exitors are consistent with the US census data used by Dunne, Roberts, and Samuelson (1988) and also the Colombian trade data used by EEKT.

2.2 Growth of firms

A large literature has emerged studying the empirical validity of Gibrat's law reviewed by Sutton (1997) and Caves (1998). Deviations from Gibrat's law have been recognized as early as Mansfield (1962). However, Hart and Oulton (1996) point out that these deviations appear to vanish when samples of large firms are considered. Thus, for initially large firms, growth rate and size are unrelated. Mansfield (1962) conjectures that the deviations maybe due to firm selection. However, using a comprehensive sample of US manufacturing firms, Evans (1987a) and Evans (1987b) show that the negative growth-size relationship is robust when correcting for sample truncation caused by the exit of smaller firms.

Similar findings are reported by EEKT for the growth of sales of firms to individual destinations. The authors examine the growth rates of sales of firms to individual destinations and find that the rates of small exporters are much larger than those of the largest exporters in the same markets. In addition, when they group firms in different quantiles according to their initial size of their sales, they find that the total sales of the smallest quantiles are those that grow the most. The latter result provides evidence that the negative relationship between growth rate and size of firms in each destination is not only due to selection of firms in the destination. Nevertheless, further empirical exploration is possible.

A robust inverse relationship between the variance of firm growth rates and the initial size of the firm has been also identified. Evidence regarding this relationship is summarized by Klette and Kortum (2004) and reviewed by Caves (1998) and Sutton (2002). A study of the relationship between the variance of growth rates and size of firm sales to individual destinations has not been performed.

2.3 Size distribution of firms

The literature on the size distribution of firms seems to have two distinct and very robust findings. The first finding is that the distribution of firm size is highly skewed and dominated by many small enterprises. The second finding is that the size distribution of larger firms is approximately Pareto. Schmalensee (1989) and Audretsch (1995) discuss the evidence on the skewness of the distribution of firms while both findings have been reported by Simon and Bonini (1958) and discussed in Ijiri and Simon (1977). Recently, Axtell (2001) reaffirmed that the distribution of sales of US manufacturing firms exhibits Pareto right-tails.

Eaton, Kortum, and Kramarz (2008) distinguish the sales of French firms by destination and verify the above facts. They also give a very clear picture of the exact shape of the size distribution of the domestic sales of French manufacturing firms. The authors find that the distribution of sales is Pareto in the right tail and also that there are many more small firms than the Pareto distribution would imply in the left tail. The shape of the distribution of sales appears to be robust over time and similar to the distribution of the sales of exporters to individual destinations.

In the next section I will lay out the main elements of a model that can account for the firm-level facts that I have reviewed above.

3 The Model

The model described extends the static version of Arkolakis (2008). It introduces a stochastic process for the entry and growth of productivities as in Reed (2001) and incorporates dynamics to a setting with heterogeneous productivity firms following Luttmer (2007).

Time is continuous and indexed by t. I will denote the importing country with an index j and the exporting country with i, where $i, j = 1, ..., N$. At each time t, country j is populated by a continuum of consumers of measure $L_{jt} = L_j e^{g_\eta t}$, where g_η is the growth rate of the population. Each consumer from country j has preferences over a consumption stream $\{C_{jt}\}_{t\geq 0}$ of a composite good from which she derives utility according to

$$\left(E\int_0^{+\infty} \rho e^{-\rho t} C_{jt}^{\frac{\iota-1}{\iota}} dt\right)^{\frac{\iota}{\iota-1}} ,$$

where $\rho > 0$ is the discount rate and $\iota > 0$ is the intertemporal elasticity of substitution.

The composite good is made from a continuum of differentiated commodities

$$C_{jt} = \left(\sum_{i=1}^{N} \int_{\omega\in\Omega_{it}} c_{ijt}(\omega)^{(\sigma-1)/\sigma} d\omega\right)^{\frac{\sigma}{\sigma-1}}$$

where $c_{ijt}(\omega)$ is the demand a consumer j has for a good from country i and $\sigma \in (1, +\infty)$ is the elasticity of substitution among different varieties of goods. I assume that firms reach consumers independently and at a given point in time t, a consumer $l \in [0, L_{jt}]$ has access to a potentially different set of goods $\Omega_{it}^l \subset \Omega_{it}$. Each good is produced by a single firm and firms differ ex-ante only in their productivity, z, and their source country i. I consider a symmetric equilibrium where all firms with the same productivity from the same country i choose to charge the same price in j, $p_{ijt}(z)$, and also reach consumers there with a certain probability, $n_{ijt}(z) \in [0, 1]$. The existence of a large number of firms implies that every consumer from country j has access to the same distribution of prices for goods of different types and thus the level of consumption

C_{jt} is the same for all consumers.[8] The existence of a large number of consumers in country j implies that the fraction of consumers reached by a firm of type z from country i is $n_{ijt}(z)$ and the total measure is $n_{ijt}(z) L_{jt}$.

Each household earns labor income w_{jt} for selling its unit labor endowment on the labor market and profit flows π_{jt} from the ownership of domestic firms. Thus, the demand for good z from country i by a consumer from country j is

$$c_{ijt}(z) = \frac{p_{ijt}(z)^{-\sigma}}{P_{jt}^{1-\sigma}} y_{jt} \ ,$$

where $y_{jt} = w_{jt} + \pi_{jt}$ and

$$P_{jt}^{1-\sigma} = \sum_{\upsilon=1}^{N} \int_0^{+\infty} p_{\upsilon jt}(z)^{1-\sigma} n_{\upsilon jt}(z) \, dM_{\upsilon jt}(z) \ . \tag{1}$$

In the equation above, $p_{\upsilon jt}(z)$ is the price at which a good, produced in country υ with productivity z, is being sold in country j. $dM_{\upsilon jt}(z)$ is the density of varieties of a given type that are actually sold to country j. Given the definition of the price index, the budget constraint for each consumer is $C_{jt} P_{jt} = y_{jt}$. The above results imply that the total demand for a good of type z from country i when selling to country j is

$$q_{ijt}(z) = n_{ijt}(z) L_{jt} \frac{p_{ijt}(z)^{-\sigma}}{P_{jt}^{1-\sigma}} y_{jt} \ . \tag{2}$$

3.1 Entry and Exit

An idea is a way to produce a good ω with productivity z. Each idea is exclusively owned and grants a monopoly over the related good. The monopoly over the goods delivers a monopolistic competition setup as in Dixit and Stiglitz (1977) and Melitz (2003). However, in my context ideas become firms only if they are materialized into production. Once ideas are born, they can only die at an exogenous rate $\delta \geq 0$. In order to consider an economy that is consistent with balanced growth, I also assume that each country innovates at an exogenous rate $g_B \geq \delta$. This rate will be specified when I construct the balanced growth path and implies that the measure of existing ideas at each time t in i is $J_i e^{(g_B - \delta)t}$, with $J_i > 0$ being the initial measure of ideas

[8] See Arkolakis (2008) for the details of this argument.

in country i.

New ideas can potentially produce with an initial productivity, $\bar{z}_{it}$, where

$$\bar{z}_{it} = \bar{z}_i \exp\left(g_E t\right) \ ,$$

and $\bar{z}_i, g_E > 0$. The parameter g_E is interpreted as the growth rate of the frontier of the productivity of new ideas. Thus, while this specification implies that all new ideas at time t enter with the same productivity, it incorporates a form of "creative destruction" since more recent ideas arrive with a higher productivity.[9] In fact, I show that, in the balanced growth path, there exists a lower productivity threshold of operation at each time t, z^*_{ijt}, and this threshold grows at a rate g_E. z^*_{ijt} is determined by the zero profit condition at each point of time: since there is no indivisible cost of production or entry, ideas with productivity higher than z^*_{ijt} are used into production and appear as firms in market j. If an idea is not productive enough to be profitable, it remains idle while waiting for the possibility to become profitable in the future (when its productivity surpasses z^*_{ijt} at a given time t).[10]

This setup for firm entry and exit into individual markets makes the model substantially more tractable than the Luttmer (2007) setup since there are no forward looking decisions for the firms. However, while in my case all the dynamics are determined by the stochastic process for the productivities, the model retains the main desirable properties of the Luttmer setup as I illustrate in section 4.

3.2 Firms and Ideas

The productivity of an idea is the same in all markets and evolves, independently across ideas, according to

$$z_{t^b,a} = \bar{z}_i \exp\left(g_E t^b + g_I a + \sigma_z W_a\right) \ , \tag{3}$$

[9] Extending this simple case to one in which new entrants arrive with different productivities drawn from a non-atomic distribution is straightforward (see, for example, Reed (2002)). In particular, unless entrants are specified to be very large with a high probability the right tails of the distribution will be unaffected. In addition, the process of growth of ideas and firms is not affected by entry.

[10] Allowing for free entry of ideas together with the new entry-exit process in the market that I propose is similar to Luttmer (2007). However, solving for a case with free entry and heterogeneous firms in a multi-country framework is not as straightforward as, for example, in the case of Arkolakis, Demidova, Klenow, and Rodriguez-Clare (2008).

where $z_{t^b,a}$ is the labor productivity of the idea at age a that was born at time t^b. $W_a \sim \mathcal{N}(0,a)$ is a Brownian motion with independent increments and the parameter σ_z regulates the volatility of the growth of ideas. Notice that the productivity of incumbent ideas is improving on average at a rate g_I. The Brownian motion assumption naturally emerges as the continuous time limit of a firm growth rate that is a discrete-random walk. This evolution process for productivities is adapted by Luttmer (2007). Similar processes have been widely used to represent firm growth since Gibrat (1931).

I assume that products markets clear every period and firms produce using a constant returns to scale production function $q\left(z_{t^b,a}\right) = z_{t^b,a}l$, where l is the amount of labor used in production. Moreover, firms pay a market penetration cost that is a function of the number of consumers reached in a given market. I model these market penetration costs using the specification of Arkolakis (2008) derived from first principles as costs of marketing. I also assume these costs are incurred by the firms at each instant of time. The specification used for the costs of marketing in each country is employed to generalize the assumption used by previous models (see for example Melitz (2003), Luttmer (2007)) that a per-country fixed cost is required to be paid at each period of time.[11]

The labor required for a firm to reach a fraction of consumers n in a market of population size L is

$$F(n,L) = \begin{cases} \frac{L^\alpha}{\psi}\frac{1-(1-n)^{-\beta+1}}{-\beta+1} & \text{for } \beta \in [0,1) \cup (1,+\infty) \\ \frac{L^\alpha}{\psi}\log(1-n) & \text{for } \beta = 1 \end{cases}.$$

where $\alpha \in [0,1]$ and $\psi > 0$. If $\alpha < 1$, the market penetration costs to reach a certain fraction of consumers decrease with the population size of the market. The parameter β governs the convexity of the marketing cost function: higher β implies more convexity and steeper increases in the marginal cost to reach more consumers. As in Arkolakis (2008), firms pay a fraction γ of the market penetration cost in terms of foreign wages and a fraction $1-\gamma$ in terms of domestic wages. This specification yields the following total market penetration cost faced by a firm from

[11] A model that considers dynamics in marketing by examining state dependence of market penetration costs on previous marketing is left for future research. Drozd and Nosal (2008) developed a model where a representative firm's demand is modeled as the firm's marketing capital that accumulates over time. This state dependence on previous marketing implies different short run versus long run elasticity of aggregate trade. My modeling of marketing is static and focused at the firm-level.

country i that reaches n fraction of consumers in country j:

$$w_j^{\gamma} w_i^{1-\gamma} F\left(n_{ij}, L_j\right) \ .$$

In addition to the cost to reach consumers, the firm has to pay a variable trade cost modeled in the standard iceberg formulation. This iceberg cost implies that a firm operating in country i and selling to country j must ship $\tau_{ij} > 1$ units in order for one unit of the good to arrive at the export destination. For simplicity, I assume that $\tau_{ii} = 1$.

Given the constant returns to scale production technology and the separability of the marketing cost function across countries, the decision of a firm to sell to a given country is independent of the decision to sell to other countries. Total profits of a particular firm are the summation of the profits from exporting activities in all countries $j = 1, ..., N$ (or a subset thereof). Thus, at a given time t, the firm's problem is the same as in Arkolakis (2008), and firm z from country i solves the following static maximization problem for each given country j:[12]

$$\pi_{ijt}\left(z\right) = \max_{n_{ijt}, p_{ijt}} \left\{ n_{ijt} L_{jt} y_{jt} \frac{p_{ijt}^{1-\sigma}}{P_{jt}^{1-\sigma}} - n_{ijt} L_{jt} y_{jt} \frac{\tau_{ij} p_{ijt}^{-\sigma} w_{it}}{P_{jt}^{1-\sigma} z} - w_{jt}^{\gamma} w_{it}^{1-\gamma} \frac{L_{jt}^{\alpha}}{\psi} \frac{1-\left[1-n_{ijt}\right]^{-\beta+1}}{-\beta+1} \right\}$$
$$\text{s.t. } n_{ijt} \in [0,1] \ \forall t \ .$$

For any β, the optimal decisions of the firm in the multi-country model are:

$$p_{ijt}\left(z\right) = \tilde{\sigma} \frac{\tau_{ij} w_{it}}{z} \ , \tag{4}$$

where

$$\tilde{\sigma} = \frac{\sigma}{\sigma - 1} \ .$$

For $z \geq z_{ijt}^*$,

$$n_{ijt}\left(z\right) = 1 - \left[L_{jt}^{1-\alpha} y_{jt} w_{jt}^{-\gamma} z^{\sigma-1} \left(\tilde{\sigma} \tau_{ij} w_{it}\right)^{1-\sigma} \psi P_{jt}^{\sigma-1} / \left(w_{it}^{1-\gamma} \sigma\right) \right]^{-1/\beta} \ , \tag{5}$$

and $n_{ijt}\left(z\right) = 0$ for $z < z_{ijt}^*$, where z_{ijt}^* is given by

$$z_{ijt}^* = \sup\left\{ z : \pi_{ijt}\left(z\right) = 0 \right\} \ , \tag{6}$$

[12] Slighly abusing the notation, I denote the decision of the firm only as a function of its productivity z, supressing time of birth and age information. Given that the optimization decision is static, the current level of productivity is the only state variable. I keep the notation parsimonious throughout the text whenever possible.

which implies that

$$z^*_{ijt} = \left[L_{jt}^{1-\alpha} y_{jt} w_{jt}^{-\gamma} \left(\tilde{\sigma}\tau_{ij} w_{it}\right)^{1-\sigma} \psi P_{jt}^{\sigma-1} / \left(w_{it}^{1-\gamma}\sigma\right)\right]^{-1/(\sigma-1)} \quad . \tag{7}$$

Equation (7) implies that apart from general equilibrium considerations, z^*_{ijt}, and therefore the entry-exit decision of the firm, will not depend on the parameter β. Substituting (4), (5) and (7) into the expression for sales per firm, (2), and multiplying by the price, the sales of firm z from country i in country j can be written as

$$r_{ijt}(z) \equiv p_{ijt}(z)\, q_{ijt}(z) = \begin{cases} L_{jt}^{\alpha} y_{jt}^{\gamma} y_{it}^{1-\gamma} \frac{1}{\tilde{\psi}} \left[e^{\bar{c}_1 \ln\left(z/z^*_{ijt}\right)} - e^{\bar{c}_2 \ln\left(z/z^*_{ijt}\right)}\right] & \text{if } z \geq z^*_{ijt} \\ 0 & \text{otherwise.} \end{cases} , \tag{8}$$

with

$$\bar{c}_1 = \sigma - 1,\ \bar{c}_2 = (\sigma - 1)\frac{(\beta - 1)}{\beta},\ \tilde{\psi} = \frac{\psi}{\sigma\left(1 - \bar{\pi}\right)} ,$$

and $\bar{\pi} \equiv \pi_{it}/y_{it}$ is the fraction of profits out of total income. In the balanced growth path equilibrium, this fraction is constant and thus I suppress its subscripts. Careful inspection of (8) reveals that for $\beta = 0$ all the firms selling from i to j sell a minimum amount, $L_{jt}^{\alpha} y_{jt}^{\gamma} y_{it}^{1-\gamma}/\tilde{\psi}$, while for $\beta > 0$ this amount is 0. Conditional on entry, more productive firms have higher sales as equation (8) indicates. These firms charge lower prices and thus sell more per consumer (intensive margin). In addition, if $\beta > 0$, equation (5) implies that they also reach more consumers (extensive margin). However, if $\beta = 0$ all entrants optimally choose $n_{ij} = 1$. Differences in β also reflect different growth patterns for firm sales as I will illustrate in section 4.

3.3 Balanced Growth Path Equilibrium

To solve for the cross-sectional distribution, I consider the stationary balanced growth path. I first define the productivity detrended by the rate of growth of the zero profit cutoff,

$$\phi_a = \bar{z}_i \exp\left\{g_E t^b + g_I a + \sigma_z W_a\right\} / \exp\left\{g_E\left(t^b + a\right)\right\}$$

$$= \bar{z}_i \exp\left\{\left(g_I - g_E\right) a + \sigma_z W_a\right\} \ .$$

Given expression (8), the dynamic behavior of ϕ_a is sufficient to characterize the relative sales of an idea at each time a in the balanced growth path.

The logarithm of ϕ_a is a Brownian motion with a drift,

$$s_a = \ln \phi_a = \bar{s}_i + (g_I - g_E)\, a + \sigma_z W_a \ , \tag{9}$$

where $\bar{s}_i = \ln \bar{z}_i$. s_a will be used as proxy for the productivity of an idea or the size of a firm after a years given that firms with larger s_a are (weakly) larger in sales, productivity and employment. The term $g_I - g_E$, i.e. the difference between the growth of incumbent ideas and the growth of the frontier of new ideas. Hereafter, I will denote this difference by μ. The probability density of $s_a = s$ for a given generation of ideas of age $a > 0$ from i is given by the normal density:[13]

$$f_i(s,a) = \frac{1}{\sigma_z \sqrt{a 2\pi}} \exp\left\{ -\left(\frac{s - \bar{s}_i - \mu a}{\sigma_z \sqrt{a}} \right)^2 /2 \right\} \ . \tag{10}$$

This distribution is not age-stationary. Continuous entry of new ideas, however, creates a stationary cross-sectional distribution of all productivities in country i, $f_i(s)$, when the size of productivities of ideas is considered across different ages. In a stationary equilibrium, with entry and exit of ideas, the dynamics of the probability density of each $s \neq \bar{s}_i$ and $\forall i$, are described by a Kolmogorov forward equation,[14]

$$-\mu f_i'(s) + \frac{1}{2}\sigma_z^2 f_i''(s) - g_B f_i(s) = 0 \ . \tag{11}$$

Intuitively, in a stationary steady state, the net changes at each point s of the distribution must equal the rate of reduction of the probability density at $s \in (-\infty, \bar{s}_i) \cup (\bar{s}_i, +\infty)$. The net changes are due to the stochastic flows of productivities in and out of that point described by

[13] See for example Harrison (1985) p. 37. $f_i(s,a)$ can be derived as the solution of the differential equation $D_a f_i(s,a) = -\mu f_i'(s,a) + \frac{1}{2}\sigma_z^2 f_i''(s,a)$, with initial condition $f_i(s,a) = \Delta(s - \bar{s}_i)$, where $\Delta(.)$ is the Dirac delta function. Additionally, the realizations of the Brownian motion over different time periods, $s_{a_1}, s_{a_2}, ..., s_{a_n}$, follow a multivariate normal distribution with means $Es_a = s_0 + \mu a$ and covariances $Cov(s_a, s_{a'}) = \sigma_z^2 [\min(a, a')]$. This feature can be used to implement further scrutiny on the model, or to pursue an alternative estimation of its parameters, by looking at the probability distribution of sales and entry exit decisions of individual firms overtime, for researchers that have access to this information.

[14] In an appendix available online, I provide a different proof by expliciltly calculating $f(s) = \int_0^{+\infty} e^{-[g_B]a} f(s,a)\, da$. This proof, though more straightforward, provides less intuition on the exact forces that give rise to the cross sectional distribution of productivities across all ideas. Reed (2001) provides another proof using moment generating functions in which the intuition is also somewhat limited.

equation (9). The reduction happens at a rate δ, due to exit, plus the rate of growth of the measure of ideas, $g_B - \delta$, given that new entry happens only at point $\bar{s}_i$.

The density of productivities, $f_i(s)$, has to satisfy a set of conditions. The first requirement is that $-\infty$ is an absorbing barrier which implies the condition

$$\lim_{s \to -\infty} f_i(s) = 0 \ . \tag{12}$$

In addition $f_i(s)$, must be a probability density which implies that

$$f_i(s) \geq 0 \ , \ \ \forall s \in (-\infty, +\infty) \tag{13}$$

and

$$\int_{-\infty}^{\bar{s}_i} f_i(s)\, ds + \int_{\bar{s}_i}^{+\infty} f_i(s)\, ds = 1 \ . \tag{14}$$

Additionally, net inflows into the distribution must equal the net outflows:[15]

$$-\mu\left[f_i(\bar{s}_i-) - f_i(\bar{s}_i+)\right] + \frac{1}{2}\sigma_z^2\left[f_i'(\bar{s}_i-) - f_i'(\bar{s}_i+)\right] = g_B \ . \tag{15}$$

The left-hand side is the net inflows into the distribution from point $\bar{s}_i$. The right-hand side is the outflows from the distribution due to new entry and random exit of ideas. By continuity, the first term in brackets will disappear. However, entry of new ideas implies that the distribution is kinked at $\bar{s}_i$. Intuitively, the rate of change of the cdf changes direction at $\bar{s}_i$ because entry happens at that point. The solution of the above system is (see appendix A.2):

$$f_i(s) = \begin{cases} \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{\theta_1(s-\bar{s}_i)} & \text{if } s < \bar{s}_i \\ \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{-\theta_2(s-\bar{s}_i)} & \text{if } s \geq \bar{s}_i \end{cases} \tag{16}$$

where

$$\theta_1 = \frac{\mu + \sqrt{\mu^2 + 2\sigma_z^2 g_B}}{\sigma_z^2} > 0 \ , \tag{17}$$

[15] This condition results by integrating (11) over all $s \in (-\infty, \bar{s}_i) \cup (\bar{s}_i, +\infty)$, i.e. considering the net inflows from point $\bar{s}_i$ to the rest of the distribution. Similar conditions are used in labor models to characterize the behavior of the distribution at a point of entry to or exit from a particular occupation (see for example Moscarini (2005) and Papageorgiou (2008)).

$$\theta_2 = -\frac{\mu - \sqrt{\mu^2 + 2\sigma_z^2 g_B}}{\sigma_z^2} > 0 \ . \tag{18}$$

The following assumption guarantees that a time-invariant distribution exists and an ever increasing fraction of ideas is not concentrated in either of the tails of the distribution:[16]

A 1 *: The rate of innovation is $g_B > 0$.*

In particular, given that $g_B \geq \delta$, $\delta = 0$ implies $g_B > \delta$. Using equation (18), A1 also implies that

$$\sigma_z^2 \frac{(\theta_2)^2}{2} + \theta_2 \mu = g_B > 0 \tag{19}$$

The resulting cross-sectional distribution of detrended productivities $\phi \in [0, +\infty)$ is the so-called double Pareto distribution (Reed (2001)) with probability density function:[17]

$$\hat{f}_i(\phi) = \begin{cases} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \frac{\phi^{\theta_1 - 1}}{\bar{z}_i^{\theta_1}} & \text{if } \phi < \bar{z}_i \\ \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \frac{\phi^{-\theta_2 - 1}}{\bar{z}_i^{-\theta_2}} & \text{if } \phi \geq \bar{z}_i \end{cases} \tag{20}$$

The double Pareto distribution is illustrated in figure 1. A closer look at the probability density of productivities (equation (20)) reveals that at each moment of time, a constant fraction of ideas $\theta_1 / (\theta_1 + \theta_2)$ is above the threshold $\bar{z}_i$. To keep all the expressions of the model as simple as possible, I assume for the rest of the paper that $1/\psi$ is sufficiently high so that $z_{ijt}^* > \bar{z}_{it}$, $\forall i$, t . Thus, the (detrended) cross-sectional distribution of operating ideas (i.e. firms) is Pareto at $[\bar{z}_i, +\infty)$ with shape parameter θ_2. Moreover, I assume that the parameters of the model are such that the distributions of firm productivities and sales have a finite mean:

A 2 *: Productivity and sales parameters satisfy*

$$g_B > \max \left\{ \mu + \sigma_z^2/2 \ , \ \mu(\sigma - 1) + \frac{\sigma_z^2}{2} (\sigma - 1)^2 \right\} \ .$$

[16] Under the assumption $\mu > 0$, Pareto distribution emerges in the right-tail of the distribution for the limit case of $\sigma_z \to 0$. However, both $\mu < 0$ and $\sigma_z > 0$ will be essential features of the model in explaining the data as I illustrate in the calibration section.

[17] This distribution can also be thought of as a limit case in the distribution of firms derived by Luttmer (2007) when the exit cutoff goes to $-\infty$. However, in his case, this assumption would imply that firms never exit and that there is no selection in the model.

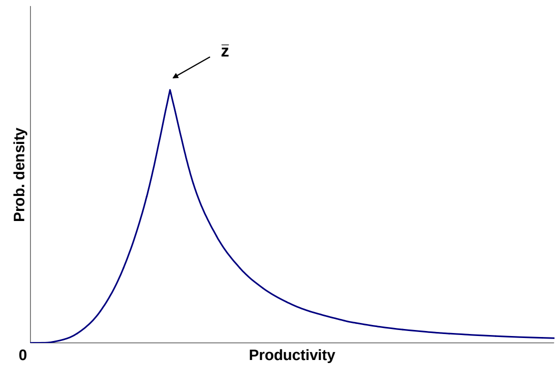

Figure 1: Double Pareto distribution

Assumption A2 implies that the entry rate of new ideas is larger than the growth rate of the productivities and sales of the most productive incumbent firms. Notice that A2 also implies the common condition that the Pareto shape coefficient, θ_2, is larger than $\sigma - 1$.

I will now construct a balanced growth path equilibrium for this economy. To do so I assume that the entry rate of new ideas is

$$g_B = g_\eta (1 - \alpha) + \delta \ , \tag{21}$$

implying that the number of ideas above the entry point will be $\theta_1 / (\theta_1 + \theta_2) J_i e^{g_\eta (1-\alpha) t}$. Aggregate variables, w_{it}, C_{it} grow at a rate g_κ where

$$g_\kappa = g_E + g_\eta (1 - \alpha) / (\sigma - 1) \, . \tag{22}$$

The growth rate of the ideas and thus the varieties adds to the growth rate of the frontier of new productivities, g_E, with a rate that is larger when goods are less substitutable.

The equilibrium also requires that the value of the aggregate endowment is finite. In order for this to happen the discount rate must exceed the rate of growth of the economy and thus:

A 3 *: Preference and technology parameters satisfy* $\rho + \frac{1}{\iota} g_\kappa > g_\kappa + g_\eta$*, with* $g_\eta \geq 0$.

Finally, notice that in the balanced growth path the cross-sectional distribution of firm sales and the bilateral trade shares, λ_{ij}, remain unchanged. This means that at each snapshot of time, this model collapses to the endogenous cost model of Arkolakis (2008) when $\beta > 0$ and the fixed cost Chaney (2008) model when $\beta \to 0$.

Proposition 1 *Given A1-A3, and the values of g_κ, g_B given by the equations (22) and (21) respectively there exists a balanced growth path for the economy.*

Proof. *By assumption we have that $L_{it} = L_i e^{g_\eta t}$ and $J_{it} = J_i e^{g_\eta (1-\alpha) t}$, and $\bar{z}_{it} = \bar{z}_i \exp(g_E t)$. Define $z^*_{ijt} = z^*_{ij} e^{g_E t}$, such that $z^*_{ij} > \bar{z}_i$, $w_{it} = w_i e^{g_\kappa t}$, $C_{it} = C_i e^{g_\kappa t}$, $P_{it} = P_i$. Given these assumptions and definitions, the cross-sectional distribution of the productivities of operating firms is Pareto. For each cross section of the model, the share of profits in total income equals $\bar{\pi} = (\sigma - 1) / (\sigma \theta_2)$ (see Arkolakis (2008)) and the market share of i to j equals to*

$$\lambda_{ij} = (\tau_{ij})^{-\theta_2} J_i (\bar{z}_i)^{\theta_2} w_i^{(1-\gamma)\left(1-\frac{\theta_2}{\sigma-1}\right)-\theta_2} / \left[\sum_{\upsilon=1}^{N} (\tau_{\upsilon j})^{-\theta_2} J_\upsilon (\bar{z}_\upsilon)^{\theta_2} w_\upsilon^{(1-\gamma)\left(1-\frac{\theta_2}{\sigma-1}\right)-\theta_2}\right]. \tag{23}$$

In turn, the equilibrium variables w_{it}, P_{it}, z^*_{ijt} are characterized by the trade balance condition $w_i L_i = \sum_\upsilon \lambda_{i\upsilon} w_\upsilon L_\upsilon$, $\forall i$, the price index given by (1), $\forall i$, and the productivity cutoff condition given by (7) for $\forall i, j$. Simply substituting the guessed values of the variables into these equilibrium equations reveals that the guess is correct since the equations hold for $\forall t$. It also allows to solve for the values of z^*_{ij}, w_i, P_i using the same equations. Finally, C_i, can be solved using the budget constraint completing the construction of the balanced growth path. ■

Moreover, although it is not necessary for the existence of a balanced growth path, I will, in general, restrict the analysis to a parameterization that will allow me to match the facts on firm growth rates as a function of firm size. This parameterization will imply that the productivity growth of firms is not too negative, so that there is positive growth, on average, in the extensive margin of consumers for the smaller firms.

A 4 *: Productivity and sales parameters satisfy $\mu(\sigma - 1) + (\sigma - 1)^2 \sigma_z^2 > 0$.*

This restriction will hold true in the calibration.

4 Theoretical Predictions of the Model

I will now proceed to describe the theoretical properties of the model based on predictions that can be characterized analytically. I also draw the connection of the main properties of the model and the stylized facts on firm turnover and growth described in section 2. The quantitative performance of the model is assessed in the next section after calibrating the model's parameters.

4.1 Entry and Exit of Firms

To facilitate exposition I will define some additional notation. Aside from the fact that there is exogenous death of ideas, the productivity of an idea can be considered at a given time $\tilde{t}$ as a new process starting from current productivity $z_{\tilde{t}}$. For convenience, I define a proxy of the relative "size" of an idea from a given origin i to a given destination j when a years have elapsed from some reference time $\tilde{t}$ as,

$$s_{ija} \equiv \ln \frac{z_{\tilde{t}+a}}{z^*_{ij\tilde{t}+a}} \ , \ a \geq 0.$$

Notice that given the expression for sales, equation (8), the variable s_{ij0} and the aggregate variables summarize current firm behavior in market j. In particular, if $s_{ij0} < 0$ the firm does not currently sell in market j. s_{ija} follows a Brownian motion with initial condition s_{ij0}, drift μ, and standard deviation σ_z.

Given that the Brownian motion is a continuous time Markov process, the larger firms in a given destination will have higher probability of selling in this market next period, in line with the evidence. Since re-entry is possible, the survival function is simply defined as the probability of selling in market j after a years conditional on initial size in the market, s_{ij0}. The expression, derived in appendix A.3.1, is given by $S_{ij}(a|s_{ij0}) = e^{-\delta a}\Phi\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)$ and is increasing in initial size. Lemma 2 characterizes the hazard rates of the survival function for a firm with productivity s_{ij0}.

Lemma 2 *Given A1-A3, the (instantaneous) hazard rate of survival for a firm of a given size s_{ij0} in market j after time a has elapsed is given by*

$$-\frac{DS_{ij}(a|s_{ij0})}{S_{ij}(a|s_{ij0})} = \delta + m\left(-\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)\frac{s_{ij0}-\mu a}{2a\sigma_z\sqrt{a}} \ , \tag{24}$$

where $m(x) = \varphi(x)/\Phi(-x)$ is the inverse Mills ratio, with $\varphi(x)$, $\Phi(x)$ are the pdf and the cdf *of the standard normal distribution. If $\mu < 0$, the long run hazard rate converges to*

$$-\lim_{a\to\infty} \frac{DS_{ij}(a|s_{ij0})}{S_{ij}(a|s_{ij0})} = \delta + (\mu/\sigma_z)^2/2.$$

Proof. See appendix A.4.1 ■

As $a \to 0$ the hazard rate is arbitrarily close to 0 for any $s_{ij0} > 0$. In the case where $\mu < 0$ and $s_{ij0} > 0$, the firm hazard rate is not monotonic in age. Initially this ratio increases with age due to the negative drift but eventually selection makes this rate to decline. Thus, the model is consistent with the stylized fact that smaller firms have higher probability of exit but conditional on survival eventually the hazard rate declines.[18] If the drift, μ, is positive, the probability of a firm selling in the market eventually increases which leads to a negative hazard rate.

The model also delivers an analytical characterization of the survival rates of a given cohort.

Lemma 3 *Given A1-A3, the fraction of firms from a given cohort selling from i to j at time 0 and continues to sell after a years is given by*

$$S_{ij}(a) = e^{-\delta a}\left[\Phi\left(\frac{\mu}{\sigma_z}\sqrt{a}\right) + e^{a\left(\frac{\sigma_z^2}{2}\theta_2^2+\mu\theta_2\right)}\Phi\left(-\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}\right)\right]. \tag{25}$$

Additionally, if $\mu < 0$, the fraction of "surviving" firms in a market is strictly decreasing in the cohort age a.

Proof. See appendix A.3.2. ■

An interesting fact is that the cohort survival function $S_{ij}(a)$ is the same independent of the destination j, the origin i, and the value of β. The expression depends on the dispersion of the distribution governed by θ_2. This parameter determines the number of firms whose productivity is close to the threshold of exit, z^*_{ijt}, at time t and thus the number of firms that will survive in the future. Finally, the long run hazard rate converges to $\delta + (\mu/\sigma_z)^2/2$ as is proved in appendix A.3.2. Although, I use a different stochastic process for the entry and exit of firms, the predictions of this model for entry and exit are consistent with those of Luttmer (2007). In

[18] Given that new entrants start small, the model is likely to match the declining hazard rate of new exporters. A model with sunk costs fails to deliver this fact as pointed out by Ruhl and Willis (2008).

addition, the new setup allows to study firm entry and exit into multiple destinations.[19] Most importantly, the model delivers analytical expressions for expected growth of firms described in the next section.

4.2 Firm Growth

Geometric Brownian motion implies that the growth rates for firm-level productivity are independent of size, i.e. Gibrat's law. If there was no other factors that affect firm growth the assumptions of constant returns to scale production and the CES demand specification (constant price elasticity) would imply identical expected growth rates of sales across all incumbent firms. However, two distinct forces act so that Gibrat's law does not hold for all firms in the model: the selection effects and the market penetration technology.

The expected size of a firm conditional on survival, for any marketing technology convexity $\beta \in [0, +\infty)$, is derived in the following lemma.

Lemma 4 *Given A1-A3, the expected sales of a firm with initial size s_0 in market j at time $\tilde{t}$ after a years are given by*

$$E\left(r_{ij\tilde{t}+a} | s_{ij0} = s_0\right) = \frac{L^{\alpha}_{j\tilde{t}+a} y^{\gamma}_{j\tilde{t}+a} y^{1-\gamma}_{i\tilde{t}+a}}{\tilde{\psi}} \sum_{i=1}^{2} (-1)^{i+1} e^{\bar{g}_i a + \bar{c}_i s_0} \frac{\Phi\left(\frac{s_0 + \mu a + \bar{c}_i \sigma_z^2 a}{\sigma_z \sqrt{a}}\right)}{\Phi\left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right)} \tag{26}$$

where $\bar{g}_i = \bar{c}_i \mu + \frac{(\bar{c}_i)^2 \sigma_z^2}{2}$, $i = 1, 2$.

Proof. See appendix A.5. ■

The term $\bar{g}_1$ is the growth rate of the intensive margin of sales per consumer and $\bar{g}_2$ is the adjustment due to the growth rate of the extensive margin of consumers. If $\bar{g}_1 > \bar{g}_2$, $\bar{g}_1$ is the main determinant of the growth of sales as $s_0 \to \infty$. Both terms include the drift and the variance of the stochastic process of productivities. The variance of the stochastic process affects the sales of the firm due to the curvature of sales as a function of uncertainty, and Jensen's

[19] Irarrazabal and Opromolla (2006) consider a two country version of the Luttmer (2007) model, and assume that entry into the foreign market requires a sunk cost of entry. This assumption implies that the average size of firms that exit is smaller than the one of entrants. In the calibrated model, indivisibilities in the marketing cost would imply that all exporters are of large size in the destination they sell. The data studied by Eaton, Eslava, Kugler, and Tybout (2008) show that the average size of entrants and exitors into individual markets is almost the same and typically very small.

inequality. The ratio of the cdfs in equation (26) captures the effects of selection in each of the two margins. Using equation (26) as a starting point, I will isolate the forces of selection and market penetration technology in order to discuss their effect into firm growth.

4.2.1 Firm selection and firm growth

I examine the effects of selection on the growth rate of firms. To do so, I study the case where $\beta \to 0$, which implies that $\bar{c}_2 \to 0$ and that the second term of $E\left(r_{ij\tilde{t}+a}|r_{ij\tilde{t}}\right)$ disappears. By considering this limit, I eliminate the effects of market penetration costs on growth since marketing convexity does not play a role. The growth rate of a firm over the period of a years in a market is $G_{ija} = \left(r_{ij\tilde{t}+a} - r_{ij\tilde{t}}\right)/r_{ij\tilde{t}}$ and therefore the expected growth rate of a firm of initial size $s_{ij0} = s_0$, conditional on survival, is given by:

$$E\left(G_{ija}|s_{ija} \geq 0, s_{ij0} = s_0\right) = \underbrace{e^{(\alpha g_\eta + g_\kappa + \bar{g}_1)a}}_{\text{intensive margin growth}} \underbrace{\frac{\Phi\left(\frac{s_0+\mu a+\bar{c}_1\sigma_z^2 a}{\sigma_z\sqrt{a}}\right)}{\Phi\left(\frac{s_0+\mu a}{\sigma_z\sqrt{a}}\right)}}_{\text{selection effect}} -1\ . \tag{27}$$

The intensive margin term is the same for all firms in a given destination and depends on growth of aggregate variables and the growth rate of sales due to the drift. The normal distribution of the growth rates implies that the selection term is of the form $\Phi\left(x+\tilde{c}\right)/\Phi\left(x\right) > 1$, $\tilde{c} > 0$, is decreasing in x, while for $x \to \infty$ it converges to 1 (see appendix A.1 property F6). Even in the case where $\beta \to 0$, if a small firm survives it means that it has grown relatively fast due to selection. Since selection plays a small role for initially large firms, their expected growth rate is roughly independent of size, delivering Gibrat's law. The selection effect is reflected in expression (27) which has a selection term and a term common to all firms, as in Klette and Kortum (2004). Finally, notice that the sign of the growth rate depends on the relative severity of the effects of the intensive margin and the selection effect. If the intensive margin growth is negative (due to intense competition from new ideas) it might turn out that even the smallest firms have a negative growth rate.

In order to be able to crisply identify the effects of selection on both the mean and the variance of firm growth, I will examine the changes in the mean and the variance of the natural logarithm of sales. [20] If $\beta \to 0$, the moments of the logarithm of sales function can be obtained using the moment generating function, as shown in appendix A.5. In this case, I define the growth over the period of a years as $\hat{G}_{ija} = \log r_{ij\tilde{t}+a}(s_{ija}) - \log r_{ij\tilde{t}}(s_{ij0})$. The expected firm growth given initial size is[21]

$$E\left(\hat{G}_{ija}|s_{ija} > 0, s_{ij0} = s_0\right) = (\alpha g_\eta + g_\kappa)\, a + (\sigma - 1)\,\mu a + (\sigma - 1)\,\sigma_z \sqrt{a} m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) . \quad (28)$$

The third term of this expression is decreasing in size, s_0, and converges to 0 for large s_0 (see appendix 1, property F4).

The variance of firm growth given initial size is

$$\mathcal{V}\left(\hat{G}_{ija}|s_{ija} > 0, s_{ij0} = s_0\right) = (\sigma - 1)^2 \sigma_z^2 a \left\{1 - m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right)\left[m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) + \frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right]\right\} .$$

The term in the brackets incorporates the effects of selection and can be shown that it is increasing in its argument, $\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}$. In turn, $\mathcal{V}$ is increasing in s_0.[22] In fact, as $s_0 \to \infty$, it is $\mathcal{V}\left(\hat{G}_{ija}|s_{ija} > 0, s_{ij0} = s_0\right) \to \sigma_z^2 a (\sigma - 1)^2$. Straightforward intuition implies that given that the normal distribution of growth rates is unimodal, censoring of the negative growth rates will reduce the variance of firm growth rates. For $\beta \to 0$, the fact that selection implies that the variance of firm growth rates is increasing in size is in sharp contrast to the empirical findings (see section 2.2).[23]

[20] The use of this statistic by applied economists is quite common in the literature. For example, Hall (1987) and Evans (1987b) empirically study the effects of selection on expected growth and variance of (primarily) employment as well as sales of firms.

[21] The correction for the selection bias is different from the specification of Heckman (1979) in that entry and sales decision are perfectly correlated in my case (both driven by productivity shocks). Partial correlation can be generated, for example, if there exists randomness in a term that would influence entry but is not perfectly correlated to sales. The obvious candidate term in this model is the parameter $1/\psi$ in the costs of entry. In such an event the second term of equation (28) would depend on the covariance of the uncertainty of the shocks to entry and sales. The econometric techiques developed to adjust for selection bias by Heckman (1979) could be appropriate for this case. Such an approach has been used by Evans (1987b).

[22] The proof can be found in Sampford (1953). More generally, the result that the left truncated variance is decreasing in the truncation point (and thus is increasing in the size of the firm) holds for all distributions with logconcave pdf (see An (1998)). This set of distributions includes the normal. Logconcavity implies unimodality of the distribution but not the other way. The sales of the firm r_{ijt} are lognormally distributed but the log normal distribution is neither logconcave nor logconvex in its entire domain. Thus, an analytical relationship for the variance of growth G_t can be obtained its variance is not in general monotonic in size.

[23] In the Klette and Kortum (2004) model, the variance unconditional on survival is inversely proportional to firm size. The decrease in the variance with firm size happens since the sales of the firm are proportional to the number of goods that the firm has. Since each good has the same variance, the total variance of firm sales is inversely proportional to firm size in that model.

4.2.2 Market Penetration Technology and Firm Growth

I now turn to study the effects of different specifications in the market penetration technology on the growth of firms. In the endogenous cost case, $\beta > 0$, the particular shape of the market penetration technology combined with the assumed stochastic process of firm productivity yield important departures from Gibrat's law for the sales of smaller firms, even after correcting for bias caused by selection. The effects of market penetration costs can be studied independently from the selection effect by focusing on the instantaneous growth rate of the firm, given that for small changes in time incumbent firms do not exit. This analysis can be performed by applying Ito's lemma to expression (8) for $s_{ija} = \ln z/z^*_{ij\tilde{t}+a} > 0$.[24] For firms with initial size $s_{ij0} > 0$, Ito's lemma implies

$$\frac{dr_{ij\tilde{t}}\left(s_{ij0}\right)}{r_{ij\tilde{t}}\left(s_{ij0}\right)} = \left[\alpha g_\eta + g_\kappa + \mu \frac{h'\left(s_{ij0}\right)}{h\left(s_{ij0}\right)} + \frac{1}{2}\sigma_z^2 \frac{h''\left(s_{ij0}\right)}{h\left(s_{ij0}\right)}\right] da + \left[\sigma_z \frac{h'\left(s_{ij0}\right)}{h\left(s_{ij0}\right)}\right] dW \qquad (29)$$

where

$$h\left(s_{ij0}\right) = e^{\bar{c}_1 s_{ij0}} - e^{\bar{c}_2 s_{ij0}}$$

In equation (29) the first and second parenthetical terms represent the (instantaneous) growth, $E\left(dr/r\right)$, and the standard deviation of growth of a firm of size s_{ij0} respectively.[25] Proposition 5 characterizes the relationship between the instantaneous growth rates of firms of size $s_{ij0} = s_0$ in a given destination for different values of β:

Proposition 5 *Given A1-A4,*

a) When $\beta \to 0$ the growth rate of all firms is the same.

b) There exist a $\beta' \in (0, +\infty)$, such that $\forall\ \beta > \beta'$, it is $\partial\left(E\left(dr/r\right)\right)/\partial s_0 < 0$, and $\forall \beta < \beta'$, $\partial\left(E\left(dr/r\right)\right)/\partial s_0 > 0$ for all firms with $s_0 > 0$. The instantaneous expected growth rate of the largest firms approaches $\alpha g_\eta + g_\kappa + \mu\left(\sigma - 1\right) + \frac{\sigma_z^2}{2}\left(\sigma - 1\right)^2$.

Proof. *The proof is somewhat instructive for the effects of market penetration technology on growth. To prove part (a) of proposition 5 notice that with the use of De l' Hospital rule the*

[24] Since the Brownian motion paths are not differentiable with probability 1 and exhibit infinite variation for any given time interval standard calculus does not apply. The application of Ito's Lemma requires the sales function to have a continuous second derivative (see for example Oksendal (2003), ch. 4). Although the function $h\left(s\right)$ is not continuous it does not attain continuous derivatives at $s = 0$.

[25] See Dixit and Pindyck (1994) chapter 3 for the details of Ito's lemma and related derivations.

terms in expression (29) for $s_{ij0} = s_0$,

$$\frac{h'(s_0)}{h(s_0)} \overset{\beta\to 0}{\longrightarrow} (\sigma - 1) \ , \tag{30}$$

$$\frac{h''(s_0)}{h(s_0)} \overset{\beta\to 0}{\longrightarrow} (\sigma - 1)^2 \, . \tag{31}$$

To prove part (b) I look at the derivative of the first parenthetical term in expression (29) with respect to s_0. In appendix A.6, I show that the sign of this derivative is negative if and only if

$$\beta \geq \frac{(\sigma-1)^2 \sigma_z^2}{2\left[\mu(\sigma-1) + (\sigma-1)^2 \sigma_z^2\right]} > 0 \ . \tag{32}$$

Thus, if A4 is not satisfied there no value of β for which the growth rates are decreasing in size. The growth rate of the largest firms can be found by considering the following two limits:

$$\lim_{s_0 \to +\infty} h'/h \longrightarrow (\sigma - 1) \ , \tag{33}$$

$$\lim_{s_0 \to +\infty} h''/h \longrightarrow (\sigma - 1)^2 \ . \tag{34}$$

■

Deviations from Gibrat's law, after accounting for selection, is not a common implication of models of firm dynamics. This model takes a step further in that the growth rate of firms is market specific since it depends on their relative size in each market. The model with endogenous market penetration costs also delivers an inverse relationship between the sales of firms in a market and the instantaneous variance of their growth rates for that market.[26] This argument is summarized in proposition 6.

Proposition 6 *a) If $\beta \to 0$, the instantaneous variance of the growth rate of sales of firms in a destination is independent of their initial sales.*

b) If $\beta > 0$, the instantaneous variance of the growth rate of sales of firms in a destination is

[26] Recent explanations that can generate this behavior include the availability of different technologies for small firms, and financial restrictions, suggested by Luttmer (2008) and Cooley and Quadrini (2001) respectively. The novelty of my approach is to suggest a demand based explanation with predictions that apply to each market. Still, the tractability and generality of the predictions of this demand-based model go beyond previous models in various dimensions that I explore.

higher the smaller their initial sales. The variance of the growth rate of the largest firms tends to $\sigma_z^2 (\sigma - 1)^2$.

Proof. *See appendix A.7.* ■

Thus, the specification of the model with $\beta > 0$ brings the predictions of the theory closer to the observed data in terms of firm dynamics. In addition, the theory suggests a precise functional form for the growth-size relationship and the correction for selection. As Evans (1987b) points out, mispecification of the growth-size relationship, heteroscedasticity, and sample selection are interrelated. In particular, a bad approximation to the functional form of growth as a function of size can manifest itself as heteroscedasticity. Wrong specifications for sample selection can appear as nonlinearity in the growth equation. While econometric techniques have been developed to address these issues future econometric investigations of the size growth relationship can benefit by exploiting the structure suggested by this paper. In the next subsection, I discuss the model predictions in terms of the cross-sectional distribution of firms.

4.3 The size distribution of sales to individual destinations

The model is able to generate the two robust findings for the distribution of sales of firms summarized in section 2.3. In particular, the model can generate the Pareto tails in the distribution of sales for the largest exporters in each country as a result of the assumptions of the Pareto tails in the distribution of productivities and the CES demand. However, the sales of the smallest firms deviate from the Pareto distribution only if $\beta > 0$.[27] When someone looks at a single cross-section of firms, rather than comparing sales of firms overtime, selection into a market does not influence the shape of the distribution. Therefore, at the heart of this result are the departures that the model predicts from Gibrat's law for the small firms in each destination due to the market penetration technology.

[27] See Arkolakis (2008). Generalizing the framework of this model in order to allow the cross-sectional distribution of productivities of operating firms to be double-Pareto, rather than Pareto, will give additional testable cross-sectional implications. It implies a skewness of the distribution of operating productivities, particularly for ideas with low productivities. The assumption of firm entry with a certain non-atomic distribution has similar effects. Both assumptions will result to a skewness of the distribution of productivities that is higher for destinations with more firms selling there. This implication is consistent with the facts reported by Eaton, Kortum, and Kramarz (2008) but cannot explain the skewness (combined with the robust shape) of the distribution of sales at each individual destination.

The implications of such a departure reflect the intuition laid out in the classic contribution of Simon and Bonini (1958):

"If we take the stochastic model seriously, then any substantial deviation of the [empirical] results from those predicted from the model is a reflection of some departure from the law of proportionate effect or from one of the other assumptions of the model. Having observed such a departure, we can then try to provide for it a reasonable economic interpretation."

In other words a unified theory of firm selection and growth can allow us to compare both the dynamic and cross-sectional predictions of the model. Thus, we can test underlying economic mechanisms by identifying when different model assumptions do well and when they do not. The next step is to investigate the predictions of such a theory quantitatively.

5 Calibration

The goal of this section is to determine the parameters of the model using as much information as possible on the domestic sales of firms and macroeconomic aggregates. I calibrate the process of productivity improvements without requiring information on the growth of firms sales. This calibration is more appropriate for exploring the extent to which the model can capture the turnover and growth of firms in the domestic and the individual exporting markets in the absence of market specific demand shocks.[28]

As a rule, I choose parameters that affect the cross-section of country trade flows and firm sales using the results of the estimation of Eaton and Kortum (2002) and information from the French exporting dataset of Eaton, Kortum, and Kramarz (2008) respectively. The parameters that affect the balanced growth path are calibrated by looking at information from the US manufacturing census and US macroeconomic aggregates. US data are easily accessible for these statistics. To calibrate the stochastic process of firm productivities, additional information on firm exit rates is also taken from the US census. Table 1 provides a summary of the model parameterization that is described below in detail and the main sources used.

[28] In an appendix available online I show that the calibration is also appropriate when productivity (or demand) shocks are market specific but the process exhibits a similar mean and variance as the domestic productivity improvements.

Parameters	Value	Source
Cross Section		**Cross-sectional exporting data**
α	0.44	Arkolakis (2008)
γ	0.31	Arkolakis (2008)
θ_2	8.28	Eaton & Kortum (2002)
σ	6.33	Sales advantage of prolific exporters in France:
β	1	Arkolakis (2008), Eaton, Kortum & Kramarz (2008)
Balanced Growth		**US macroeconomic aggregates**
g_η	0.0122	US population growth
g_κ	0.02	US GDP growth
g_E	0.0187	US GDP growth
Idiosyncratic Productivity		**US manufacturing Census data**
δ	0.0089	Death rate of firms with 500+ employees
g_I	0.0024	Exit rates of 1963 cohort from Dunne, Roberts
σ_z	0.0664	& Samuelson (1988) (& the value of θ_2)

Table 1: Model Parameterization

5.1 Parameters from the static model

For the calibration of the static parameters I follow Arkolakis (2008) given that each cross-section of the dynamic model is identical to that setup. In this paragraph I briefly describe his procedure. Parameters α, γ govern the firm entry as a function of income per capita and population of the market and are estimated using the dataset of French firms of Eaton, Kortum, and Kramarz (2008). Looking at the cross-section of countries, $\alpha = .44$ and $\gamma = .31$ closely predict the ratio of the number of French entrants to their market share in a given destination. The cross-sectional predictions of the model also allow me to calibrate the Pareto shape parameter, θ_2, the elasticity of substitution, σ, and the marketing convexity, β. In particular, the bilateral trade market shares are essentially the same as in Eaton and Kortum (2002) (equation (23)) and thus I use their estimate of $\theta_2 = 8.28$. Given this parameter, it is $\sigma = 6.33$ and $\beta = 1$ that allow the model to generate the size advantage in France of prolific exporters compared to firms that export little or not at all.[29] Essentially, the ability of firms to penetrate the markets is used as an "objective" measure of productivity advantage of exporters. Given θ_2, the parameters σ and β determine how the productivity advantage is translated to sales advantage in the domestic market.

[29] This value for the elasticity of substitution is in the ballpark of the estimates of Broda and Weinstein (2006).

5.2 Parameters governing dynamics

To determine the value of the parameters that govern the dynamics of the model I use census information and macroeconomic data for the US. The parameters g_η, g_κ, g_E govern primarily aggregate dynamics. The growth of the population from 1960 onwards in the US is around 1.22% and the growth rate of real GDP per capita is around 2%. Thus, I set $g_\eta = .0122$ and $g_\kappa = .02$. Given the definition of g_κ, the growth of the technological frontier of new ideas is

$$\begin{aligned} g_E &= g_\kappa - g_\eta \frac{(1-\alpha)}{(\sigma-1)} \\ &= 0.0187 \; . \end{aligned}$$

The parameters δ, g_I and σ_z that govern firm dynamics must also to be specified. In the model, δ regulates the exogenous death rate. Given that the probability of endogenous exit for firms with large size is (practically) 0, I calibrate δ looking at the death rate of these firms. This information is obtained by the US Manufacturing Census during the period 1996-2004, where the tabulation of the largest manufacturing firms is those with 500 or more employees. The data indicate an average exit rate of 0.89% per year for these firms and thus δ is set at 0.0089.[30]

The two remaining parameters that govern productivity and thus firm dynamics are g_I and σ_z. However, θ_2, which is an explicit function of these two parameters (equation (18)), was calibrated to the value of 8.28. Thus, to jointly calibrate g_I and σ_z I need one moment from the data which is a function of these two parameters in the model. I get this information from the data by looking at the cohort exit rates of US manufacturing firms as reported by Dunne, Roberts, and Samuelson (1988). I use the benchmark exit rate of 42% per year in the first 4 years for the first cohort analyzed by Dunne, Roberts, and Samuelson (1988). The results though are not affected by the choice of the cohort. In the model this moment is represented by equation (25). Using equations (25) and (18) together with the empirical values for the elasticity of trade and the cohort exit rates delivers $g_I = 0.24\%$ and $\sigma_z = 6.64\%$. This parameterization implies that $\mu = g_I - g_E = -1.63\%$ for the incumbent firms. Notice that μ, σ_z, and δ are present in equations (25) and (18) while $\sigma - 1$ and β do not affect these relationships.

[30] More details about the data and discussion on the parameterization are provided in appendix B and in an online appendix.

6 Quantitative Results

In the previous section, the model was calibrated so that it is consistent with a balanced growth of the economy. The cross-sectional predictions of the calibrated model and its ability to match cross-sectional facts have been extensively discussed by Arkolakis (2008) and Eaton, Kortum, and Kramarz (2008). In this section, I study how aggregate sales in a market are accounted by firm turnover and the evolution of incumbent and new entrants market shares.

6.1 Firm Turnover

In the rest of the analysis, I will define as a "cohort of firms," all the firms that were in the market at year 0. The survivors of that cohort at year t are the active firms at year 0 which also sell in the market at year t. "Entry cohorts" are the firms that enter in the market after year 0. Figure 2 illustrates the turnover of 4 cohorts of the US manufacturing census. I use statistics from Dunne, Roberts, and Samuelson (1988) to illustrate the fraction of active firms at year 0 as a percentage of all active firms at year t. This fraction is by construction 100% in year 0 of a cohort and declines as more firms exit the market and new firms enter. Figure 2 shows that firm behavior is roughly independent of what year we consider as year 0. It is so, due to the large number of firms that are included in each census cohort (more than 250,000). Two facts clearly emerge. First, there is a large turnover overall in the data. After 2 decades, only a fifth of the operating firms were also active at year 0. Second, almost 50% of the turnover takes place in the first 4-5 years. The predictions of the calibrated model are also illustrated in figure 2. Although the model is calibrated to match the exit of the first 4 years of the 1963 cohort, it can also predict the overall turnover of US firms over the two decades.

Using statistics provided by EEKT, figure 3 illustrates the fraction of active firms at year 0 as a percentage of all active firms at year t for Colombian firms that sell in (each of) 9 different destinations. This is a comprehensive dataset that covers all the export transactions of Colombian manufacturing firms for the years 1996-2007 (but not the domestic sales). In order to reduce the variance of these fractions I take their averages in each market over 10 different year 0 cohorts for their year 1, over 9 cohorts for their year 2 and so on. By doing so, I reduce the volatility that could be caused by some factors that are not included in the model such as a decrease in tariffs, exchange rate shocks etc. The two facts analyzed in the previous paragraph for the turnover of the firms in the US census are also present in these data. The difference is that the turnover is much more prevalent in the very first years. On average, around 1/3 of the firms, that sell in a market, were not selling in the same market the previous year. Overall, the model with idiosyncratic productivity improvements can account for up to 2/3 of the turnover of exporters to individual destinations in the first year. It also predicts that, over a decade, there is a turnover of 60% in exporting markets. This number is close to what is observed in the Colombian data.[31] Thus, when the model with productivity improvements is calibrated to match the early exit rates of US census firms, it explains a large part of the turnover of exporting firms to individual destinations. However, it cannot match their high turnover rates for the first years of a cohort. It calls for some factor that could explain this higher turnover and is specific to exporting. This factor could be, for example, higher volatility of foreign demand shocks or learning.[32]

6.2 Cohort Market Share

I now turn to study the contribution of surviving firms from a cohort versus the share of new entrants in explaining aggregate sales. In figure 4 I plot the sales of surviving firms from four US census cohorts after t years as a fraction of total sales of all firms producing after t years. These sales do decline over time but this decline is relatively small in the first years of the cohort. In the same figure, I illustrate the predictions of the model obtained from analytical relationships that are described in the appendix A.8. The model can match the small contribution of entering cohorts in their first years. This small contribution is an implication of the fact that sales of small firms are arbitrarily close to 0 for $\beta > 0$. As the theoretical analysis indicates, and in particular proposition 5, β is a also a crucial parameter for predictions on growth rates of firms and cohorts.[33]

Figure 5 illustrates the market share of active firms at year 0 as a percentage of the sales

[31] Irarrazabal and Opromolla (2009) complement the findings of this section by illustrating that a similar model with productivity advances is consistent with the intertemporal entry-exit patterns of exporters into individual destination as reported by EEKT.

[32] Productivity (or demand) shocks, imperfectly correlated shocks accross markets are consistent with the first possible explanation and are discussed in the online appendix. Learning has been recently considered into a model of international trade by Eaton, Eslava, Krizan, Kugler, and Tybout (2008) and Ruhl and Willis (2008).

[33] In an online appendix, I illustrate the predictions of the model with low β. The results there illustrate how low β's imply that the entering firms are too large and grow too slow compared to what is observed in the census data.

of all active firms at year t for Colombian firms that sell in (each of) 9 different destinations. The model (in general) overpredicts the market shares of exporting cohorts over time since it underpredicts exit, as described above. These data have much more variance than the census data because there is a much smaller number of firms in some destinations, and their sales are possibly affected by exchange rate devaluations, trade liberalizations etc. The analysis of these factors is beyond the scope of this paper. Instead, I attempt to understand the underlying forces that drive the predictions of the model.

6.3 Average Sales of Survivors and of New Entrants

This section discusses the predictions of the model regarding the average sales of surviving and new firms. Figure 6 shows the average sales of surviving firms from a cohort divided by the average sales of all firms in a market for two decades for the US census data. This ratio increases in a consistent way as the time elapses. The model predicts an increase in the average sales of surviving firms for three reasons: the growth in the intensive margin of sales, the effects of selection, and the effects of the marketing technology which arise only for high βs. Eventually, after 2 decades, the predictions of the model diverge from what is in the data. It is so because the sales of the surviving largest firms in a market depend ultimately in the intensive margin growth of firms which turns out to be of small magnitude in the calibrated model. Additionally, I look the average sales of entry cohorts versus the average sales of all firms (figure 7). New firms in a market start small while they grow fast as implied by proposition 5. The dynamic behavior of entrants explains their small initial market share compared to that of the existing firms as figure 4 shows.

6.4 Median Age

An outstanding question is whether the firm size is attained in a reasonable time. Luttmer (2008) argues that in a model that implies the Gibrat's law and is calibrated to match the right tail of the distribution and the exit rate of firms, growth is too slow. Given that the model delivers a good approximation of the distribution of firm sales and a relatively faster growth for the smallest firms in a market, I find that the transition to higher percentiles of firms happens at a reasonable time. In particular, I look at the median time that a firm at the bottom of the distribution needs to move at a certain percentile, conditional on ever reaching there. For the highest percentile that I could obtain relevant information from the US data this time is less than 70 years (this is the 99.9842 percentile for firms with 10,000+ employees. The information is available by the Small Business Administration, www.sba.gov).[34] This number is reasonable and is partly due to the negative drift, μ: a firm that does not reach a very large size early on is very unlikely to be of that size after a long time has elapsed.

7 Conclusion

This paper develops a simple unified framework for studying firm selection and growth. The framework is based on idiosyncratic firm productivity improvements and has the potential to be used in applied empirical work. It closely matches the key cross-sectional observations on firm domestic and exporting sales data, and is quantitatively consistent with some of the key observations related to the time dimension of the domestic or exporting sales of firms.

The underlying Total Factor Productivity (TFP) process of the firm and the modeling framework can be seen as a (highly abstract) benchmark for firm-level theory, with the same role as the TFP plays in the growth model. Future related research should try to explain what are the real determinants of this TFP process while being quantitatively consistent with the main predictions of the model. What the productivity dynamics model is silent about is age dependent turnover and growth of firms. A relationship of the age of the firm, conditional on size, with turnover and growth has been reported for the gross-sales of firms but has received only limited attention in economic modeling. Theoretical and empirical investigation of this relationship would complement but also extend the results of this paper. While a lot remains to be done, this paper has taken important steps towards the construction of a "unified" theory of firm dynamics consistent with the macroeconomic regularities.

[34] Analytical expressions and the proof for this result are available in the online appendix. The proof derives the distribution of the first hitting time of a Brownian motion as in Harrison (1985), p. 14.

A Model Appendix

A.1 Preliminary definitions and facts

In the various proofs and derivations of this appendix I am going to use the following definitions and well known facts for the Normal distribution quoted as **properties F**.

F 1 *The simple normal distribution with mean 0 and variance 1 is given by* $\varphi(x) = \frac{e^{-x^2}/2}{\sqrt{2\pi}}$.

F 2 *The cdf of the normal is given by* $\Phi\left(\frac{x-\mu}{\sigma_z}\right) = \frac{1}{\sigma_z\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left\{-\frac{(\tilde{x}-\mu)^2}{2\sigma_z^2}\right\}d\tilde{x}$. *Using change of variables* $\upsilon = (\tilde{x}-\mu)/\sigma_z$ *which implies* $d\upsilon = d\tilde{x}/\sigma_z$ *it is also true that*

$$\Phi\left(\frac{x-\mu}{\sigma_z}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma_z}} e^{-\frac{(\tilde{x})^2}{2}}d\tilde{x}$$

F 3 *Because of the symmetry of the normal distribution,* $\varphi(x) = \varphi(-x)$ *and* $\Phi(x) = 1-\Phi(-x)$.

F 4 *The inverse mill's ratio of the Normal,* $\varphi(x)/\Phi(-x)$, *is increasing in* x, $\forall x \in (-\infty, +\infty)$.

F 5 $\varphi(x)/\Phi(-x)/x$ *is decreasing in* x, $\forall x \in (0, +\infty)$ *with* $\lim_{x\to\infty}\varphi(x)/\Phi(-x)/x = 1$. *This implies that* $\varphi(x)/(1-\Phi(x)) > x$ *for* $\forall x \in (-\infty, +\infty)$

F 6 $\Phi(x+\tilde{c})/\Phi(x)$, *with* $\tilde{c} > 0$, *is decreasing in* x, $\forall x \in (-\infty, +\infty)$.

F 7 *The error function is defined by:* $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-(\tilde{x})^2}d\tilde{x}$.

F 8 $\Phi(x) = \frac{1}{2}\left[1+\text{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$, *where* $\Phi(x)$ *is the cdf of the standard normal cdf*

F 9 *The error function is odd:* $\text{erf}(-x) = -\text{erf}(x)$. *Also* $\lim_{x\to+\infty}\text{erf}(x) = 1$.

F 10 $\int e^{-\tilde{c}_1x^2+\tilde{c}_2x}dx = e^{(\tilde{c}_2)^2/4(\tilde{c}_1)}\sqrt{\pi}\,\text{erf}\left(\frac{2\tilde{c}_1x-\tilde{c}_2}{2\sqrt{\tilde{c}_1}}\right)/\left(2\sqrt{\tilde{c}_1}\right)$, *for some constants* $\tilde{c}_1$, $\tilde{c}_2 > 0$

A.2 Deriving the Stationary Distribution of Productivities

A simple guess for the solution of the Kolmogorov equation (11) is $f(s) = A_1e^{\theta_1 s} + A_2e^{-\theta_2 s}$ where θ_1 and $-\theta_2$ are given by the two solutions of the quadratic equation $\frac{1}{2}\sigma_z^2\theta_i^2 - (g_I - g_E)\theta_i - g_\eta(1-\alpha) = 0$, where $i = 1, 2$. Using condition (12) set $A_2 = 0$ for $s < \bar{s}_i$ and using the requirement that $f(s)$ is a probability density set $A_1 = 0$ for $s \geq \bar{s}_i$.

Finally, from the characterization of the flows at the entry point (15), I pick A_1, A_2 such that

$$\frac{1}{2}\sigma_z^2\left(A_1\theta_1 e^{\theta_1\bar{s}_i} + A_2\theta_2 e^{-\theta_2\bar{s}_i}\right) = g_\eta\left(1-\alpha\right) \ ,$$

which in combination with (14) that gives

$$\int_{-\infty}^{\bar{z}_i'} A_1 e^{\theta_1 s} ds + \int_{\bar{z}_i'}^{+\infty} A_2 e^{-\theta_2 s} ds = 1 \ ,$$

imply that

$$A_1 = \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{-\theta_1\bar{s}_i} \ , \ A_2 = \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{\theta_2\bar{s}_i} \ .$$

Notice, that the solutions also satisfy the first term in the LHS of (15) since the above solutions imply that $f\left(\bar{s}_i-\right) = f\left(\bar{s}_i+\right)$. In other words the distribution is continuous, but the derivative has a kink at $\bar{s}_i$.

A.3 Survival Rates

A.3.1 Firm Survival in a market

The objective is to compute the probability a firm will be selling in a market after a years (so that $s_{ija} \geq 0$), conditional on the initial productivity of the firm today, $s_{ij0} = s_0$. I denote this probability by $S\left(a|s_0\right)$ and thus, using expression (10),

$$S\left(a|s_0\right) = \Pr\left(s_{ija} > 0|s_{ij0} = s_0\right) = e^{-\delta a}\int_0^{+\infty} \frac{e^{-\frac{\left(\frac{s_a - s_0 - \mu a}{\sigma_z\sqrt{a}}\right)^2}{2}}}{\sigma_z\sqrt{a2\pi}} ds_a$$

which using change of variables implies:

$$S\left(a|s_0\right) = e^{-\delta a}\Phi\left(\frac{s_0+\mu a}{\sigma_z\sqrt{a}}\right) \ . \qquad (35)$$

A.3.2 Cohort Survival Rates

The expression to be derived is the probability that a firm which is currently operating, $s_{ij0} \geq 0$, also operates after time a has elapsed, $s_{ija} \geq 0$. If I denote this probability a $\Pr\left(s_{ija} \geq 0|s_{ij0} \geq 0\right)$, then taking account random death the cohort survival rate is $S_{ij}\left(a\right) =$

$e^{-\delta a}\Pr\left(s_{ija}\geq 0|s_{ij0}\geq 0\right)$. I first derive the probability,

$$\Pr\left(s_{ija}\geq 0|s_{ij0}\geq 0\right)=\int_{0}^{+\infty}\int_{0}^{+\infty}\Pr\left(s_{ija}=s_{a}|s_{ij0}=s_{0}\right)\frac{\Pr\left(s_{ij0}=s_{0}\right)}{\Pr\left(s_{ij0}\geq 0\right)}ds_{a}ds_{0}$$

$$=\int_{0}^{+\infty}\frac{\Pr\left(s_{ij0}=s_{0}\right)}{\Pr\left(s_{ij0}\geq 0\right)}\int_{0}^{+\infty}\Pr\left(s_{ija}=s_{a}|s_{ij0}=s_{0}\right)ds_{a}ds_{0} \tag{36}$$

Using equation (16) the conditional density of productivities is given by

$$\frac{\Pr\left(s_{ij0}=s_{0}\right)}{\Pr\left(s_{ij0}\geq 0\right)}=\theta_{2}e^{-\theta_{2}(s_{0}-0)}\ . \tag{37}$$

The inner integral of expression (36) is given by equation (35). Thus, by replacing expressions (35), (37) in (36) and using integration by parts,

$$\Pr\left(s_{ija}\geq 0|s_{ij0}\geq 0\right)=\Phi\left(\frac{\mu\sqrt{a}}{\sigma_{z}}\right)+\int_{0}^{+\infty}e^{-\theta_{2}s_{0}}\frac{1}{\sigma_{z}\sqrt{a}}\varphi\left(\frac{s_{0}+\mu a}{\sigma_{z}\sqrt{a}}\right)ds_{0}\ .$$

Using property F10 the integral of the last expression equals to

$$\frac{e^{-\frac{1}{2}\frac{\mu^{2}}{\sigma_{z}^{2}}a}}{\sigma_{z}\sqrt{a2\pi}}\left|e^{\frac{\left(\frac{\mu}{\sigma_{z}^{2}}+\theta_{2}\right)^{2}}{4\frac{1}{2\sigma_{z}^{2}a}}}\frac{\sqrt{\pi}}{\sqrt{2\frac{1}{2\sigma_{z}^{2}a}}}\operatorname{erf}\left(\frac{2\frac{1}{2\sigma_{z}^{2}a}x+\frac{\mu}{\sigma_{z}^{2}}+\theta_{2}}{2\sqrt{\frac{1}{2\sigma_{z}^{2}a}}}\right)\right|_{x=0}^{x=+\infty}$$

$$=e^{\frac{\sigma_{z}^{2}a}{2}(\theta_{2})^{2}+\theta_{2}\mu a}\Phi\left(-\left(\frac{\mu}{\sigma_{z}}+\theta_{2}\sigma_{z}\right)\sqrt{a}\right)\ .$$

Combining the expressions with the random death term gives the survival function, $S_{ij}\left(a\right)$, in lemma 3.

In the online appendix I show that $S_{ij}\left(a\right)$ is increasing in μ, and if $\mu<0$, $S_{ij}\left(a\right)$ is decreasing in a, $DS_{ij}\left(a\right)<0$. The results are applications of the properties of the normal distribution.

A.4 Hazard rates

A.4.1 Firm Hazard Rates

From expression (35) I can compute the instantaneous (conditional) hazard rate which is defined as the rate of change of the survivor function, $-DS_{ij}\left(a|s_{ij0}\right)/S_{ij}\left(a|s_{ij0}\right)$. Simple substitution for the definition of $S_{ij}(a|s_{ij0})$, equation (35), gives expression (24). Notice that the limit of the expression (24) $s_{ij0} \to \infty$ (for a given a), is given by

$$\lim_{s_{ij0}\to\infty}\left[-\frac{DS_{ij}(a|s_{ij0})}{S_{ij}(a|s_{ij0})}\right] = \delta + \frac{1}{2\sigma_z\sqrt{a}a}\frac{\lim_{s_{ij0}\to\infty}\varphi\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)(s_{ij0}-\mu a)}{\lim_{s_{ij0}\to\infty}\Phi\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)} = \delta + 0\ ,$$

which implies that the instantaneous hazard rate for large firms is only the exogenous death rate of ideas.[35]

I will also consider two limits of the hazard rate $a \to \infty$ and $a \to 0$, for $\mu < 0$. Since the first term of equation (24) is always δ I will derive what happens to the second term in these two cases. First, consider the limit of the second term of the hazard function for $a \to \infty$. This limit is 0/0 and thus applying De l' Hospital:

$$\lim_{a\to+\infty}\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)^2}\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\left(\frac{s_{ij0}-\mu a}{a2\sigma_z\sqrt{a}}\right)^2+\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)^2}\frac{a\mu-3s_{ij0}}{4a^{\frac{5}{2}}\sigma_z}}{\frac{-s_{ij0}+\mu a}{a2\sigma_z\sqrt{a}}\varphi\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)} =$$

$$\lim_{a\to+\infty}\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\frac{-s_{ij0}+\mu a}{a2\sigma_z\sqrt{a}} + \lim_{a\to+\infty}\frac{a\mu-3s_{ij0}}{(-s_{ij0}+\mu a)\,4a\sigma_z} = \frac{1}{2}\left(\frac{\mu}{\sigma_z}\right)^2 + 0$$

This derivation gives the result of lemma 2.

Also it is easy to derive that the hazard rate goes to zero when $a \to 0$, i.e. derive the hazard rate for very small ages: Notice that $\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}$ always declines with age and thus also $\Phi\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)$. Thus,

$$-\lim_{a\to 0}\frac{\varphi\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)\frac{-s_{ij0}+\mu a}{a2\sigma_z\sqrt{a}}}{\Phi\left(\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}\right)} = \frac{0}{1} = 0$$

Finally, remains to show that in the case where $\mu < 0$, $s_{ij0} > 0$, the firm hazard in expression (24) ends up not being monotonic in age a. Simply notice that F4 implies that the term $m(\cdot)$ is decreasing in its argument, $\frac{s_{ij0}+\mu a}{\sigma_z\sqrt{a}}$, where the latter is initially decreasing and eventually increasing with age. The term $\frac{s_{ij0}-\mu a}{a2\sigma_z\sqrt{a}}$ is always decreasing in age.

[35] In this result and other results of the appendix I use the fact that exponential growth is faster than polynomial growth without further discussion.

A.4.2 Cohort Hazard Rates

In the online appendix I show that the cohort hazard rate is given by:

$$-\frac{DS_{ij}(a)}{S_{ij}(a)} = \delta + \frac{\frac{\theta_2\sigma_z \frac{\mu+\theta_2\sigma_z^2}{\sigma_z}}{2\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}} \frac{\varphi\left(\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}\right)}{\Phi\left(-\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}\right)} - \frac{\sigma_z^2}{2}\theta_2^2 - \mu\theta_2}{1+\exp\left\{-a\left(\frac{\sigma_z^2}{2}\theta_2^2+\mu\theta_2\right)\right\}\frac{\Phi\left(\frac{\mu\sqrt{a}}{\sigma_z}\right)}{\Phi\left(-\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}\right)}} \tag{38}$$

The limit $\frac{\varphi(x)}{\Phi(-x)x}$ as $x \to \infty$ is 1 (see F5) so that the numerator in the expression is given by $-\mu\theta_2/2$. If $\mu < 0$, I have to use De l' Hospital to compute the same limit for the denominator. It is

$$1 + \frac{\mu^2 + \sigma_z^2\theta_2^2\sigma_z^2 + 2\mu\theta_2\sigma_z^2}{-\left(\mu+\theta_2\sigma_z^2\right)\mu} = \frac{\sigma_z^2\theta_2^2\sigma_z^2 + \mu\theta_2\sigma_z^2}{-\left(\mu+\theta_2\sigma_z^2\right)\mu} = -\frac{\theta_2}{\mu}\sigma_z^2$$

and this implies the result stated in the paper:

$$-\lim_{a\to\infty}\frac{DS_{ij}(a)}{S_{ij}(a)} = \delta + \frac{1}{2}\left(\frac{\mu}{\sigma_z}\right)^2$$

To prove monotonicity of the hazard rate as a function of a, if $\mu < 0$, notice that by property F5 the numerator of expression (38) decreases in time. For the denominator we simply have to prove that it increases. The derivative of the denominator wrt to a:

$$\frac{e^{-a\left(\frac{\sigma_z^2}{2}\theta_2^2+\mu\theta_2\right)}\Phi\left(\frac{\mu\sqrt{a}}{\sigma_z}\right)}{\Phi\left(-\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}\right)} = \frac{e^{-a\left(\frac{\sigma_z^2}{2}\theta_2^2+\mu\theta_2+\left(\frac{\mu}{\sigma_z}\right)^2\right)}}{-a\left(\frac{\mu}{\sigma_z}\right)^2}\frac{\Phi\left(\frac{\mu\sqrt{a}}{\sigma_z}\right)}{\Phi\left(-\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}\right)}$$

$$= \frac{m\left(\frac{\mu+\theta_2\sigma_z^2}{\sigma_z}\sqrt{a}\right)}{m\left(-\frac{\mu}{\sigma_z}\sqrt{a}\right)} \tag{39}$$

where m is again the inverse mills ratio. Define $\tilde{c}_1 \equiv \frac{\mu+\theta_2\sigma_z^2}{\sigma_z} > \tilde{c}_2 \equiv -\frac{\mu}{\sigma_z}$. For $\mu < 0$ and (19) $\tilde{c}_1, \tilde{c}_2 > 0$. In order for the derivative of (39) wrt to $\sqrt{a}$ to be positive it must be:

$$\frac{m'\left(\tilde{c}_1\sqrt{a}\right)}{m\left(\tilde{c}_1\sqrt{a}\right)}\tilde{c}_1\sqrt{a} > \frac{m'\left(\tilde{c}_2\sqrt{a}\right)}{m\left(\tilde{c}_2\sqrt{a}\right)}\tilde{c}_2\sqrt{a}\ .$$

This condition is equivalent to the lemma below which completes the proof.

Lemma 7 *Let $y > x > 0$. Then*

$$y\frac{m'(y)}{m(y)} > x\frac{m'(x)}{m(x)} \ ,$$

Proof. Detailed derivations can be found in an online appendix. Here, I sketch the proof. Notice that $m' = m(m-x)$. Using that, $\left(x\frac{m'}{m}\right)' = mx(m-x) - 2x + m$, it suffices to show

$$mx(m-x) - 3m > 2x - 4m \ . \tag{40}$$

It is also true that $m - x > 0$ from property F5 and that (see a lenghty proof by Barrow and Cohen (1954) p. 406 and online appendix) $mx(m-x) - 3m > -\frac{2}{m-x}$. In combination with 40 the last two inequalities imply that it suffices to show

$$-\frac{2}{m-x} > 2x - 4m \implies 0 < (2m-x)(m-x) - 1.$$

This inequality has been proven by Sampford (1953), which completes the proof. ∎

A.5 Expected Growth Rates and Sales

A.5.1 Expected Sales

The purpose is to compute the expected value of expression (8) where I will use the definition of $s_{ija} = \ln z_{\tilde{t}+a}/z^*_{ij\tilde{t}+a}$ (for simplicity denoted by s_a). I focus on deriving the expected value of the two terms inside the brackets since the terms outside the brackets are deterministic. Of course since the term s_{ija} follows a simple Brownian motion with a drift μ and a volatility σ_z I can consider each term separately and calculate $E\left(e^{\bar{c}_i s_{ija}} | s_{ija} \geq 0, s_{ij0} \geq 0\right)$ where $\bar{c}_1 = \sigma - 1$, $\bar{c}_2 = (\sigma - 1)/\tilde{\beta}$ with $\tilde{\beta} = \beta/(\beta - 1)$. Then the terms can be combined and multiplied by the values of the deterministic parameters. Regarding the expected values for $\bar{c}_i$, $i = 1, 2$,

$$E\left(e^{\bar{c}_i s_{ija}} | s_{ija} \geq 0, s_{ij0} = s_0\right) = \int_0^{+\infty} \frac{e^{\bar{c}_i s_a}}{\sigma_z\sqrt{a2\pi}} \frac{e^{-\frac{\left(\frac{s_a - s_0 - \mu a}{\sigma_z\sqrt{a}}\right)^2}{2}}}{\Phi\left(\frac{s_0+\mu a}{\sigma_z\sqrt{a}}\right)} ds_a$$

$$= \frac{e^{\frac{-(s_0)^2 - 2\mu a s_0 - \mu^2 a^2}{\sigma_z^2 a2}}}{\Phi\left(\frac{s_0+\mu a}{\sigma_z\sqrt{a}}\right)\sigma_z\sqrt{a2\pi}} \int_0^{+\infty} \exp\left\{-\frac{1}{\sigma_z^2 a2}(s_a)^2 + \frac{2s_0 + 2\mu a + \bar{c}_i\sigma_z^2 a2}{\sigma_z^2 a2} s_a\right\} ds_a$$

using property F10 and F9 the above expression equals to:

$$\frac{e^{\frac{-(s_0)^2-2\mu a s_0-\mu^2a^2}{\sigma_z^2 a2}}}{\Phi\left(\frac{s_0+\mu a}{\sigma_z\sqrt{a}}\right)\sigma_z\sqrt{a2\pi}}e^{\frac{\left(\frac{2s_0+2\mu a+\bar{c}_i\sigma_z^2 a2}{\sigma_z^2 a2}\right)^2\sigma_z^2 a2}{4}}\frac{\sqrt{\pi}}{2\sqrt{1/\left(\sigma_z^2 a2\right)}}\left[1+\text{erf}\left(\frac{s_0+\mu a+\bar{c}_i\sigma_z^2 a}{\sqrt{\sigma_z^2 a2}}\right)\right]$$

Using F8 this last expression gives

$$E\left(e^{\bar{c}_i s_{ija}}|s_{ija}\geq 0, s_{ij0}=s_0\right)=\exp\left\{\frac{\left(\bar{c}_i\right)^2\sigma_z^2}{2}a+\bar{c}_i s_0+\bar{c}_i a\mu\right\}\frac{\Phi\left(\frac{s_0+\mu a+\bar{c}_i\sigma_z^2 a}{\sigma_z\sqrt{a}}\right)}{\Phi\left(\frac{s_0+\mu a}{\sigma_z\sqrt{a}}\right)} \tag{41}$$

A.5.2 Expected Log Sales

The natural logarithm of sales of the firm for $\beta\to 0$ is given by

$$\ln L_{j\tilde{t}}^{\alpha}y_{j\tilde{t}}^{\gamma}y_{i\tilde{t}}^{1-\gamma}\frac{1}{\tilde{\psi}}+(\sigma-1)\ln s_{ij\tilde{t}}\ .$$

The growth rate of the first term is deterministic and equal to $(\alpha g_\eta+g_\kappa)\,a$. To compute the expected value of the second term I can make use of the moment generating function for some variable $\tilde{s}_a=(\sigma-1)\,s_a$ (MGF) with mean $\tilde{\mu}=(\sigma-1)\,\mu$, variance denoted $\tilde{\sigma}^2=\sigma_z^2 a\,(\sigma-1)^2$ and lower threshold $\tilde{x}=x\,(\sigma-1)$. The MGF is (for some $\tilde{c}\in R$)

$$E\left(e^{\tilde{c}s_a}|s_0,s_a\geq 0\right)=\frac{1}{\tilde{\sigma}\sqrt{2\pi}}\frac{\int_{\tilde{x}}^{\infty}e^{\tilde{c}x}e^{-\frac{1}{2}\left(\frac{x-\tilde{\mu}}{\tilde{\sigma}}\right)^2}dx}{1-\Phi\left(\frac{\tilde{x}-\tilde{\mu}}{\tilde{\sigma}}\right)}$$

$$=e^{-\frac{(\tilde{\mu})^2-\left[(\tilde{\sigma})^2\tilde{c}+\tilde{\mu}\right]^2}{2(\tilde{\sigma})^2}}\frac{\int_{\tilde{x}}^{\infty}\frac{1}{\tilde{\sigma}\sqrt{2\pi}}e^{-\frac{\left[x-(\tilde{\sigma})^2\tilde{c}-\tilde{\mu}\right]^2}{2(\tilde{\sigma})^2}}dx}{1-\Phi\left(\frac{\tilde{x}-\tilde{\mu}}{\tilde{\sigma}}\right)}=e^{\tilde{\mu}\tilde{c}+\frac{\tilde{\sigma}^2\tilde{c}^2}{2}}\frac{1-\Phi\left(\frac{\tilde{x}-\tilde{\mu}-\tilde{\sigma}^2\tilde{c}}{\tilde{\sigma}}\right)}{1-\Phi\left(\frac{\tilde{x}-\tilde{\mu}}{\tilde{\sigma}}\right)}$$

where in the last equality I used the definition of the normal distribution F2.

The MGF can be used to compute the moments of the distribution by computing the successive derivatives of MGF wrt to $\tilde{c}$. I derive the results illustrated in section 4.2.1 by setting the mean, $\tilde{\mu}=(\sigma-1)\,s_0+(\sigma-1)\,\mu a$, the standard deviation, $\tilde{\sigma}=(\sigma-1)\,\sigma_z\sqrt{a}$, and the lower threshold $\tilde{x}=0$.

A.6 Proof of Proposition 5

To prove the proposition I need to show that $\partial\left(\left(g_I - g_E\right)\frac{h'(s)}{h(s)} + \frac{\sigma_z^2}{2}\frac{h''(s)}{h(s)}\right)\Big/\partial s \leq 0$. Extended derivations for this proposition given in an online appendix imply that it is equivalent to show that

$$\left(g_I - g_E\right)(\sigma - 1)\left(\frac{1-\tilde{\beta}}{\tilde{\beta}}\right)e^{-s\frac{(\sigma-1)}{\beta}} + \frac{\sigma_z^2}{2}(\sigma-1)^2\frac{1-\tilde{\beta}^2}{\tilde{\beta}^2}e^{-s\frac{(\sigma-1)}{\beta}} \leq 0$$

so that for $\tilde{\beta} = \beta/(\beta - 1)$ I need to show (notice that $e^{s\frac{(\sigma-1)}{\beta}} \geq 1$, for $s \geq 0$)

$$-\left(\tilde{\beta}\right)^2\left[\left(g_I - g_E\right)(\sigma-1) + (\sigma-1)^2\frac{\sigma_z^2}{2}\right] + \left(g_I - g_E\right)(\sigma-1)\tilde{\beta} + (\sigma-1)^2\frac{\sigma_z^2}{2} \leq 0\ .$$

This expression after some manipulations gives the condition in equation (32). Notice that if $\left(g_I - g_E\right)(\sigma-1) + (\sigma-1)^2\sigma_z^2 < 0$ there does not exist a $\beta \in [0, +\infty)$ that satisfies the inequality.

A.7 Proof of Proposition 6

The proof of the proposition uses Ito's Lemma. In particular, the variance of the instantaneous growth rate of firms is given by the square of the second bracketed term in expression (29). Given (30) this term equals to $\sigma_z^2(\sigma-1)^2$ for $\beta \to 0$. For the second part of the proposition, given $\beta > 0$, the derivative of the term is always negative. Thus, the instantaneous variance of growth rates of firms selling to a destination is inversely related to their size there. In the limit for $s_{ij0} \to +\infty$ the term tends to $\sigma_z^2(\sigma-1)^2$ completing the proof of the proposition.

A.8 Cohort Market Shares

To compute the cohort market share I compute the expected sales of a cohort and divide by the total sales in this market by all firms from a given exporting country. This ratio at year 0 is 1. The growth rate of the total sales of all firms equals the rate of change of nominal GDP, $\kappa + \eta$.

The expected sales of the cohort can be computed by finding the expected sales of firms of different sizes, weighted by the density of initial sizes of these firms effectively using (16) and (41) (the second unconditional on survival). To compute the implied integral I split it again into two parts as I did in the previous section and compute for $i = 1, 2$:

$$\int_0^\infty \theta_2 e^{-\theta_2 (s_0)} e^{a\left(\frac{\bar{c}_i^2 \sigma_z^2}{2} + \bar{c}_i \mu + \bar{c}_i s_0\right)} \Phi\left(\frac{s_0 + \mu a + \bar{c}_i \sigma_z^2 a}{\sigma_z \sqrt{a}}\right) ds_0$$

$$= \frac{\theta_2 e^{a\left(\frac{\bar{c}_i^2 \sigma_z^2}{2} + \bar{c}_i \mu\right)}}{\theta_2 - \bar{c}_i} \left\{ \Phi\left(\frac{\mu a + \bar{c}_i \sigma_z^2 a}{\sigma_z \sqrt{a}}\right) + \int_0^\infty \frac{e^{(\bar{c}_i - \theta_2) s_0}}{\sigma_z \sqrt{a}} \varphi\left(\frac{s_0 + \mu a + \bar{c}_i \sigma_z^2 a}{\sigma_z \sqrt{a}}\right) ds_0 \right\} \quad (42)$$

where the second expression is derived using integration by parts. The integral inside the brackets equals to

$$\frac{e^{\frac{-(\mu a)^2 - \sigma_z^4 (\bar{c}_i a)^2 - 2\mu a \bar{c}_i \sigma_z^2 a}{2\sigma_z^2 a}}}{\sigma_z \sqrt{2\pi} \sqrt{a}} \int_0^\infty \exp\left\{ -\frac{1}{2\sigma_z^2 a} (s_0)^2 - \frac{\theta_2 \sigma_z^2 + \mu}{\sigma_z^2} s_0 \right\} ds_0 =$$

$$\frac{e^{\frac{-(\mu a)^2 - \sigma_z^4 (\bar{c}_i a)^2 - 2\mu a \bar{c}_i \sigma_z^2 a}{2\sigma_z^2 a}}}{\sigma_z \sqrt{2\pi} \sqrt{a}} e^{\left(\frac{\theta_2 \sigma_z^2 + \mu}{\sigma_z^2}\right)^2 / \left[4\left(\frac{1}{2\sigma_z^2 a}\right)\right]} \frac{\sqrt{\pi}}{2\sqrt{\frac{1}{2\sigma_z^2 a}}} \left[1 - \text{erf}\left(\frac{\frac{\theta_2 \sigma_z^2 + \mu}{\sigma_z^2}}{2\sqrt{\frac{1}{2\sigma_z^2 a}}}\right)\right] =$$

$$e^{\left[\frac{\sigma_z^2}{2}(\theta_2 + \bar{c}_i) + \mu\right](\theta_2 - \bar{c}_i) a} \Phi\left(-\frac{\left(\theta_2 \sigma_z^2 + \mu\right) \sqrt{a}}{\sigma_z}\right)$$

where I used the definition of the normal for the first line, the property F10 for the first equality, and property F8 for the second equality. The expression is derived by replacement in expression (42). Notice that the assumption $\theta_2 \geq (\sigma - 1)$ is required that derives from assumption A2. Expression (42) will be added for $i = 1$ and substracted for $i = 2$. Finally, I have to multiply by $e^{g_\kappa + g_\eta \alpha}$ to capture the rate of growth of the average sales of the incumbents and $e^{-\delta a}$ to discount by the exogenous death rate of ideas. This completes the derivation.

B Data Appendix

B.1 Census and Colombian exporting Data

The statistics for the data on US manufacturing firms are constructed using data provided by the census and the paper of Dunne, Roberts, and Samuelson (1988). See also online excel file for details.

The destinations of Colombian exporterting firms include 9 from the top 10 destinations for Colombian exporters: USA, Venezuela, Ecuador, Peru, Mexico, Dominican Republic, Puerto, Panama, and Costa Rica. I exclude the EU as a destination because the data treat EU as one destination instead of separate destination countries. Including the data for the EU would not change the results of my analysis.

B.2 Exit and Entry in the Data and the Model

In the data, firms that stop to operate oftentimes seize to exist. Also the reverse is true: firms that may be recorded as selling in consecutive censuses years do not necessarily sell in all the years between the censuses, which is closer to the interpretation of exit given in this paper. Additionally, reentry is also frequent in the trade data. Using some estimates kindly provided by Maurice Kugler for the Colombian exporting data around 15% of the times that a firm exits a given destination it sells back in the destination in some or all the three consecutive years after the year of exit. This pattern of re-entry is consistent with my model but possibly not with a model with an absorbing barrier as the one of Luttmer (2007) or Irarrazabal and Opromolla (2006).

In addition to the above, there are at least two reasons why the entry-exit process of ideas can be as good or better an approximation as the entry-exit process of firms with an absorbing barrier. First, the way panel information on the sales of firms is usually constructed (see for example Dunne, Roberts, and Samuelson (1988) footnote 13) does not correspond exactly in either model specification. Second, in the calibration that I consider, the growth rate of the frontier of ideas, g_E is much higher than g_I which implies that firms that do exit are likely not to sell again. Finally, both the process that I consider and an absorbing barrier deliver similar predictions in calibrated exercises.[36] Therefore, possible bias is rather small.

[36] The numbers for the calibrated parameters, g_I, g_E and σ_z are slightly different in my exercise compared to those of Luttmer (2007). The difference lies in the fact that the elasticity of the Pareto distribution is much lower in the case of Luttmer (1.06 compared to 8.28 that I use). This low number will not provide a good fit for a variety of facts on international trade and especially for the responses of trade flows to differences in tariffs and prices (see Eaton and Kortum (2002) and Ruhl (2005)).

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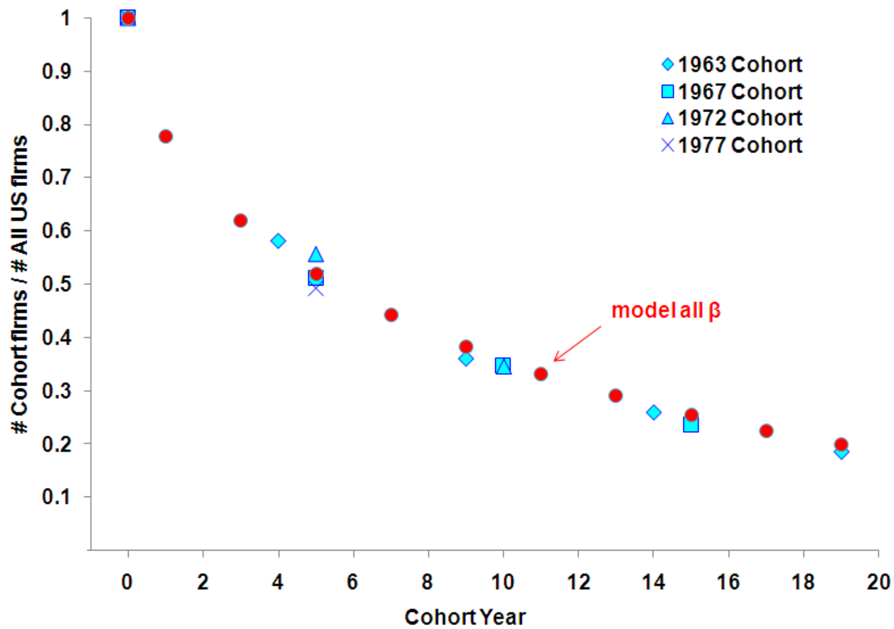

Figure 2: Cohort Survivors and new entry in US manufacturing Census and Model. Data: Dunne, Roberts, and Samuelson (1988)

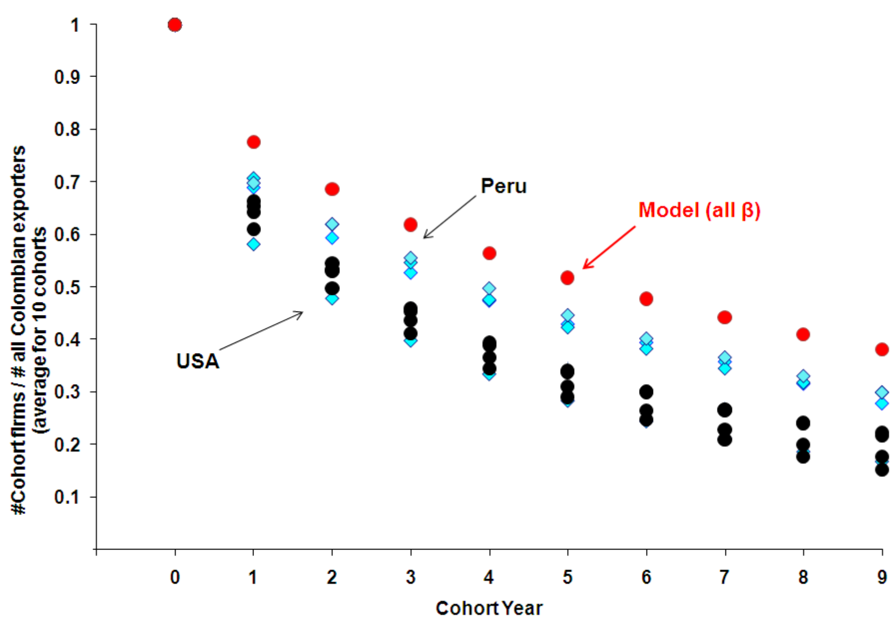

Figure 3: Cohort Survivors and new entry in Colombian exporting data for the most popular exporting destinations for Columbian exporters. Data provided by Eaton, Eslava, Kugler, and Tybout (2008), see appendix B for details.

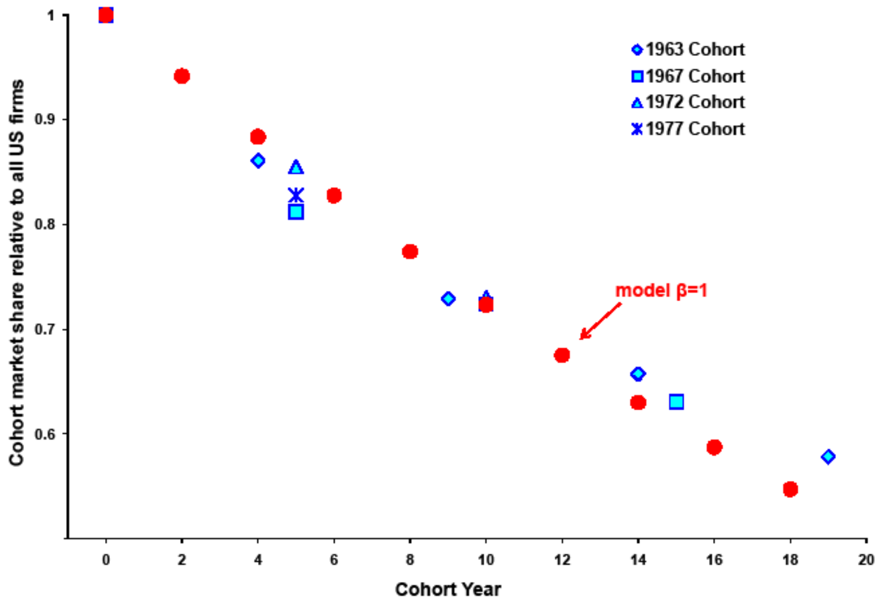

Figure 4: Market share of Cohort Survivors and new entrants in US manufacturing Census and Model. Data: Dunne, Roberts, and Samuelson (1988)

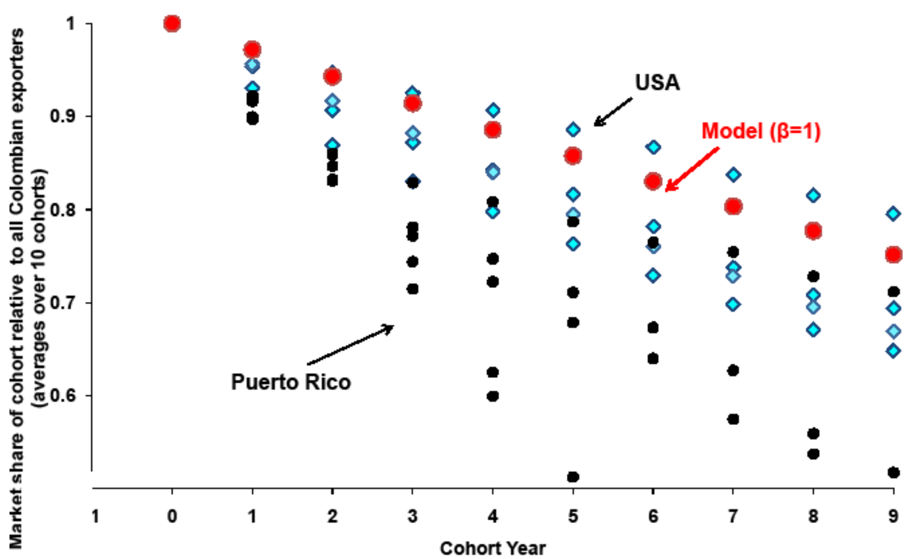

Figure 5: Market share of Cohort Survivors and new entrants into individual destinations for Colombian exporters. Data provided by Eaton, Eslava, Kugler, and Tybout (2008), see appendix B for details.

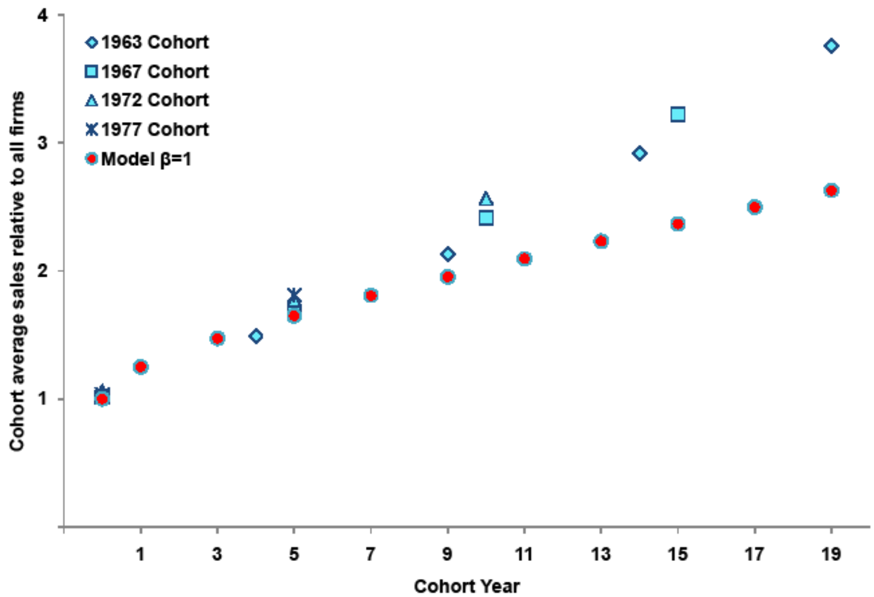

Figure 6: Average sales of Cohort Survivors and all census firms in US manufacturing Census and Model. Data: Dunne, Roberts, and Samuelson (1988), combining table 9 and 10. (Numbers may contain small aproximation error 1%-2% due to rounding).

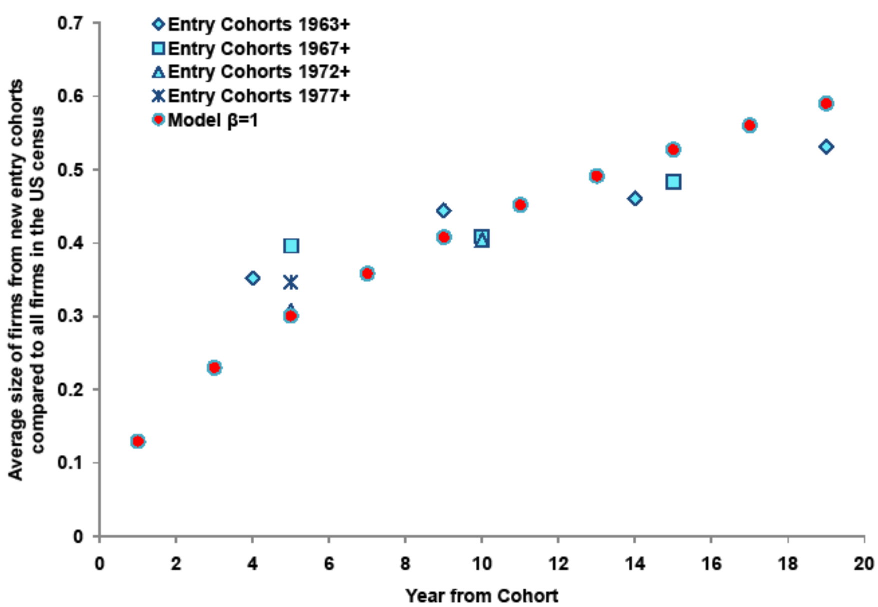

Figure 7: Average sales of firms that are new compared to starting Cohort Survivors and average sales of all census firms in US manufacturing Census and Model. Data: Dunne, Roberts, and Samuelson (1988), combining table 9 and 10. (Numbers may contain small aproximation error 1%-2% due to rounding).