

Capital and Dynamics in the “Universal Gravity” framework

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Abstract

In this note, we show how the steady state equilibrium of a dynamic multi-country trade model with capital markets in financial autarky, a special case of the model of [Takahashi \(2015\)](#), falls into the class of models considered in the “Universal Gravity” framework of [Allen, Arkolakis, and Takahashi \(2014\)](#).

1 Related literature

In this note, we consider a dynamic many country trade model with domestic capital investment and show how it relates to the “Universal Gravity” framework. We first discuss several dynamic trade models which are related with this note.

[Takahashi \(2015\)](#) analyzes a dynamic Armington model with investment. On top of capital investment that paper introduces incomplete financial markets, which are subject to bilateral financial frictions. Such frictions generate bilateral *capital* flows between countries. [Eaton, Kortum, Neiman, and Romalis \(2011\)](#) is also related with this note. The authors add durable goods and capital investment to [Eaton and Kortum \(2002\)](#) and investigate a reason for the trade collapse during the great recession. In their model, Arrow-Debreu state-contingent claims are traded across countries and countries endogenously borrow or lend. Put differently, trade flows need not to be balanced. [Eaton, Kortum, Neiman, and Romalis \(2011\)](#) and [Takahashi \(2015\)](#) analyze a perfect foresight equilibrium; namely, there are aggregate shocks, but no aggregate uncertainty.

[Backus, Kehoe, and Kydland \(1992, 1994\)](#) analyze an economy with Armington aggregation and capital investment under complete financial markets. They show that their model cannot account for consumption correlations among countries; observed correlations are much higher than one predicted by their model. [Heathcote and Perri \(2002\)](#) try to resolve this puzzle by assuming (exogenously) incomplete market assumption instead of complete markets. All these papers feature aggregate shocks and aggregate uncertainty in contrast to [Eaton, Kortum, Neiman, and Romalis \(2011\)](#), [Takahashi \(2015\)](#), and this note.

2 The Universal Gravity framework

We now remind the reader of the universal gravity framework from [Allen, Arkolakis, and Takahashi \(2014\)](#). For a given set of bilateral frictions $\{K_{ij}\}$, income shifters $\{B_i\}$ and gravity constants α and β , the universal gravity framework satisfies the following equilibrium conditions:

1. The value of trade flows between any two locations satisfy the **gravity equation**:

$$X_{ij} = K_{ij}\gamma_i\delta_j, \quad (1)$$

where $\mathbf{K} \equiv \{K_{ij}\}$ is assumed to be exogenous (i.e. it is a model parameter), while $\{\gamma_i\}$ and $\{\delta_j\}$ are endogenous.

2. In all locations, the **goods market clears**:

$$Y_i = \sum_{j \in S} X_{ij}; \quad (2)$$

3. In all locations, **trade is balanced**:

$$Y_i = \sum_{i \in S} X_{ji}; \quad (3)$$

4. In all locations, the **generalized labor market clearing condition** holds:

$$Y_i = B_i\gamma_i^\alpha\delta_i^\beta, \quad (4)$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are the **gravity constants**.

3 Model with Capital

Consider a simple Armington model with N locations.¹ Each location $i \in \{1, \dots, N\}$ is endowed with its own differentiated variety and in period $t \in \mathbb{N}$, L_i^t workers supply their unit labor inelastically and consume varieties from all locations with CES preferences and an elasticity of substitution $\sigma \geq 1$. There are two factors of production, labor L_i^t and capital K_i^t that are combined in a Cobb-Douglas production function to produce output:

$$Y_i^t = (a_i^t L_i^t)^\zeta (K_i^t)^{1-\zeta}.$$

The agents in country i can consume C_i or invest I_i , where I_i is equal to $K_i^t - (1 - \rho_i) K_i^{t-1}$ and ρ_i is the country-specific depreciation rate.

Rather than providing the full optimization problem here, we begin with the following condition which relates the capital levels inter-temporally. In the appendix, we provide a micro-foundation for this condition based on logarithmic utility function and financial autarky.

¹We consider an Armington model in order to have an explicit welfare function, the results that follow will hold for any general equilibrium gravity model where $\beta = 0$.

Condition 1. The consumption for country i is given by

$$P_i^t C_i^t = (1 - \nu_i) (r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1} + w_i^t L_i^t$$

for some constant ν_i . Also the capital for country i is given by

$$P_i^t K_i^t = \nu_i (r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1}. \quad (5)$$

Here ν_i can be interpreted as a time preference for country i . The higher ν_i is, the more patient households in country i are. If ν_i is closed to 0, then the consumption in country i is close to 0. In the limit case as $\nu_i \rightarrow 0$, agents consume immediately.

The total expenditure E_j is expressed as

$$E_j^t = r_j^t K_j^{t-1} + w_j^t L_j^t = \frac{1}{\zeta} w_j^t L_j^t.$$

Perfect competition implies the relative wage is equal to the marginal product of labor:

$$\frac{w_i^t}{p_i^t} = \zeta (a_i^t)^\zeta \left(\frac{K_i^{t-1}}{L_i^t} \right)^{1-\zeta} \triangleq A_i^t, \quad (6)$$

where p_i^t is the price of good i .² The price of good i in country j , p_{ij} , is

$$p_{ij} = \tau_{ij} p_i = \tau_{ij} \frac{w_i}{A_i}.$$

Note that the RHS is the aggregate labor productivity in country i and denoted by A_i^t . Also because K_i^{t-1} is a state variable, A_i^t is determined at time t . Then the trade flow from i to j satisfies equation (1) of the universal gravity framework:

$$X_{ij}^t = \underbrace{(\tau_{ij})^{1-\sigma}}_{=K_{ij}} \underbrace{\left(\frac{w_i^t}{A_i^t} \right)^{1-\sigma}}_{=\gamma_i} \underbrace{(P_j^t)^{\sigma-1} \frac{w_j^t L_j^t}{\zeta}}_{=\delta_j}.$$

Then given A_i , the equilibrium conditions are written as (2), (3), and (4), where

$$Y_i^t = \frac{1}{\zeta} w_j^t L_j^t = \underbrace{\zeta^{-1} A_i^t L_i}_{=B_i} (\gamma_i^t)^{\frac{1}{1-\sigma}} (\delta_i^t)^0,$$

²The price good i is equal to the marginal cost.

$$p_i^t = MC_i^t = \zeta^{-\zeta} (1 - \zeta)^{-(1-\zeta)} \left(\frac{w_i^t}{a_i^t} \right)^\zeta (r_i^t)^{(1-\zeta)}.$$

Also we have

$$\frac{w_i^t}{r_i^t} = \frac{\zeta}{1 - \zeta} \frac{K_i^{t-1}}{L_i^t}.$$

Combining these equations, we get (6). Or we can derive (6) from the FONC w.r.t. L_i^t for the profit maximization problem.

i.e. equation (4) is satisfied with $B_i^t = \zeta^{-1} A_i^t L_i$, $\alpha = \frac{1}{1-\sigma}$ and $\beta = 0$. The goods market clearing (equation (2)) and balanced trade (equation (3)) are also satisfied in every period. Hence, at every date t , the model above satisfies the conditions of the universal gravity framework given A_i^t . However, because B_i^t (or A_i^t) is now endogenous, an additional condition is required to characterize the dynamics This is given by (5):

$$K_i^t = \nu_i \left[\frac{r_i^t}{P_i^t} + (1 - \rho_i) \right] K_i^{t-1}.$$

We can rewrite the rental rate by the labor income $w_i L_i$. Then

$$K_i^t = \nu_i \left[\frac{1 - \zeta}{\zeta} \frac{w_i^t L_i^t}{P_i^t} + (1 - \rho_i) K_i^{t-1} \right]. \quad (7)$$

(3) implies that if the relative price $\frac{w_i^t}{P_i^t}$ is high, then the capital stock tends to increase. Given the complementarity between K_i^{t-1} and L_i^t , lower capital implies the lower wage, and the higher rental rate of capital. Therefore if country i has a lower real wage, then country i 's rental rate of capital is high, which promotes investment.

Now define an equilibrium in this economy. The equilibrium is a sequence of $(w_i^t, P_i^t, A_i^t, K_i^t)_{i,t}$ satisfying equation (2), (3), (4), (6), and (7).

Proposition 2. *Given the initial distribution of capital $(K_i^{-1})_i$, the equilibrium (the sequence of the allocations) is unique if*

$$\sigma > \frac{1}{\gamma}.$$

Proof. Suppose that the equilibrium is not unique. Then there exists A_i^t such that there are multiple solutions to equation (2), (3), and (4). Since the conditions in Theorem 1 in [Allen, Arkolakis, and Takahashi \(2014\)](#) are satisfied, this is a contradiction. \square

4 Steady state

At the steady state, it turns out that this model is isomorphic to an Armington model with intermediate inputs, where the capital at the steady state (K^{SS}) acts like an intermediate input, and the steady state system with capital is included in the class of the problems analyzed in [Allen, Arkolakis, and Takahashi \(2014\)](#).

To derive the equilibrium system at the steady state, note that (7) is reduced to:

$$K_i^{SS} = \frac{\nu_i}{\rho_i} \frac{1 - \zeta}{\zeta} \frac{w_i^{SS} L_i^{SS}}{P_i^{SS}}.$$

Substituting this expression into (6), we get

$$A_i^{SS} = \alpha \left(\frac{\nu_i}{\rho_i} \frac{1 - \zeta}{\zeta} \frac{w_i^{SS}}{P_i^{SS}} \right)^{1-\alpha}. \quad (8)$$

Then it is easy to show that the trade flow is written as follows:

$$X_{ij}^{SS} = \underbrace{\left[\zeta^{-\zeta} (1 - \zeta)^{-(1-\zeta)} \right]^{(1-\sigma)} (\tau_{ij})^{1-\sigma} \left(\frac{V_i}{\rho_i} \right)^{-(1-\gamma)(1-\sigma)}}_{=K_{ij}^{SS}} \underbrace{(w_i^{SS})^{\gamma(1-\sigma)} (P_i^{SS})^{(1-\gamma)(1-\sigma)}}_{=\gamma_i^{SS}} \underbrace{(P_j^{SS})^{\sigma-1} \frac{w_j^{SS} L_j^{SS}}{\zeta}}_{=\delta_j^{SS}}.$$

Therefore, at the steady state, the allocations are characterized by different gravity constants α and β .

$$\begin{aligned} (\alpha^{SS}, \beta^{SS}) &= \left(\frac{1}{1 - \zeta\sigma}, \frac{1 - \zeta}{1 - \zeta\sigma} \right) \\ B_i^{SS} &= \left(\frac{L_i}{\zeta} \right)^{\zeta \frac{1-\sigma}{1-\zeta\sigma}}. \end{aligned}$$

Theorem 1 in [Allen, Arkolakis, and Takahashi \(2014\)](#) implies that the steady state equilibrium is unique if

$$\sigma > \frac{1}{\gamma} > 1. \quad (9)$$

The condition for uniqueness (9) is the same as one in the Armington model with intermediate input. Since the steady state is written in the framework of [Allen, Arkolakis, and Takahashi \(2014\)](#), all theoretical and empirical results in that paper hold at the steady state.

5 Conclusion

This brief note shows how it is possible to incorporate capital into the framework of [Allen, Arkolakis, and Takahashi \(2014\)](#). Off the steady state, the transition dynamics is characterized by the equilibrium conditions of the universal gravity framework along with a simple equation governing the transition of capital. In the steady state, the dynamic model is fully isomorphic to an Armington model with intermediate inputs and all the results presented in [Allen, Arkolakis, and Takahashi \(2014\)](#) hold. While this model is deliberately simple and tractable, this setup is easily generalized to analyze other interesting dynamic phenomena.

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A A micro-foundation of Condition 1

In this section, we present a micro-foundation which delivers Condition 1. Suppose there are two types of agents in each country; workers and entrepreneurs. Workers are hand-to-mouth agents; namely, they consume their labor income immediately. Entrepreneurs own the capital, and make investment. Therefore their maximization problem is as follows:

$$\begin{aligned} \max \quad & \sum_t \nu_i^t \log(C_i^t) \\ \text{s.t.} \quad & P_i^t (C_i^t + K_i^t) \leq (r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1}, \end{aligned}$$

where r_i^t is the rental rate of capital minus depreciation. The associated Lagrangian is

$$\mathcal{L} = \sum_t \nu_i^t [\log(C_i^t) + \lambda_i^t ((r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1} - P_i^t (C_i^t + K_i^t))].$$

Taking the FONCs, we get

$$1 = \beta_i \frac{P_i^t C_i^t}{P_i^{t+1} C_i^{t+1}} \left(\frac{r_i^{t+1} - P_i^{t+1} (1 - \rho_i)}{P_i^t} \right).$$

Now guess the policy functions:

$$P_i^t C_i^t = (1 - \nu_i^t) (r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1} \quad (10)$$

$$P_i^t K_i^t = \nu_i^t (r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1}. \quad (11)$$

Then we can verify that the FONCs are solved by the guessed policy functions.

$$\begin{aligned} & \nu_i \frac{P_i^t C_i^t}{P_i^{t+1} C_i^{t+1}} \left(\frac{r_i^{t+1} - P_i^{t+1} (1 - \rho_i)}{P_i^t} \right) \\ = & \nu_i \frac{(1 - \nu_i) (r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1}}{(1 - \nu_i) (r_i^{t+1} - P_i^{t+1} (1 - \rho_i)) K_i^t} \left(\frac{r_i^{t+1} - P_i^{t+1} (1 - \rho_i)}{P_i^t} \right) \\ = & \frac{\nu_i (r_i^t - P_i^t (1 - \rho_i)) K_i^{t-1}}{P_i^t K_i^t} = 1, \end{aligned}$$

which is desired.

Therefore the consumption policy for entrepreneurs is given by (1), and the total consumption is

$$(1 - \nu_i) (r_i^t - P_i^t (1 - \rho_i)) K_i^t + w_i L_i,$$

which is desired.